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TORI WITH ONE-PARAMETER GROUPS OF MOTIONS

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1. Introduction. A general theory of geodesics was developed by one of the authors in [2]. The spaces considered there are G-spaces, a generalization of Finsler spaces, which in turn comprise the Riemann spaces. A G-space is a metric space satisfying additional postulates which essentially amount to requiring 1) that the space be finitely compact (i.e., complete as this term is used in the foundations of differential geometry) and 2) that the space have locally unique geodesics, where a geodesic is defined as a locally isometric image of an entire euclidean straight line. The geodesics which are isometric in the large to euclidean straight lines are particularly important and are called straight lines. If each geodesic of a G-space is a straight line, then the space is called straight.

The G-spaces whose universal covering spaces are straight are, in the language of the calculus of variations, the spaces without conjugate points. In [2] the general theory of geodesics is applied principally to these spaces and other spaces with a simple behavior of the conjugate points. It is the purpose of the present paper to show that the methods of [2] can also be very effective in establishing the distribution of conjugate points in given cases. We concentrate here mainly on tori with one-parameter groups of motions (a motion of a metric space is a distance preserving mapping of the space on itself). Special tori of this type have been treated in the literature with the standard methods of the calculus of variations, so that the two approaches can be compared.

We prove: Let R be a torus, metrized as a G-space, which possesses a one-parameter group of motions Γ_R. If no orbit of Γ_R is a closed geodesic, then the metric is Minkowskian, i.e., the metric of R allows a transitive abelian group of motions.

If, on the other hand, Γ_R possesses orbits which are closed geodesics, then the identity component of the group of motions Γ corresponding to Γ_R in the universal covering plane P of R possesses orbits which are straight lines. Every geodesic through a point p of such a straight orbit is a straight line. When a point p of P does not lie on a straight

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orbit it lies in a narrowest strip bounded by two such orbits A, B. Then there are two straight lines L and M through p which approach A and B asymptotically. The lines L and M bound the closed angular domain D' containing A and B and the remaining open domain D''. A geodesic K through p and a point of D' is a straight line. Any half geodesic H issuing from p and through a point of D'' lies (except for the point p) in D'' and contains no ray, i.e., no isometric image of a euclidean half line. These terms mean in the language of the calculus of variations that no point of K has a conjugate point on K, and that every point q of H is followed by a conjugate point to q on H. As a simple illustration of this theorem, consider the case where R is a torus in E^3 obtained by revolving about the Z-axis an arbitrary closed curve C lying in the XZ-plane and not intersecting the Z-axis. Then the group Γ_R is the group of rotations with axis Z, and the straight lines which are orbits of Γ correspond to those points of C which have minimal distance from the Z-axis.

The special case of this theorem to which we alluded above is found in Kimball [4]. There R is a torus in E^3 obtained by rotating an analytic, closed, convex curve C about an axis Z (in the plane of C, but not intersecting C), with the additional requirement that R be symmetric with respect to some plane perpendicular to Z. The restrictive assumptions are introduced to make the equations for the conjugate points manageable, whereas our result not only applies to tori obtained by revolving curves of arbitrary shape, but also to tori with non-Riemannian metrics. For the ordinary torus, where C is a circle, our theorem is already found in Bliss [1], who determines the geodesics explicitly in terms of elliptic functions and gives the equations for the conjugate points. Our purely qualitative results do, of course, not compete with those of Bliss.

Hedlund [3], guided by the methods of Morse [5], studies the existence and behavior of straight lines on arbitrary Riemannian tori. These methods are also purely geometric and somewhat related to ours. In fact, in the preliminary discussion of Section 2, where we deal with more general situations, we generalize some of Morse's and Hedlund's results and simplify their proofs; we refer the reader, in particular, to Theorem (2.6) which we believe to be interesting in itself. Throughout the paper we shall freely use the results and terminology of [2].

2. Motions of G-planes. We begin with some facts which can be applied to a much wider class of surfaces than tori and are therefore important for other investigations.

In a G-space R, let G be a straight line and Φ a motion of R. If Φ, reduced to G, is a proper translation of G, then Φ is called an *axial motion* of R and G an *axis* of Φ.

(2.1) *If G is an axis of Φ then for any point x of G*

$$xx\Phi = \inf_{y \in R} yy\Phi .$$

In fact, for any positive integer n

$$xx\Phi^n = n \cdot xx\Phi \leqq xy + \sum_{\nu=0}^{n-1} y\Phi^\nu y\Phi^{\nu+1} + y\Phi^n x\Phi^n$$

$$= 2xy + n \cdot yy\Phi ,$$

so that

$$xx\Phi \leqq (2/n)\, xy + yy\Phi .$$

If G is an axis of Φ then G is also an axis of Φ^ν, where ν is some integer $(\neq 0)$, so that we have the immediate corollary:

(2.2) *Under the assumptions of* (2.1)

$$xx\Phi^\nu = \inf_{y \in R} yy\Phi^\nu .$$

(2.3) *Two axes of the same axial motion are parallel.*

This means: If G^+ and H^+ are two oriented axes of Φ then a co-ray from a point q of H^+ to a positive sub-ray S of G^+ is a positive sub-ray of H^+, and similarly for the opposite orientations G^-, H^- of G^+, H^+; for the terminology compare [2, Section 22 and p. 207].

PROOF OF (2.3): We first observe that for the limit-sphere $K_\infty(q,S)$ the following holds: For ν integral positive $K_\infty(q,S)\Phi^\nu = K_\infty(q\Phi^\nu, S)$, because $K_\infty(q\Phi^\nu, S\Phi^\nu) = K_\infty(q\Phi^\nu, S)$. Now let $K_\infty(q,S)$ intersect G^+ in p. Then by (2.2)

$$p\Phi^{-1}p\Phi^\nu = q\Phi^{-1}q\Phi^\nu \geqq q\Phi^{-1}K_\infty(q\Phi^\nu, S) \geqq p\Phi^{-1}p\Phi^\nu$$

which shows that $q\Phi^\nu$ is a foot of $q\Phi^{-1}$ on $K_\infty(q\Phi^\nu, S)$. A similar argument shows that $q\Phi^\nu$ is also a foot af q on $K_\infty(q\Phi^\nu, S)$. If follows (see [2, (20.6)]) that $q\Phi^\nu$ is the unique foot of q on $K_\infty(q\Phi^\nu, S)$. This implies that the co-ray to G^+ through q is the positive sub-ray of H^+ beginning at q.

(2.4) *In a G-space R let Φ be a motion and p a point for which $0 < pp\Phi = \inf_{x \in R} xx\Phi$. If T is any segment from p to $p\Phi$, then the curve $\bigcup_{\nu=-\infty}^{\infty} T\Phi^\nu$ is a geodesic.*

PROOF: If y is an interior point of T then we will prove $(yp\Phi y\Phi)$, so that (see [2, (6.7)]) $T' = T(y, p\Phi) \cup T(p\Phi, y\Phi)$ is a segment; this implies that $\bigcup_{\nu=-\infty}^{\infty} T\Phi^{\nu}$ is a geodesic. Thus it suffices to prove that $(yp\Phi y\Phi)$ (and $yy\Phi = pp\Phi$), which is seen as follows:

$$pp\Phi = py + yp\Phi = yp\Phi + p\Phi y\Phi \geqq yy\Phi \geqq pp\Phi ,$$

hence we have

$$yp\Phi + p\Phi y\Phi = yy\Phi = pp\Phi .$$

A G-space homeomorphic to a plane will briefly be called a G-plane.

(2.5) *Let Φ be an orientation preserving motion of a G-plane. If G is an axis of Φ^{λ}, where λ is some integer greater than* 1, *then G is also an axis of Φ.*

PROOF: Assume G were not an axis of Φ, then

$$G' = G\Phi \neq G$$

and

$$G'\Phi^{\lambda} = G\Phi\Phi^{\lambda} = G\Phi^{\lambda}\Phi = G\Phi = G' ;$$

hence G' is an axis of Φ^{λ} and, according to (2.3), G' is parallel to G. Let $\mathscr{H}$ be the half-plane bounded by G and containing G'. Since Φ preserves the orientation, $\mathscr{H}\Phi$ is the half-plane bounded by G' and not containing G. Repeating this argument we conclude that $\mathscr{H}\Phi^{\lambda}$ is a half-plane not containing $G, G\Phi, \ldots, G\Phi^{\lambda-1}$, hence $\mathscr{H}\Phi^{\lambda} \neq \mathscr{H}$, which contradicts $G\Phi^{\lambda} = G$. We will see in the next section that this simple argument can be used to generalize some results of Morse and Hedlund, whose original proofs are rather involved.

The next result strengthens (2.4) in case Φ is an orientation preserving motion of a G-plane and is basic for all that follows.

(2.6) THEOREM: *If Φ is an orientation preserving motion of a G-plane P, p a point for which*

$$0 < pp\Phi = \inf_{x \in P} xx\Phi ,$$

and T a segment $T(p, p\Phi)$, then

$$G = \bigcup_{\nu=-\infty}^{\infty} T\Phi^{\nu}$$

is a straight line (and Φ is axial with axis G).

The proof proceeds in several steps. Put generally $p\Phi^{\nu} = p^{\nu}$, $T\Phi^{\nu} = T^{\nu}$, etc.

(a) *T is the only segment from p^0 to p^1.*

For, if S is another segment joining p^0 and p^1, then

$$\bigcup_{\nu=-\infty}^{\infty} S^\nu \quad \text{and} \quad \bigcup_{\nu=-\infty}^{\infty} T^\nu$$

are, by (2.4), geodesics, hence *cross* each other at p^1. Consequently, T traversed from p^0 to p^1 followed by S traversed from p^1 to p^0 would be an oriented, simple, closed curve whose image under Φ is an oriented curve with the opposite orientation, so that Φ would not preserve the orientation.

(b) *G has no multiple points.*

For otherwise G would contain a simple monogon M. Because of (2.4) we may assume that p is the vertex of M. There are two possibilities:

1) $p = p^0 = p^\nu$ for some $\nu \geqq 2$. Then

$$T^{\nu-1} = T(p^{\nu-1}, p^\nu), \qquad T^\nu = T(p^\nu, p^{\nu+1}) = T(p^0, p^1) = T\,,$$

hence Φ maps $M = \bigcup_{j=0}^{\nu-1} T^j$ (which is really a closed geodesic) on itself and the interior of M on the interior of M, but then Φ would, according to a well-known result of Brouwer, have a fixed point inside or on M, which contradicts $\inf_{x \in P} xx\Phi > 0$.

2) If $p = p^0$ is not a p^ν, then $(p^{\nu-1}p^0p^\nu)$ for some ν. But, according to (2.4) and (a) above, the only segment from p^0 to p^1 is $T(p^0, p^\nu) \cup T(p^\nu, p^1)$ which, therefore, coincides with T, so that G is again a closed geodesic, and this leads to the same contradiction as before.

(c) *The arc A^ν of G from p^0 to p^ν is a segment.*

We shall prove this by induction. So assume the assertion to be true for $r-1$. If A^r were not a segment, draw a segment S from p^0 to p^r, so that $p^0p^r = \text{length of } S < r \cdot p^0p^1$.

S has no other point in common with A^r than p^0 and p^r. For S cannot contain *interior* points of the arcs of G from p^0 to p^{r-1} or from p^{r-1} to p^r, because these are segments, so that S would coincide with A^r. The same argument holds for the point p^{r-1}, since $T(p^0, p^{r-1})$ and $T(p^{r-1}, p^r)$ are unique (see (a)), hence would lie on S.

Since G has no multiple points and Φ preserves orientation, we have (at least locally) a definite side of G defined, hence the segment $S = S^0$ and the curve S^1 intersect at some point q^0 for which

$$q^0\Phi = q^1 = S^1 \cap S^2\,.$$

Finally, using the inductive assumption $p^1p^r = (r-1) \cdot p^0p^1$, we get

$$r\cdot p^0p^1 > p^1p^{r+1} = p^1q^0+q^0q^1+q^1p^{r+1} = p^1q^0+q^0q^1+q^0p^r$$
$$\geqq p^1p^r+q^0q^1 = (r-1)\cdot p^0p^1+q^0q^1$$

or $p^0p^1 > q^0q^1$ which contradicts

$$p^0p^1 = \inf_{x\in P} xx\Phi .$$

(2.7) COROLLARY: *If $\Gamma=\{\psi_t\}$ (where $-\infty<t<\infty$, $\psi_{t_1+t_2}=\psi_{t_1}\psi_{t_2}$ are always understood) is a one-parameter group of motions of a G-plane, and if a number t_0 and a point p exist such that*

$$0 < pp\psi_{t_0} = \inf_{x\in P} xx\psi_{t_0}$$

then the orbit $\{p\psi_t, -\infty<t<\infty\}$ is a straight line.

PROOF: The motions ψ_t preserve orientation because this is true for small t. By (2.6) the points $p\psi_{\nu t_0}$, $-\infty<\nu<\infty$, lie on a straight line G; the same holds by (2.5) for $p\psi_{\nu t_0/\lambda}$ for any positive integer λ and any integer ν; hence $p\psi_t$ lies for any t on G.

In the next proof we shall use the following well-known lemma (see [2, (39.9)])

(2.8) *If Φ is an axial motion of a G-space with axis A, and ψ is any motion, then $\psi^{-1}\Phi\psi$ is axial with axis $A\psi$.*

(2.9) *Let Φ be an axial motion with axis A of the G-plane P. Let $\Gamma=\{\psi_t\}$ be a one-parameter group of motions of P which commute with Φ and such that $A\psi_{t_0}\neq A$ for at least one t_0. Then the lines $A_t=A\psi_t$ are all distinct and are axes of Φ, hence parallels. They cover P simply. No ψ_t, $t\neq 0$, has a fixed point.*

PROOF: The hypothesis implies $A_{t_0/\nu}\neq A$ for positive integral ν. The line A_t is an axis of Φ, for it is, by lemma (2.8), an axis of $\psi_t^{-1}\Phi\psi_t=\Phi$ (since $\Phi\psi_t=\psi_t\Phi$). Denote by $\mathscr{H}_0$ the half-plane bounded by A and containing A_{t_0}, and put $\mathscr{H}_t=\mathscr{H}_0\psi_t$. Because ψ_{t_0} preserves the orientation

$$\mathscr{H}_{t_0}\subset\mathscr{H}_0, \qquad \mathscr{H}_{\nu t_0}\subset\mathscr{H}_{\mu t_0}$$

for any integers $\nu>\mu$. Hence

$$A_{\mu t_0}\neq A \quad \text{for} \quad \mu\neq 0, \qquad \text{and} \qquad A_{\mu t_0/\nu}\neq A \quad \text{for} \quad \mu\neq 0,$$

and any positive integer ν. Consequently

$$A_t\neq A \qquad \text{for} \qquad t\neq 0 .$$

If $p\in A$ put $T=T(p,p\Phi)\subset A$. Then $AA_t=TA_t$ which shows that

$AA_t > 0$ for $t \neq 0$. Similarly $A_{\nu t_0}A_{(\nu+1)t_0} = AA_{t_0} > 0$. Since $\mathscr{H}_{(\nu+1)t_0} \subset \mathscr{H}_{\nu t_0}$ the lines $A_{\nu t_0}$ form a "monotone" sequence. If $A_{\nu t_0}$ converges as $\nu \to \infty$ then $\bigcap_{\nu=1}^{\infty} \mathscr{H}_{\nu t_0} \neq 0$ is a half-plane bounded by the line $B = \lim_{\nu\to\infty} A_{\nu t_0}$. This is, however, impossible, because in that case

$$BA_{\nu t_0} \to 0 \quad \text{as} \quad \nu \to \infty ,$$

but

$$BA_{\nu t_0} \geqq A_{(\nu+1)t_0}A_{\nu t_0} = AA_{t_0} > 0 .$$

Hence $\bigcap_{\nu=1}^{\infty} \mathscr{H}_{\nu t_0} = 0$ and $A_{\nu t_0}$ diverges. Obviously the same is true of $A_{\nu t_0/\mu}$, where μ is any fixed positive integer, $-\infty < \nu < \infty$. This implies that A_t, $-\infty < t < \infty$, cover the whole plane P. The A_t cover P simply because they are parallel; this also implies that no ψ_t has a fixed point.

We conclude this section with the following observation.

(2.10) *If $\Gamma = \{\psi_t\}$ is a one-parameter group of motions of a G-plane P, and C denotes an orbit $\{p_t\} = \{p\psi_t\}$, then any segment T either lies on C or has at most two points in common with C.*

Proof: If any proper subsegment T' of the segment T lies on the orbit C, put $T' = T(a, a\psi_{t_0})$. Then $C = \bigcup_{-\infty<\tau<\infty} T'\psi_{\tau t_0}$. Hence C is a geodesic which must coincide with the geodesic determined by the segment T, so that $T \subset C$.

For an indirect proof of the second part of the assertion assume that T contains three distinct points a, b, c. Then u, v with (aub) and (bvc) and not on C exist. Traversing T from u towards a and b we reach first points a', b' on C, similarly b'', c'' are the first points of C reached when traversing T towards b and c, respectively. If t is sufficiently small, then $T(a', b')$ and $T(a_t', b_t')$ as well as $T(b'', c'')$ and $T(b_t'', c_t'')$ intersect at interior points. But this leads to a contradiction since it would imply that T and $T\psi_t$ have two common interior points.

3. Geodesics on a G-surface. In this section we shall briefly state some general implications of the results of Section 2.

Let R be an orientable, not simply connected G-surface whose universal covering space is a G-plane P (not necessarily straight), that is, R is topologically any orientable surface except the plane or sphere. We shall interpret the fundamental group $\mathfrak{F}$ of R as the group of covering motions of P, and use the correspondence between free homotopy classes of R and classes of conjugate elements of $\mathfrak{F}$. The motions in $\mathfrak{F}$ preserve orientation because R is orientable.

If the free homotopy class of the closed curve K in R contains a shortest curve K', then K' is a closed geodesic (see [2, (32.1)]). When Φ is an element of the class of conjugate elements in $\mathfrak{F}$ determined by the closed curve K, then a point p of P with $pp\Phi = \inf_{x\in P} xx\Phi$ exists if and only if the free homotopy class of K contains a shortest curve. By (2.6) the geodesics in P lying over this curve are straight lines.

Under the present assumptions it then follows from (2.6) that if the homotopy class determined by K contains a shortest curve, then the corresponding motion Φ is axial, and from (2.1) and (2.2) that a closed geodesic belonging to K traversed ν times is the shortest curve belonging to νK. Finally from (2.5): if K is freely homotopic to $\lambda \overline{K}$ then a shortest geodesic in K is the shortest geodesic in $\overline{K}$ traversed λ times. This was proved for compact Riemannian surfaces of genus greater than one by Morse [5], and for those of genus one by Hedlund [3].

For applications in the next section we also mention that, if R is compact, then every motion in $\mathfrak{F}$ is axial (compare (2.6)).

4. Tori with one-parameter groups of motions. Let R be a torus metrized as a G-space which possesses a one-parameter group $\Gamma_R = \{\psi_t'\}$ of motions. Denote by P the universal covering plane of R; then the motions of P over $\{\psi_t'\}$ form a group (see [2, (28.9)]). The identity component of this group is a one-parameter group $\Gamma = \{\psi_t\}$ of motions of P. The elements of Γ commute with the motions of $\mathfrak{F}$ (see [2, (28.11)]).

Since the axes of motions in $\mathfrak{F}$ are not all parallel there is at least one axis of a motion in $\mathfrak{F}$ which does not go into itself under ψ_t. By (2.9) no ψ_t, $t \neq 0$, has a fixed point. For each t a point $p \in P$ with

$$0 < pp\psi_t = \inf_{x\in P} xx\psi_t$$

exists, because there is a p with

$$pp\psi_t = \inf_{x\in F} xx\psi_t ,$$

where F is a bounded, closed, fundamental set (e.g. a parallelogram) of R in P. If y is an arbitrary point in P then a motion $\Phi \in \mathfrak{F}$ with $y\Phi \in F$ exists, and

$$yy\psi_t = y\Phi y\psi_t\Phi = y\Phi y\Phi\psi_t \geqq pp\psi_t .$$

The points $p_t = p\psi_t$ then form a straight line G. Using (2.9) again we can say that, if G is not an axis of the motion Φ in $\mathfrak{F}$, then the axes of Φ (namely the images under ψ_t of an axis of Φ) cover the plane P simply and G intersects these axes. Thus $xx\Phi$ is constant, since there is an axis of Φ through x and $x\Phi$ and $xx\Phi = \inf_{y\in P} yy\Phi$.

Two cases are now possible:

(1) G is not an axis of any motion in $\mathfrak{F}$,

(2) G is an axis of some element in $\mathfrak{F}$.

We shall first discuss the implications of (1).

Obviously, G cannot pass through any pair of points x and $x\Phi$, where Φ is any motion in $\mathfrak{F}$, because this would imply the existence of two different straight lines through x and $x\Phi$.

Choose any point $p \in G$ and two motions Φ_1 and Φ_2 in $\mathfrak{F}$ with different axes. Denote by A and B the axes of Φ_1 and Φ_2 through p. Form the intersections $p_\nu = G \cap B\Phi_1^\nu$, $\nu = \pm 1, \pm 2, \ldots$, and put $p_0 = p$. Finally, let the axes $B\Phi_1^\nu$ intersect A in q_ν.

For each p_i there is a motion of the form $\Phi_1^{-i}\Phi^{\pm\mu}$ which carries p_i into a point p_i' of the segment $T(p, p\Phi_2)$. No two such points, p_i' and p_j', $i \neq j$, can coincide, because then a suitable motion $\Phi_1^{-i}\Phi_2^{\mu'}\Phi_1^{j} \in \mathfrak{F}$ would carry $p_i \in G$ into $p_j \in G$, which was just seen to be impossible.

But for any p_i' and p_j', $i \neq j$, there is a translation along B (namely $\Phi_1^{i}\psi_t\Phi_1^{-j}\Phi_2^{\mu}$, t and μ depending on i, j) which carries p_i' into p_j'. The set $\{p_i'\}$ has at least one accumulation point on $T(p, p\Phi_2)$, which implies that arbitrarily small translations along B exist. It follows that all translations of P along B exist. In addition, all translations of P along $G \neq B$ exist. These translations generate together a transitive abelian group of motions of P, so that P is a Minkowski plane (see [2, (50.1)]). Thus in case (1) we have proved:

(4.1) THEOREM: *If a G-torus R possesses a one-parameter group of motions and no orbit of this group is a closed geodesic, then the metric of R is Minkowskian.*

Now consider the case (2), where G is the axis of some $\Phi \in \mathfrak{F}$. Choose a point p not on an axis of Φ. Since the axes of Φ form a closed set, there is an (oriented) axis A^+ of Φ closest to p "from above" and an axis B^+ of Φ closest to p "from below".

(4.2) *Every point p between A and B lies on two lines L^+ and M^+ such that every positive sub-ray of L^+ is a co-ray to A^+ and every negative sub-ray of L^+ is a co-ray to B^-; for M^+ the roles of A^+, A^- and B^+, B^- are interchanged; $LA = LB = MA = MB = 0$.*

PROOF: Choose a motion $\Phi_1 \in \mathfrak{F}$ whose axes are not axes of Φ (A, B are axes of Φ) and such that the points $q_\nu = p\Phi_1\Phi^\nu$ lie on the same side of B as A does. There is an axis through p and q_ν and, as $\nu \to \infty$, the line $g(p, q_\nu)$ converges to a line L. For each ν the line $g(p, q_\nu)$ intersects A and all parallels to A, in particular B (compare (2.9)). The limit line L intersects neither A nor B, because it does not intersect A, and A and B

are parallel. If we choose r_ν on $g(p,q_\nu)$ such that $(r_\nu p q_\nu)$ and $pr_\nu =$ constant then the sub-ray of L from $r = \lim_{\nu\to\infty} r_\nu$ through p is a co-ray to A^+ (for proper orientation of A) and hence the sub-ray from p of this co-ray is the unique co-ray from p to L (see [2, (22.19)]). Similarly the opposite sub-ray of L from p is the co-ray to B^-. The construction of M is analogous.

If for increasing t, $u\psi_t$, $u \in A$, traverses A^+ in the positive sense, then $L\psi_t$ lies for all negative t between L and A, moreover $L\psi_{t_1}$ lies for $t_1 < t$ between $L\psi_t$ and A. Therefore

$$A' = \lim_{t\to-\infty} L\psi_t \quad \text{exists}.$$

A' is a straight line, which goes into itself under all ψ_t, hence A' is an axis of the ψ_t and therefore parallel to A. Since A was an axis of Φ (or ψ_t) closest to p it follows that $A' = A$. This implies

$$LA = L\psi_t A\psi_t = L\psi_t A \to 0 \quad \text{as} \quad t \to -\infty .$$

Hence $LA = 0$; similarly for $LB = MA = MB = 0$. This finishes the proof of (4.2).

We now consider a motion $\Omega = \Phi_1 \psi_{t_0}$, where $t_0 \neq 0$ and $\Phi_1 \in \mathfrak{F}$, but, as in (4.2), $\Phi_1 \neq \psi_t$ for all t. Obviously Ω preserves the orientation, and it has no fixed point, because $y\Phi_1 = y\psi_{-t_0}$ would, in contradiction to (2.9), imply that the axes of Φ_1 through y and $y\psi_{-t_0} \neq y$ have the common point $y\Phi_1$. Moreover, Ω commutes with all elements of $\mathfrak{F}$. It follows as in the beginning of the section, that a point q with $0 < qq\Omega = \inf_{x\in P} xx\Omega$ exists. The theorems (2.6) and (2.9) yield then, that Ω has an axis G_Ω and that the lines $G_\Omega \psi_t$ are also axes of Ω and cover P. Therefore there is an axis of Ω through a given pair of points z, $z\Omega$.

It should be noticed that it follows from the preceding remarks that for each $t \neq 0$ we can construct a family of parallel, straight lines covering P, namely the axes of the motion $\Phi_1\psi_t$.

Now let a point p lie on an orbit of ψ_t which is a straight line L. The axis of $\Phi_1\psi_t$ through p depends continuously on t and tends for $t \to \infty$ or $t \to -\infty$ to the line L, because the point $p\Phi_1\psi_t$ traverses a parallel to L. Hence, every point y not on L lies, for a suitable t, on an axis of $\Phi_1\psi_t$ through p. Thus all geodesics through p are straight lines and axes of motions $\Phi_1\psi_t$, where $\Phi_1 \in \mathfrak{F}$. Thus:

(4.3) *If p lies on an orbit of ψ_t which is a straight line, then every geodesic through p is a straight line and axis of a suitable motion $\Phi_1\psi_t$, where $\Phi_1 \in \mathfrak{F}$.*

We next turn to the situation of (4.2) and preserve the notations of (4.2). Denote by D' the closed, angular domain bounded by L and M which contains A and B. Let again $\Phi_1 \in \mathfrak{F}$, but $\Phi_1 \neq \psi_t$ for all t. By the construction of A and B the points $p\Phi_1\psi_t$ lie all in D'. The axis of $\Phi_1\psi_t$ through p intersects A (and B), depends continuously on t, and tends for $t \to \infty$ to one of the lines L, M and for $t \to -\infty$ to the other. Hence, given any interior point y of D' there is a t such that y lies on an axis of $\Phi_1\psi_t$ through p.

Any geodesic which contains a proper segment in D' with center p is therefore a straight line which lies entirely in D'. A geodesic containing a proper segment with center p which does not lie on D' has no common point with D' other than p. Otherwise it would contain a sub-curve C beginning at p with endpoint on L or M, say L, such that C lies, except for its endpoints, in $D'' = P - D'$. Then a $t \neq 0$ would exist such that $L\psi_t$ contains a point c of C and the closed strip bounded by L and $L\psi_t$ contains C. Then C would not cross $L\psi_t$ at c.

Thus any geodesic through p which contains a point of D'' lies, except for p, entirely in D'', in fact, a half geodesic H with origin p and containing a point of D'' stays (except for p) entirely in one component of D''; otherwise a suitable axis of Φ_1 would touch a sub-curve of this half geodesic without crossing it. Moreover, such a half geodesic H cannot contain a ray. Since $H - p \subset D''$ it suffices to show that H is no ray, and this may be seen as follows:

Assume $H - p$ lies in the component of D'' bounded by co-rays from p to A^+ and B^+. Then H has positive distance from A and B, because otherwise it would be a co-ray to A^+ or B^+, see [2, (22.22)], whereas the co-rays from p to A^+ and B^+ are sub-rays of L and M, see [2, (22.19)]. As in the proof of (4.2) we see that $H' = \lim_{\nu \to \infty} H\Phi^{-\nu}$ exists. It would be a straight line with positive distance from A and B which goes into itself under Φ, and hence would be an axis of Φ, contrary to the definition of A and B.

(4.4) *Under the hypothesis and with the notations of* (4.2) *every point in the interior of the closed angular domain D' bounded by L and M, and containing A and B, lies on a straight line through p which is an axis of a suitable motion $\Phi_1\psi_t$, where $\Phi_1 \in \mathfrak{F}$ and $\Phi_1 \neq \psi_t$.*

A half geodesic with origin p which contains a point of $D'' = P - D'$ lies, except for p, entirely in one component of D'' and contains no ray.

We notice the following corollary of (4.1) and (4.3):

(4.5) *If a G-torus possesses a one-parameter group of motions and all* *orbits of this group are geodesics, then the universal covering space of the torus is straight.*

PROOF: If no orbit is a closed geodesic then (4.5) follows from (4.1). If all orbits are geodesics and one is a closed geodesic, then it suffices, because of (4.3), to show that in the universal covering space all orbits of ψ_t are straight lines. But this is obvious: if a segment T connecting two points of an orbit did not lie entirely on this orbit, then we could find another orbit of ψ_t which touches T in an interior point without crossing it, but the orbit is, by hypothesis, a geodesic.

If the metric of R is Riemannian then it is euclidean, because the orbits are equidistant geodesics. But the metric need not be Minkowskian, in fact, the geodesics need not even satisfy Desargues' theorem if the metric is *not* Riemannian, see [2, (33.5)].

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