MATH. SCAND. 3 (1955), 261–263

ON NON-CONSTRUCTIVE THEOREMS OF ANALYSIS AND THE DECISION PROBLEM

R. L. GOODSTEIN

In this note we exhibit a method for strengthening some known results on the impossibility of proving certain classical theorems in recursive analysis. For the concepts and nomenclature of recursive analysis the reader is referred to [3].

We denote by $\mathscr{R}$ some (unspecified) formalisation of recursive arithmetic, and by $\mathscr{R}^*$ an extension of $\mathscr{R}$ to rational numbers and functions, adequate for recursive analysis.

1. If $f(n, x)$ is any rational recursive function, recursively convergent in n, and differentiable in x, relative to n, with relative derivative $f^1(n, x)$ for $0 \leqq x \leqq 1$, and if

$$f(n, 0) = f(n, 1) = 0$$

then we say that $f(n, x)$ satisfies the conditions of the relative Rolle's theorem and write $f \in RT$.

It was proved in [1] that if $f \in RT$ then there is a recursive ν_k and a recursive c_k such that $n \geqq \nu_k \to f^1(n, c_k) = 0(k)$ is provable in $\mathscr{R}^*$.

If there exists a recursive sequence c_k, a recursive ν_k and an integer p such that $1/p < c_k < 1 - 1/p$ and

$$\text{(i)} \qquad n \geqq \nu_k \to f^1(n, c_k) = 0(k)$$

is provable in $\mathscr{R}^*$ (with free variable n) then we say that $f(n, x)$ satisfies the conditions of the uniform Rolle's theorem and write $f \in URT$; if however condition (i) is provable, not necessarily for a variable n, but for each positive integral value of n, then we write $f \in IRT$.

We gave in [2] an example of a function f such that $f \in RT$ but $f \notin URT$. We shall now prove the stronger result that there exists an f such that $f \in RT$ but $f \notin IRT$. We shall in fact show that a proof in $\mathscr{R}^*$ of

$$f \in RT \to f \in IRT$$

Received August 6, 1955.

provides a decision method for the class of equations $\varrho(n)=0$, where ϱ is a recursive function which takes only the values 0 and 1, but, as is well known (see [4, pp. 417–418]), this class of equations is undecidable.

2. Given any recursive function $\varrho(n)$ which takes only the values 0 and 1 we define (as in [2])

$$e_0 = 0, \qquad e_{n+1} = e_n + \prod_{r=0}^{n} \left(1 \dot{-} \varrho(r)\right)$$

$$d_0 = 1, \qquad d_{n+1} = 1/e_{n+1}$$

and, for $0 \leqq x \leqq 1$ and $n \geqq 3$,

$$f(n, x) = \frac{d_n{}^4 x(1 \dot{-} x)}{d_n{}^2 + (1 \dot{-} 2d_n)x}.$$

It is supposed that $\varrho(0)=0$.

The following properties of these functions are readily provable in $\mathscr{R}^*$ (for details, see [2, pp. 228–230]).

(2.1) $e_n \leqq n$.

(2.2) $e_n < n \rightarrow (Er)\left(r \leqq n \ \&\ \varrho(r)=1\right)$.

(2.3) If $N > n \geqq 1$ then $0 \leqq d_n - d_N < 1/n$.

(2.4) For $n \geqq 3$ and $0 \leqq x \leqq 1$, we have

$$0 \leqq f(n, x) \leqq d_n{}^4 .$$

(2.5) If $3 \leqq n < N$ and $0 \leqq x \leqq 1$ then

$$0 \leqq f(n, x) - f(N, x) < 1/n^4 ,$$

from which it follows that $f(n, x)$ converges uniformly in x for $0 \leqq x \leqq 1$.

(2.6) For $0 \leqq x \leqq 1$, $f(n, x)$ is differentiable in x uniformly in x and n, so that $f(n, x)$ is differentiable in x relative to n, and the relative derivative $f^1(n, x)$ converges uniformly in x, and $f \in RT$.

(2.7) If there is a recursive $V(k)$ such that

$$n \geqq V(k) \rightarrow d_n = 0(k)$$

is provable in $\mathscr{R}^*$ for all integers n, then $\varrho(n)=0$ is provable in $\mathscr{R}$ for all integers n.

3. If $f \in IRT$ then (by definition) there exists a recursive $V(k)$, a recursive c_k and an integer p such that $c_k \geqq 1/p$ and

$$n \geqq V(k) \rightarrow f^1(n, c_k) = 0(k)$$

is provable in $\mathscr{R}^*$ for all integers n.

Since

$$f^1(n, c_k) = \frac{d_n^4(d_n - c_k)\{d_n + (1-2d_n)c_k\}}{\{d_n^2 + (1-2d_n)c_k\}^2}$$

and

$$d_n + (1-2d_n)c_k > d_n^2 + (1-2d_n)c_k ,$$

$$d_n^2 + (1-2d_n)c_k \leqq (1-d_n)^2 < 1 ,$$

it follows that

$$\boldsymbol{n} \geqq V(k) \rightarrow d_{\boldsymbol{n}}^4(d_{\boldsymbol{n}} - c_k) = 0(k)$$

is provable in $\mathscr{R}^*$ for all $\boldsymbol{n}$.

By (2.1), either $d_{\boldsymbol{p}+1} = 1/(\boldsymbol{p}+1)$ or $d_{\boldsymbol{p}+1} > 1/(\boldsymbol{p}+1)$.

If $d_{\boldsymbol{p}+1} = 1/(\boldsymbol{p}+1)$ then $d_n \leqq 1/(\boldsymbol{p}+1)$ for $n \geqq (\boldsymbol{p}+1)$ so that

$$|d_n - c_k| > 1/[\boldsymbol{p}(\boldsymbol{p}+1)] ,$$

and therefore

$$\boldsymbol{n} \geqq V(4k+\boldsymbol{p}) \rightarrow d_{\boldsymbol{n}} = 0(k)$$

is provable in $\mathscr{R}^*$ for all $\boldsymbol{n}$, whence, by (2.7), $\varrho(\boldsymbol{n}) = 0$ is provable in $\mathscr{R}$ for all integers $\boldsymbol{n}$.

If however $d_{\boldsymbol{p}+1} > 1/(\boldsymbol{p}+1)$ then by (2.2) there is an $\boldsymbol{r}$ between 0 and $\boldsymbol{p}+1$ for which $\varrho(\boldsymbol{r}) = 1$ is provable in $\mathscr{R}$, and so the hypothesis $f \in IRT$ implies the existence of a decision procedure for the undecidable class of equations $\varrho(n) = 0$.

BIBLIOGRAPHY

1. R. L. Goodstein, *Mean-value theorems in recursive function theory. Part 1. Differential mean-value theorems*, Proc. London Math. Soc. 52 (1950), 81–106.
2. R. L. Goodstein, *A problem in recursive function theory*, J. Symbolic Logic 18 (1953), 225–232.
3. R. L. Goodstein, *The relatively exponential, logarithmic and circular functions in recursive function theory*, Acta Math. 92 (1954), 171–190.
4. D. Hilbert and P. Bernays, *Grundlagen der Mathematik* II, Berlin, 1939.

UNIVERSITY COLLEGE, LEICESTER, ENGLAND