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ON POLYNOMIAL SOLUTIONS OF A DIFFERENTIAL EQUATION

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1. The problem of finding all area-preserving, analytic functions $f(z)$ leads to the equation $|f'(z)| \equiv 1$. The solutions are trivial, namely, $f(z)=az+b$ where a and b are constants and $|a|=1$. In the corresponding problem in two variables z_1 and z_2 we seek volume-preserving pairs of analytic functions $u=u(z_1, z_2)$ and $v=v(z_1, z_2)$, that is, solutions of the partial differential equation

$$(1) \qquad u_{z_1} v_{z_2} - u_{z_2} v_{z_1} = 1 .$$

This differential equation has solutions [1] other than the trivial ones

$$\begin{aligned} u &= a+bz_1+cz_2, \\ v &= d+ez_1+fz_2 , \end{aligned} \qquad \begin{vmatrix} b & c \\ e & f \end{vmatrix} = 1 .$$

We note that, if (u, v) satisfies (1), so do $(u+F(v),\ v)$ and $(u,\ v+F(u))$ where F is arbitrary. This may be utilized to construct chains of solutions starting with the identity mapping $u_1=z_1$, $v_1=z_2$. As an example let $\alpha\beta \neq 0$, and set

$$\begin{aligned} u_2 &= u_1 + \beta\alpha^{-1} v_1, & v_2 &= v_1 , \\ u_3 &= u_2, & v_3 &= v_2 - \beta^{-1}\sin\alpha u_2 , \\ u_4 &= u_3 - \beta\alpha^{-1} v_3, & v_4 &= v_3 , \end{aligned}$$

which gives

$$\begin{aligned} u_4 &= z_1 + \alpha^{-1}\sin(\alpha z_1+\beta z_2) , \\ v_4 &= z_2 - \beta^{-1}\sin(\alpha z_1+\beta z_2) . \end{aligned}$$

The author believes that all polynomial solutions of (1) may be obtained from the identity mapping by means of such chains where the F's involved are polynomials. If m and n denote the degrees of u and v, respectively, then the above conjecture is equivalent to showing that $m|n$ or $n|m$ [1, p. 263]. In this paper we settle the question in part by showing that (1) has no polynonial solutions with $(m, n)=1$ and $m \geqq 2$, $n \geqq 2$.

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2. Theorem. *Let $u=u(z_1, z_2)$ and $v=v(z_1, z_2)$ denote two polynomials of degrees $m \geqq 2$ and $n \geqq 2$, respectively, in the two complex variables z_1 and z_2. If the Jacobian*

$$u_{z_1}v_{z_2} - u_{z_2}v_{z_1} = k = \text{constant}$$

and m and n are relatively prime, then $k=0$ and there exists a polynomial h of first degree in z_1 and z_2 such that u and v are polynomials in h.

Without loss of generality we may normalize any pair (u, v) of polynomial solutions of $u_{z_1}v_{z_2}-u_{z_2}v_{z_1}=k$ so that $u(0, 0)=v(0, 0)=0$ and $m \geqq n$. We assume throughout the paper that this has been done.

Remark I. The equation

$$u_{z_1}v_{z_2} - u_{z_2}v_{z_1} = 0$$

has polynomial solutions of any degrees m and n as is seen by the example $u=(z_1+z_2)^m$ and $v=(z_1+z_2)^n$.

Grouping terms of the same degrees, we may write

$$u = \sum_{i=1}^{m} f_i(z_1, z_2) \qquad \text{and} \qquad v = \sum_{i=1}^{n} \varphi_i(z_1, z_2) ,$$

where the f_i's and φ_i's are homogeneous polynomials of ith degree in z_1 and z_2, and $f_m \not\equiv 0 \not\equiv \varphi_n$. For $m \geqq n > 1$, we then have [1]

$$(2) \qquad f_{m-\mu} = \sum_{\gamma=0}^{\mu} C_\gamma \sum \binom{(m-\gamma)/n}{\nu} \frac{\nu!}{\prod_{\alpha=1}^{n-1} \nu_\alpha!} \varphi_n^{(m-\gamma)/n-\nu} \prod_{\alpha=1}^{n-1} \varphi_\alpha^{\nu_\alpha},$$

$$\mu = 0, 1, \ldots, m+n-3$$

where the C_γ's are suitable constants, the sum without limits is to be extended over all combinations of non-negative integers ν_α satisfying

$$(3) \qquad \sum_{\alpha=1}^{n-1} (n-\alpha)\nu_\alpha = \mu-\gamma ,$$

and ν is defined by $\nu=\nu_1+\nu_2+\ldots+\nu_{n-1}$. For $\mu \geqq m$ the left-hand side of (2) shall be set equal to zero, and, if $\varphi_\alpha \equiv 0$ for some α, then we set $\varphi_\alpha^{\nu_\alpha}=1$ when $\nu_\alpha=0$.

Remark II. If $1=n \leqq m$, we may insert $u=f_1+f_2+\ldots+f_m$ and $v=\gamma z_1+\delta z_2$ in $u_{z_1}v_{z_2}-u_{z_2}v_{z_1}=k$ and compare homogeneous polynomials of equal degrees on both sides of the equation. This gives

$$u = \alpha z_1 + \beta z_2 + \sum_{\mu=2}^{m} C_{m-\mu}(\gamma z_1+\delta z_2)^\mu, \qquad \begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix} = k ,$$

where the sum is missing if $m=1$.

Proceeding to the proof of the theorem, we have by (2)

$$f_m = C_0 \varphi_n{}^{m/n}$$

and, since m/n is in lowest terms, we easily see that there exists a homogeneous polynomial h of first degree such that

$$\varphi_n = h^n, \qquad f_m = C_0 h^m \quad \text{and} \quad C_0 \neq 0 . \tag{4}$$

Next we shall show that

$$\varphi_\alpha = k_\alpha h^\alpha, \qquad \alpha = 1, 2, \ldots, n-1 , \tag{5}$$

where the k_α's are constants. Once (5) is established, formula (2) shows that

$$f_\mu = l_\mu h^\mu, \qquad \mu = 1, 2, \ldots, m ,$$

where the l_μ's are constants. Thus u and v are polynomials in h and therefore $u_{z_1} v_{z_2} - u_{z_2} v_{z_1} = 0$.

Equation (5) is proved by contradiction. To this end we factor out the greatest possible power h^{p_α} of φ_α and write

$$\varphi_\alpha = \eta_\alpha h^{p_\alpha}, \qquad \alpha = 1, 2, \ldots, n-1 , \tag{6}$$

where $0 \leqq p_\alpha \leqq \alpha$, the η_α's are homogeneous polynomials of degree $\alpha - p_\alpha$ and $h \nmid \eta_\alpha$ if $\eta_\alpha \neq 0$. We assume that $p_\alpha < \alpha$ for some value of α. Clearly the corresponding η_α's are not identically zero. We insert (4) and (6) in (2) and find a lower bound for the exponent e of h in the resulting equation,

$$\begin{aligned} e = m - \gamma - n\nu + \sum_{\alpha=1}^{n-1} p_\alpha \nu_\alpha &= m - \gamma - \sum_{\alpha=1}^{n-1} (n-p_\alpha)\nu_\alpha \\ &= m - \gamma - \sum_{\alpha=1}^{n-1} \frac{n-p_\alpha}{n-\alpha}(n-\alpha)\nu_\alpha \\ &\geqq m - \gamma - \left(\max_{1 \leqq \alpha \leqq n-1} \frac{n-p_\alpha}{n-\alpha}\right) \sum_{\alpha=1}^{n-1} (n-\alpha)\nu_\alpha . \end{aligned}$$

By (6) we see that

$$\frac{n-p_\alpha}{n-\alpha} = 1 + \frac{\alpha - p_\alpha}{n-\alpha} \geqq 1$$

and by our assumption

$$\frac{n-p_\alpha}{n-\alpha} > 1$$

for at least one value of α. We may write

$$\max_{1 \leqq \alpha \leqq n-1} \frac{n-p_\alpha}{n-\alpha} = \frac{p}{q} = \frac{\varrho p}{\varrho q} > 1,$$

where p and q are relatively prime,

$$(7) \qquad (p, q) = 1.$$

We introduce the notation

$$(8) \qquad \alpha_\varrho = n - \varrho q, \qquad \varrho = 1, 2, \ldots, [n/p],$$

and see that

$$(9) \qquad n - p_{\alpha_\varrho} \leqq \varrho p, \qquad \varrho = 1, 2, \ldots, [n/p],$$

and

$$(10) \qquad \frac{n-p_\alpha}{n-\alpha} < \frac{p}{q}, \quad \alpha \neq \alpha_\varrho, \quad \varrho = 1, 2, \ldots, [n/p].$$

Let r denote the largest value of ϱ for which there is equality in (9) and write

$$(6') \qquad \varphi_{\alpha_\varrho} = \chi_\varrho h^{n-\varrho p}, \qquad \varrho = 1, 2, \ldots, [n/p],$$

and observe that

$$(11) \qquad h \nmid \chi_r = \eta_{\alpha_r} \qquad \text{(by assumption)}.$$

By (3) and the above notation

$$e \geqq m - \gamma - \frac{p}{q} \sum_{\alpha=1}^{n-1} (n-\alpha)\nu_\alpha = m - \gamma - \frac{p}{q}(\mu - \gamma)$$
$$= m - \mu\frac{p}{q} + \gamma\left(\frac{p}{q} - 1\right) \geqq m - \mu\frac{p}{q}.$$

This lower bound, $m - \mu p/q$, for e is attained if and only if

$$(12) \qquad \mu \equiv 0 \pmod{q} \quad \bigl(\text{in view of (7)}\bigr),$$

$$(13) \qquad \gamma = 0 \qquad (\text{since } p/q - 1 > 0),$$

$$(14) \qquad \nu_\alpha = 0 \quad \text{for} \quad \alpha \neq \alpha_\varrho, \quad \varrho = 1, 2, \ldots, r, \quad \bigl(\text{by (10)}\bigr),$$

and

$$(15) \qquad \nu_{\alpha_\varrho} = 0 \qquad \text{when} \qquad n - p_{\alpha_\varrho} < \varrho p.$$

It will be shown later that we may choose $\mu \equiv 0 \pmod{q}$ so that $m - \mu p/q < 0$ and $\mu \leqq m + n - 3$. The right-hand side of (2) will then contain fractions whose denominators are powers of h, the largest such power being $h^{\mu p/q - m}$. If we multiply both sides of (2) by $h^{\mu p/q - m - 1}$, we obtain

$$\text{a polynomial} = \text{a polynomial} + C_0 N_\mu / h.$$

Thus h divides N_μ. By (14) and (15) the polynomials η_α that appear in N_μ are exactly those which have the subscripts α_ϱ defined in (8) and for which $n-p_{\alpha_\varrho}=\varrho p$. For the sake of convenience, we enlarge the sum N_μ by removing the restriction (15). The resulting sum is denoted by P_μ. By (2), (14) and (6′)

$$(16)\qquad P_\mu = \sum \binom{m/n}{\nu} \frac{\nu!}{\prod_{\varrho=1}^{r} \nu_{\alpha_\varrho}!} \prod_{\varrho=1}^{r} \chi_\varrho^{\nu_{\alpha_\varrho}}, \qquad \mu \equiv 0 \pmod{q},$$

where the sum—by (3) and (8)—is to be extended over all combinations of non-negative integers ν_{α_ϱ} satisfying

$$(3')\qquad \sum_{\varrho=1}^{r} \varrho \nu_{\alpha_\varrho} = \frac{\mu}{q}.$$

Each term in P_μ which does not also belong to N_μ contains at least one factor χ_ϱ for which $n-p_{\alpha_\varrho}<\varrho p$. By (6) and (6′) these factors are divisible by h. Thus h divides P_μ when

$$(17)\qquad m\,q/p < \mu \leqq m+n-3\,.$$

Setting $r=2$, $q=1$, $\chi_1=-2x$, $\chi_2=1$ and replacing m/n by $-1/2$ in (16), it may be shown that the P_μ's reduce to the Legendre polynomials. Like the Legendre polynomials, the polynomials P_μ have a generating function from which we may deduce a recurrence formula. To show this let

$$T = 1 + \sum_{\varrho=1}^{r} \chi_\varrho t^{\varrho q}$$

and expand $T^{m/n}$ binomially

$$T^{m/n} = \sum_{\nu=0}^{\infty} \binom{m/n}{\nu} \left[\sum_{\varrho=1}^{r} \chi_\varrho t^{\varrho q}\right]^\nu.$$

Then we expand the brackets multinomially and obtain

$$T^{m/n} = \sum_{\nu=0}^{\infty} \binom{m/n}{\nu} \left[\sum \frac{\nu!}{\prod_{\varrho=1}^{r} \nu_{\alpha_\varrho}!} \left(\prod_{\varrho=1}^{r} \chi_\varrho^{\nu_{\alpha_\varrho}}\right) t^\mu\right] = \sum_{\substack{\mu=0 \\ \mu\equiv 0 \,(\mathrm{mod}\, q)}}^{\infty} P_\mu t^\mu$$

where

$$\mu = q \sum_{\varrho=1}^{r} \varrho \nu_{\alpha_\varrho}$$

and the P_μ's are given by (16) and (3′) but not restricted by (17). Comparing coefficients on both sides of the identity

$$T\frac{\partial T^{m/n}}{\partial t} = \frac{m}{n}T^{m/n}\frac{\partial T}{\partial t} = \frac{m}{n}\frac{\partial T}{\partial t}\sum P_\mu t^\mu \equiv T\sum \mu P_\mu t^{\mu-1}$$

and setting $\chi_0 = 1$, we obtain the recurrence formula

$$(18) \qquad \sum_{\varrho=0}^{r}\left[\frac{\mu}{q} - \varrho\left(\frac{m}{n}+1\right)\right]\chi_\varrho P_{\mu-\varrho q} \equiv 0, \qquad \mu = q, 2q, \dots ,$$

where

$$P_{\varrho q} \equiv 0, \qquad \varrho = -1, -2, \dots, -r.$$

Since m/n is in lowest terms, $r < n$ and μ/q is an integer, we see that

$$(19) \qquad \mu/q - r(m/n+1) \neq 0 .$$

Next we show that the interval (17) contains at least r consecutive multiples μ of q, that is, $m+n-3-mq/p > (r-1)q+(q-1)$. Using the facts that $r \leqq n/p$, $m \geqq n+1$ and $n \geqq p \geqq q+1$, we see that

$$\begin{aligned} m+n-3-m\frac{q}{p} - rq+1 &\geqq m\frac{p-q}{p} + n - 2 - \frac{n}{p}q \\ &\geqq \frac{1}{p}[(n+1)(p-q)+p(n-2)-nq] \\ &= \frac{1}{p}[2n(p-q-1)+2n-p-q] \geqq \frac{1}{p} > 0 . \end{aligned}$$

We now choose $\mu \equiv 0 \pmod{q}$ so that

$$mq/p < \mu-(r-1)q \leqq \mu \leqq m+n-3$$

and see by (18), (19) and (11) that $h|P_{\mu-rq}$. Repeating this argument, decreasing μ by q units at a time, we finally obtain $h|P_0 \equiv 1$, a contradiction. Thus the assumption that $p_\alpha < \alpha$ for some α is wrong, which proves (5) and completes the proof of the theorem.

REFERENCE

1. Arne Magnus, *Volume-preserving transformations in several complex variables*, Proc. Amer. Math. Soc. 5 (1954), 256–266.

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