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ON GROUPS WITH FULL BANACH MEAN VALUE

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1. Introduction. In a recent paper [3, pp. 14–15] the author proved the following

THEOREM 1. *Let L be any right-translation invariant linear space of bounded real functions on a group G. A necessary and sufficient condition that there exist a real functional Mf on L with the properties*

$$\begin{aligned} \inf_x f(x) \leqq Mf &\leqq \sup_x f(x) \\ M\{f(xa)\} &= M\{f(x)\} \\ M\{\lambda f\} &= \lambda Mf \qquad (\lambda \text{ real}) \\ M\{f+g\} &= Mf+Mg \end{aligned}$$

is that $\sup_x H(x) \geqq 0$ *for every function* $H(x)$ *of the form*

$$H(x) = h_1(x) - h_1(xa_1) + \ldots + h_n(x) - h_n(xa_n) \tag{0}$$

where $h_1, \ldots, h_n$ are arbitrary functions from L and $a_1, \ldots, a_n$ are arbitrary elements from G.

The functional Mf is called a right-invariant Banach mean value on L.

We shall say that G has a full Banach mean value if there exists a bi-invariant Banach mean value on the space of all bounded (real) functions on G.

It is known [1, p. 116] that if there exists a right-invariant Banach mean value M on all bounded functions on G, there exists also a bi-invariant Banach mean value on all bounded functions on G, viz.

$$M_1 f = \underset{x}{M}\,\underset{s}{M}\{f(x^{-1}s)\}\,.$$

In fact,

$$M_1\{f(xa)\} = \underset{x}{M}\,\underset{s}{M}\{f(x^{-1}sa)\} = \underset{x}{M}\,\underset{s}{M}\{f(x^{-1}s)\} = M_1 f$$

and

$$M_1\{f(ax)\} = \underset{x}{M}\,\underset{s}{M}\{f(ax^{-1}s)\} = \underset{x}{M}\,\underset{s}{M}\{f(x^{-1}s)\} = M_1 f\,.$$

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The aim of the present paper is to characterize those groups G which have a full Banach mean value.

We note (see for instance [3, pp. 7–9]) that every abelian group has a full Banach mean value. For further orientation we shall prove now three simple theorems. Only the first of them will be used in the sequel.

THEOREM 2. *If a group G has a full Banach mean value M, then every subgroup G^* of G has also a full Banach mean value.*

PROOF. We choose an arbitrary, but fixed, element x_0 in each left coset of G^* different from G^*. An arbitrary bounded function $f(x)$ on G^* can then be extended to a function $f_1(x)$ on G by defining $f_1(x)$ at the element $x=x_0y$, $y\in G^*$ by $f_1(x)=f(y)$ (while $f_1(x)=f(x)$ for x in G^*). If a is in G^*, the function $f(xa)$ extends in this way to the function $f_1(xa)$. Thus $M_1f=Mf_1$ is a right-invariant Banach mean value defined on all bounded functions $f(x)$ on G^*. Hence G^* has a full Banach mean value.

THEOREM 3. *If a group G has a full Banach mean value M and H is a normal subgroup of G, then the factor group G/H has also a full Banach mean value.*

PROOF. Let $h(x)$ be the natural mapping of G on G/H. Then

$$M_1f = M\{f(h(x))\}$$

can be used as a full Banach mean value on G/H.

THEOREM 4. *Let H be a normal subgroup of the group G. A necessary and sufficient condition that G have a full Banach mean value is that both H and the factor group G/H have a full Banach mean value.*

PROOF. It follows from Theorem 2 and Theorem 3 that the condition is necessary. In order to prove that the condition is sufficient we assume that H has a full Banach mean value M and G/H has a full Banach mean value M_1. Let again $h(x)$ be the natural mapping of G on G/H. For a bounded function $f(x)$ on G we put

$$M_2f = \underset{y\in G/H}{M_1}\{\underset{x\in H}{M}\{f(xa_y)\}\}$$

where a_y is an arbitrary element in G for which $h(a_y)=y$. The inner mean value in the expression for M_2f depends only on y, and not on a_y, since M is an invariant Banach mean value on H. Further

$$M_2\{f(xa)\} = \underset{y\in G/H}{M_1}\{\underset{x\in H}{M}\{f(xa_ya)\}\}$$
$$= \underset{y\in G/H}{M_1}\{\underset{x\in H}{M}\{f(xa_{yh(a)})\}\} = \underset{y\in G/H}{M_1}\{\underset{x\in H}{M}\{f(xa_y)\}\} = M_2\{f(x)\},$$

since M_1 is an invariant Banach mean value on G/H. Thus $M_2 f$ is a right-invariant Banach mean value defined on all bounded functions $f(x)$ on G. Hence G has a full Banach mean value.

2. Statement of the Main Theorem. As mentioned above, our aim is to characterize the groups which have a full Banach mean value. This is done in the following theorem and remark.

MAIN THEOREM. *A necessary condition that a group G have a full Banach mean value is that for every k in the interval $0<k<1$, and arbitrary, finitely many, elements $a_1, \ldots, a_n$ from G, there exists a finite subset E of G such that*

$$N(E \cap Ea_i) \geqq kN(E) \quad \text{for} \quad i = 1, \ldots, n, \tag{1}$$

where $N(\ \ldots\ )$ denotes the number of elements in the set between the brackets.

A sufficient condition that a group G have a full Banach mean value is that there exists a number k_0 in the interval $0<k_0<1$ such that for arbitrary, finitely many, not necessarily different, elements $a_1, \ldots, a_n$ from G there exists a finite subset E of G such that

$$n^{-1} \sum_{i=1}^{n} N(E \cap Ea_i) \geqq k_0 N(E) . \tag{2}$$

REMARK. Obviously the inequality (1) implies the inequality (2) with $k_0 = k$. Hence it follows from the Main Theorem *that either of the two conditions in the Main Theorem is actually both necessary and sufficient that G have a full Banach mean value.*

3. Proof of the sufficiency part of the Main Theorem. In this section we shall prove the second part of the Main Theorem, i.e., we shall prove that the condition (2) is a sufficient condition that G have a full Banach mean value. Thus we assume the condition (2) fulfilled: There exists a k_0 in the interval $0<k_0<1$ with the property that for any, finitely many, not necessarily different, elements $a_1, \ldots, a_n$ from G there exists a finite subset E of G with

$$n^{-1} \sum_{i=1}^{n} N(E \cap Ea_i) \geqq k_0 N(E) . \tag{2}$$

On account of Theorem 1 and the remark following it we have to show that $\sup_x H(x) \geqq 0$ for every $H(x)$ of the form (0) where now $h_1, \ldots, h_n$ are arbitrary bounded functions on G.

We assume, to the contrary, that there exists an $\varepsilon_0 > 0$ such that

$H(x) \leqq -\varepsilon_0$ for all x. To each expression of the form (0) we associate the number $K = K(H) = 2nJ$, where J is the largest one of the n numbers $\sup_x |h_i(x)|$ $(i = 1, \ldots, n)$. Obviously $|H(x)| \leqq K$ for all x.

We shall show that starting with $H(x)$ it is possible by a certain procedure, to be indicated below, to arrive at a new function $H_1(x)$ of the type (0) with $H_1(x) \leqq -\varepsilon_0$ for all x and $K(H_1) \leqq (1-k_0)K(H)$.

By using the same procedure on $H_1(x)$ instead of $H(x)$ we arrive at a new function $H_2(x)$ of the type (0) with $H_2(x) \leqq -\varepsilon_0$ for all x and $K(H_2) \leqq (1-k_0)K(H_1) \leqq (1-k_0)^2K(H)$.

Continuing in this way we obtain functions $H_m(x)$ of type (0) satisfying $H_m(x) \leqq -\varepsilon_0$ for all x, and $K(H_m) \leqq (1-k_0)^m K(H)$. Since $|H_m(x)| \leqq K(H_m)$ for all x we have

$$-(1-k_0)^m K(H) \leqq H_m(x) \leqq -\varepsilon_0$$

for all x. This is a contradiction for m sufficiently large, as $(1-k_0)^m \to 0$ for $m \to \infty$.

We shall now indicate the procedure by which we get from $H(x)$ to $H_1(x)$ (and more generally from $H_{m-1}(x)$ to $H_m(x)$). To the elements $a_1, \ldots, a_n$ which occur in the expression (0) for $H(x)$ we determine the set E as indicated in (2) and put

$$H_1(x) = N(E)^{-1} \sum_{y \in E} H(xy) = N(E)^{-1} \sum_{i=1}^{n} \sum_{y \in E} \left(h_i(xy) - h_i(xya_i)\right).$$

By computation for fixed i of the inner sum $\sum_{y \in E}$, many terms cancel. Indeed, $N(E \cap Ea_i)$ of the terms $h_i(xy)$ will cancel $N(E \cap Ea_i)$ of the terms $-h_i(xya_i)$. The reduced expression for $H_1(x)$ is also of the type (0). Further

$$K(H_1) \leqq 2 \sum_{i=1}^{n} \left(N(E) - N(E \cap Ea_i)\right) J N(E)^{-1}$$

$$= 2 \sum_{i=1}^{n} \left(1 - N(E \cap Ea_i) N(E)^{-1}\right) J \leqq 2(n - k_0 n) J = (1-k_0)K(H).$$

Finally, from the definition of $H_1(x)$ we see that $H_1(x) \leqq -\varepsilon_0$ for all x. This completes the proof of the sufficiency part of the Main Theorem.

4. Finite systems of linear inequalities with a finite number of variables. For the proof of the necessity part of the Main Theorem we need the following

LEMMA. *The finite system of linear inequalities with a finite number of variables*

$$
(3) \qquad \begin{aligned} L_1(X) &= a_{11}x_1 + \dots + a_{1n}x_n \geqq b_1 \\ & \cdots\cdots\cdots\cdots \\ L_m(X) &= a_{m1}x_1 + \dots + a_{mn}x_n \geqq b_m \end{aligned}
$$

has at least one solution $X_0 = (x_1^0, \ldots, x_n^0)$ *if there does not exist an "obvious contradiction" between the inequalities, i.e., if there does not exist non-negative constants* $d_1, \ldots, d_m$ *such that both* $d_1 L_1(X) + \dots + d_m L_m(X)$ *is identically equal to* 0 *and* $d_1 b_1 + \dots + d_m b_m > 0$.

The analogous theorem for a system of linear inequalities with the sign $>$, instead of $\geqq$, is due to W. B. Carver [2, Theorem 3]. The proof is simple and algebraical. The Lemma can be proved in a similar way. The analogous theorem for a system of linear relations with arbitrary signs $=, <, >, \leqq, \geqq$ (not necessarily the same for all relations) is given by Th. Motzkin [4, Theorem D 2, p. 48]. The Lemma is a special case of this theorem.

5. Proof of the necessity part of the Main Theorem. In this final section we shall prove the first part of the Main Theorem which states that (1) is a necessary condition that G have a full Banach mean value.

Let G have a full Banach mean value. We shall prove that given a k in the interval $0 < k < 1$ and arbitrary, finitely many, elements $a_1, \ldots, a_n$ from G, there exists a finite subset E of G such that

$$
N(E \cap Ea_i) \geqq kN(E) \qquad \text{for} \quad i = 1, \ldots, n\,.
$$

The proof is indirect. We assume that for a certain k, $0 < k < 1$, there exist finitely many elements $a_1, \ldots, a_n$ from G with the property that for every finite subset E of G we have

$$
(4) \qquad N(E \cap Ea_i) < kN(E) \qquad \text{for some } i.
$$

From this assumption we shall prove that there exist n functions $h_1, \ldots, h_n$ on the denumerable subgroup G^* of G, which is generated by the elements $a_1, \ldots, a_n$, such that

$$
(5) \qquad |h_i(x)| \leqq A, \qquad i = 1, \ldots, n\,,
$$

and

$$
(6) \qquad H(x) = h_1(x) - h_1(xa_1) + \dots + h_n(x) - h_n(xa_n) \leqq -1
$$

for all x in G^*, where $A = (1-k)^{-1}$. Thus by Theorem 1 the group G^* will not have a full Banach mean value, and by Theorem 2 we have arrived at a contradiction.

We shall prove below that if F is an arbitrary finite subset of G we can find n functions $f_1, \ldots, f_n$ such that $f_i(x)$ is defined on $F \cup Fa_i$ with

$$|f_i(x)| \leqq A, \qquad i = 1, \ldots, n,$$

and such that the relation

$$f_1(x)-f_1(xa_1)+\ldots+f_n(x)-f_n(xa_n) \leqq -1$$

holds for all x in F.

This will be the salient point of our proof. With the result at our disposal it is easy to find functions $h_1, \ldots, h_n$ on G^* which satisfy (5) and (6) for all x in G^*. We choose a sequence of finite sets

$$F_1 \subset \ldots \subset F_m \subset \ldots$$

which exhaust G^*. By using the above-stated result we choose a function $f_i^{(m)}(x)$ on $F_m \cup F_m a_i$ with

$$|f_i^{(m)}(x)| \leqq A, \qquad i = 1, \ldots, n,$$

and such that the relation

$$f_1^{(m)}(x)-f_1^{(m)}(xa_1)+\ldots+f_n^{(m)}(x)-f_n^{(m)}(xa_n) \leqq -1$$

holds for all x in F_m, $m=1, 2, \ldots$. Next we choose a subsequence $m_1 < \ldots < m_p < \ldots$ of the sequence of natural numbers so that the sequences $f_i^{(m_p)}(x)$ converge for every x in G^*. The limit functions $h_i(x)$ are defined on G^* and satisfy (5) and (6) for all x in G^*.

Let F be an arbitrary finite subset of G. We shall prove that there exist n functions $f_1, \ldots, f_n$ such that $f_i(x)$ is defined on $F \cup Fa_i$ with

$$|f_i(x)| \leqq A, \qquad i = 1, \ldots, n$$

and such that the relation

$$f_1(x)-f_1(xa_1)+\ldots+f_n(x)-f_n(xa_n) \leqq -1$$

holds for all x in F.

In order to do this we shall introduce some notions.

Let E be an arbitrary finite subset of G. By an *open a_i-chain in E* we understand a finite sequence of different elements of the form $x, xa_i, \ldots, xa_i^{\lambda-1}$ which all belong to E and so that xa_i^{-1} and xa_i^{λ} do not belong to E. The first element of the sequence is called the *origin of the open a_i-chain in E*. By a *subchain of the open a_i-chain in E* we understand a finite sequence of the form $x, xa_i, \ldots, xa_i^{\nu-1}$ where $\nu \leqq \lambda$. They all have the origin of the open a_i-chain in E as first element.

If a_i has the finite order μ, there may also exist closed a_i-chains in E.

By a *closed a_i-chain in E* we understand a finite sequence of elements of the form $x, xa_i, \ldots, xa_i^{\mu-1}$ which all belong to E. On each closed a_i-chain we choose an element as *origin of the closed a_i-chain.* Let the origin of a closed a_i-chain in E be x_0. By a *subchain of the closed a_i-chain in E* we understand a finite sequence of the form $x_0, x_0a_i, \ldots, x_0a_i^{\nu-1}$ where this time $\nu < \mu$. They all have the origin of the closed a_i-chain in E as first element.

Let $p_i(E)$ denote the number of open a_i-chains in E. Plainly, any closed a_i-chain in E will as a whole pass into itself under right-translation by a_i. Thus, for a closed a_i-chain C_i in E we have $N(C_i \cap C_i a_i) = N(C_i)$. For an open a_i-chain O_i in E we have $N(O_i \cap O_i a_i) = N(O_i) - 1$. Now E is the union, for fixed i, of the open and the closed a_i-chains in E, and these chains are disjoint; further, when one of the open a_i-chains in E is right-translated by a_i, the translated chain cannot intersect any of the other a_i-chains in E. Hence we get

$$N(E \cap Ea_i) = N(E) - p_i(E), \qquad i = 1, \ldots, n .$$

Combining this with our assumption (4) we get

$$N(E) - (1-k)^{-1} p_i(E) < 0 \qquad \text{for some } i.$$

Since $A = (1-k)^{-1}$, we see that for every finite subset E of G we have

$$N(E) - A\big(p_1(E) + \ldots + p_n(E)\big) < 0 . \tag{7}$$

This relation plays a decisive role in the sequel.

In order to prove the existence of the above-mentioned functions $f_1, \ldots, f_n$ we introduce the $nN(F)$ unknowns

$$Y_{x,i} = f_i(x) - f_i(xa_i), \qquad i = 1, \ldots, n;\, x \in F .$$

Our functions $f_1, \ldots, f_n$ will exist if it is possible to find a solution $Y_{x,i}$ $(i = 1, \ldots, n;\, x \in F)$ of the system of relations

$$\sum_{i=1}^{n} Y_{x,i} \leqq -1, \qquad x \in F, \tag{8}$$

$$\sum_{x \in C_i} Y_{x,i} = 0 , \tag{9}$$

$$\Big| \sum_{x \in S_i} Y_{x,i} \Big| \leqq A , \tag{10}$$

where (9) is to hold for all closed a_i-chains C_i in F, $i = 1, \ldots, n$, and (10) is to hold for all subchains S_i of all open and closed a_i-chains in F, $i = 1, \ldots, n$. In fact, if we can find a solution

$$Y_{x,i}, \qquad i = 1, \ldots, n;\, x \in F ,$$

of the system of relations (8), (9), (10), we choose $f_i(x)$ equal to 0 at all the origins of the open and closed a_i-chains in F and determine $f_i(x)$ recursively from the $Y_{x,i}$ by using the relation $f_i(xa_i)=f_i(x)-Y_{x,i}$. This gives no contradiction for x on the closed a_i-chains in F on account of (9), and $f_i(x)$ will be defined on $F\cup Fa_i$, $i=1, \ldots, n$. On account of (10) we shall have $|f_i(x)|\leqq A$ on $F\cup Fa_i$, and finally, on account of (8) we shall have

$$f_1(x)-f_1(xa_1)+\ldots+f_n(x)-f_n(xa_n) \leqq -1$$

for all x in F.

We write the system of relations (8), (9), (10) as a system of inequalities of the type occurring in the Lemma.

$$(8) \qquad \sum_{i=1}^{n}(-Y_{x,i}) \geqq 1, \qquad x\in F,$$

$$(9') \qquad \sum_{x\in C_i} Y_{x,i} \geqq 0, \qquad \text{all } C_i, \text{ all } i,$$

$$(9'') \qquad \sum_{x\in C_i}(-Y_{x,i}) \geqq 0, \qquad \text{all } C_i, \text{ all } i,$$

$$(10') \qquad \sum_{x\in S_i} Y_{x,i} \geqq -A, \qquad \text{all } S_i, \text{ all } i,$$

$$(10'') \qquad \sum_{x\in S_i}(-Y_{x,i}) \geqq -A, \qquad \text{all } S_i, \text{ all } i.$$

On account of the Lemma, a necessary and sufficient condition that this system of inequalities has at least one solution is that there exists no "obvious contradiction" between the inequalities. Thus we have to show that whenever with non-negative coefficients d, l, r, q, s

$$\sum_{x\in F} d_x \sum_{i=1}^{n}(-Y_{x,i}) +$$

$$(11) \qquad +\sum_{i=1}^{n}\sum_{C_i} l(C_i)\sum_{x\in C_i} Y_{x,i} + \sum_{i=1}^{n}\sum_{C_i} r(C_i)\sum_{x\in C_i}(-Y_{x,i})+$$

$$+\sum_{i=1}^{n}\sum_{S_i} q(S_i)\sum_{x\in S_i} Y_{x,i} + \sum_{i=1}^{n}\sum_{S_i} s(S_i)\sum_{x\in S_i}(-Y_{x,i})$$

is identically equal to 0, then

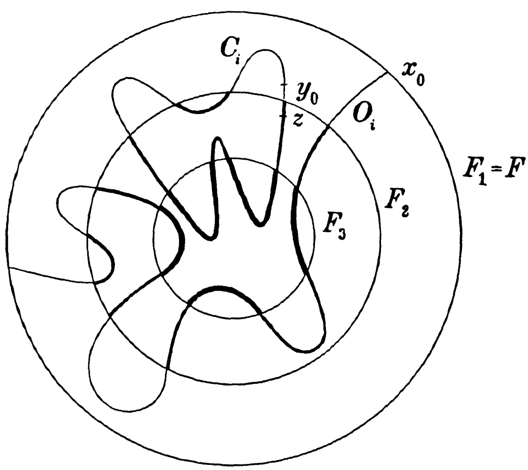
$$(12) \qquad \sum_{x\in F} d_x - A\sum_{i=1}^{n}\sum_{S_i} q(S_i) - A\sum_{i=1}^{n}\sum_{S_i} s(S_i) \leqq 0.$$

In the proof we may assume that $r(C_i)=0$. In fact, if z is the last element of C_i before its origin, the variable $Y_{z,i}$ in (11) gets the total coefficient $-d_z+l(C_i)-r(C_i)=0$. Thus $l(C_i)-r(C_i)\geqq 0$, and we can replace $l(C_i)$ in (11) by $l(C_i)-r(C_i)$ and $r(C_i)$ by 0.

In the proof we may further assume that for each S_i either $q(S_i)=0$ or $s(S_i)=0$. In fact, if $q(S_i)\geqq s(S_i)>0$ we replace $q(S_i)$ in (11) by $q(S_i)-s(S_i)$ and $s(S_i)$ by 0, and if $s(S_i)>q(S_i)>0$ we replace $q(S_i)$ in (11) by 0 and $s(S_i)$ by $s(S_i)-q(S_i)$; if (12) is satisfied after this change has been made in (11), then (12) must be valid also before the change, for the change has diminished $q(S_i)$ and $s(S_i)$.

Let $D_1 < \ldots < D_Q$ be the different values of d_x which occur in (11). Let F_P denote the set of those x for which $d_x\geqq D_P$, $P=1, \ldots, Q$, and F_{Q+1} the empty set. In particular $F_1=F$.

By using the fact that the expression (11) vanishes identically, we shall see in the course of the proof of (12) that the coefficients $q(S_i)$, $s(S_i)$, and $l(C_i)$ can be determined from the coefficients d_x.



We first determine the coefficients $q(S_i)$ and $s(S_i)$ for all subchains S_i of a given open a_i-chain O_i in F (see the figure). Let x_0 be the origin of O_i, and let x be an arbitrary element of O_i. By x' we denote the element which follows x in O_i if such an element exists (i.e., if x is not the last element of O_i). By (x_0,x) we denote the subchain S_i of O_i which has x as last element.

If x' does not exist, and x is in F_M, but not in F_{M+1}, then

$$q(x_0,x) = D_M = (D_M-D_{M-1})+\ldots+(D_2-D_1)+D_1 .$$

In fact, when x is the last element of O_i, the total coefficient of $Y_{x,i}$ in

(11) is $-d_x+q(x_0,x)-s(x_0,x)=0$, and taking into account that either $q(x_0,x)=0$ or $s(x_0,x)=0$ we get $q(x_0,x)=d_x=D_M$ (and $s(x_0,x)=0$).

If x' exists and is in F_N, but not in F_{N+1}, and x is in F_M, but not in F_{M+1}, then

$$q(x_0,x) = D_M-D_N = (D_M-D_{M-1})+\ldots+(D_{N+1}-D_N) \quad \text{if} \quad M > N,$$

$q(x_0,x)=s(x_0,x)=0$ if $M=N$, and $s(x_0,x)=D_N-D_M$ if $M<N$. In fact, the total coefficient of $Y_{x,i}$ and the total coefficient of $Y_{x',i}$ in (11) are 0, that is,

$$-d_x + \sum_{S_i\supset(x_0,x)} \left(q(S_i)-s(S_i)\right) = 0$$

and

$$-d_{x'} + \sum_{S_i\supset(x_0,x')} \left(q(S_i)-s(S_i)\right) = 0$$

(where $S_i\supset(x_0,x)$ indicates that S_i is a subchain of O_i which contains x). Now $d_x=D_M$ and $d_{x'}=D_N$, so that by subtracting the above relations we get $q(x_0,x)-s(x_0,x)=D_M-D_N$, and our statement follows.

We find in particular that the above expressions for $q(x_0,x)$ contain the term D_P-D_{P-1}, where $P=1,\ldots,Q$ is a given number and $D_0=0$, if and only if x is in F_P and x' is not in F_P (or does not exist). The number of such elements x is equal to the number of open a_i-chains in F_P which are contained in O_i.

We next determine the coefficients $q(S_i)$ and $s(S_i)$ for all subchains S_i of a given closed a_i-chain C_i in F. Also the coefficient $l(C_i)$ will be determined. Let R be chosen so that the last element z of C_i before the origin y_0 of C_i lies in F_R, but not in F_{R+1} (see the figure). Plainly $l(C_i)=D_R$, for the total coefficient of $Y_{z,i}$ in (11) is $-d_z+l(C_i)=0$, and $d_z=D_R$. Let x be an arbitrary element of C_i different from z, and let x' denote the element which follows x in C_i. By (y_0,x) we denote the subchain S_i of C_i which has x as last element.

If $x'=z$, and $x\neq z$ is in F_M, but not in F_{M+1}, we get

$$q(y_0,x) = D_M-D_R = (D_M-D_{M-1})+\ldots+(D_{R+1}-D_R) \quad \text{if} \quad M > R,$$

$$q(y_0,x) = s(y_0,x) = 0 \quad \text{if} \quad M = R,$$

$$s(y_0,x) = D_R-D_M = (D_R-D_{R-1})+\ldots+(D_{M+1}-D_M) \quad \text{if} \quad M < R.$$

In fact, the total coefficient of $Y_{x,i}$ in (11) is

$$-d_x+q(y_0,x)-s(y_0,x)+l(C_i) = 0,$$

where $d_x=D_M$ and $l(C_i)=D_R$. Hence $q(y_0,x)-s(y_0,x)=D_M-D_R$, and our statement follows.

If $x' \neq z$ is in F_N, but not in F_{N+1}, and $x \neq z$ is in F_M, but not in F_{M+1}, then

$$q(y_0,x) = D_M - D_N = (D_M - D_{M-1}) + \ldots + (D_{N+1} - D_N) \quad \text{if} \quad M > N,$$

$$q(y_0,x) = s(y_0,x) = 0 \quad \text{if} \quad M = N,$$

$$s(y_0,x) = D_N - D_M = (D_N - D_{N-1}) + \ldots + (D_{M+1} - D_M) \quad \text{if} \quad M < N.$$

In fact, the total coefficient of $Y_{x,i}$ and the total coefficient of $Y_{x',i}$ in (11) are 0, that is,

$$-d_x + \sum_{S_i \supset (y_0, x)} \big(q(S_i) - s(S_i)\big) + l(C_i) = 0$$

and

$$-d_{x'} + \sum_{S_i \supset (y_0, x')} \big(q(S_i) - s(S_i)\big) + l(C_i) = 0\,.$$

Now $d_x = D_M$ and $d_{x'} = D_N$, so that by subtracting the above relations we get $q(y_0,x) - s(y_0,x) = D_M - D_N$, and our statement follows.

We find in particular that the above expressions for $q(y_0,x)$, $x \neq z$, contain the term $D_P - D_{P-1}$, where $P = 1, \ldots, Q$ is a given number and $D_0 = 0$, if and only if x is in F_P and x' is not in F_P. The number of such elements x is equal to the number of open a_i-chains in F_P which are contained in C_i, except when z is in F_P and y_0 is not in F_P.

Furthermore, the above expressions for $s(y_0,x)$, $x \neq z$, contain the term $D_P - D_{P-1}$, where $P = 1, \ldots, Q$ is a given number and $D_0 = 0$, if and only if x' is in F_P and x is not in F_P. The number of such elements x is equal to the number of open a_i-chains in F_P which are contained in C_i, except when y_0 is in F_P and z is not in F_P.

Since every open a_i-chain in F_P, $P = 1, \ldots, Q$, is contained in some a_i-chain in F, open or closed, and since these a_i-chains are disjoint for fixed i, we obtain the following relation from the results stated in italics:

$$\begin{aligned}
&\sum_{x \in F} d_x - A \sum_{i=1}^{n} \sum_{S_i} q(S_i) - A \sum_{i=1}^{n} \sum_{S_i} s(S_i) \\
&\leqq \{D_1 N(F_1) + (D_2 - D_1) N(F_2) + \ldots + (D_Q - D_{Q-1}) N(F_Q)\} - \\
&- \big\{D_1 A\big(p_1(F_1) + \ldots + p_n(F_1)\big) + (D_2 - D_1) A\big(p_1(F_2) + \ldots + p_n(F_2)\big) + \\
&+ \ldots + (D_Q - D_{Q-1}) A\big(p_1(F_Q) + \ldots + p_n(F_Q)\big)\big\} \\
&= D_1\big\{N(F_1) - A\big(p_1(F_1) + \ldots + p_n(F_1)\big)\big\} + (D_2 - D_1)\big\{N(F_2) - A\big(p_1(F_2) + \\
&+ \ldots + p_n(F_2)\big)\big\} + \ldots + (D_Q - D_{Q-1})\big\{N(F_Q) - A\big(p_1(F_Q) + \ldots + p_n(F_Q)\big)\big\}\,.
\end{aligned}$$

By applying (7) to this, with $E = F_1, \ldots, F_Q$, the inequality (12) follows.

This completes the proof of the Main Theorem.

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