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SETS OF PRIMES WITH INTERMEDIATE DENSITY

SOLOMON W. GOLOMB

1. Introduction. It is a surprising fact that all the well-known sets of prime numbers belong to one of two very distinct classes. In the first of these classes are sets of primes which are known not merely to be infinite, but to have an asymptotic distribution of $Ax/\log x$ for a suitable constant $A > 0$. The set of all primes; the primes in any arithmetic progression $an+b$ with $(a, b) = 1$; primes of the form $a^2+b^2+c^2+1$; all these belong to this first class.

The second class contains sets of primes whose distribution does *not* exceed $Ax/\log x$ for all x (or even, for all sufficiently large x), for *any* $A > 0$. Typical illustrations here are the twin primes, primes of the form a^2+1, and even primes of the form a^2+b^2+1. For this class, not only the conjectured asymptotic formulas remain unproven, but none of the sets of primes in question has even been proved infinite. Brun's sieve method succeeds in proving that all of these sets are indeed $o(x/\log x)$, but no significant *lower* bound has ever been obtained.

Also in the second class are the well-known Fermat primes and Mersenne primes, each surely $o(x/\log x)$—in fact, it is quite elementary that the distribution of the Fermat primes is $O(\log\log x)$, and of the Mersenne primes, $O(\log x/\log\log x)$. But the infinitude of these sets not merely remains unproven—it is a matter of legitimate doubt. The "regular primes" of Kummer furnish yet another example of a set of primes whose infinitude remains in doubt.

It is certainly reasonable to ask whether there is a middle ground—whether there are sets of primes, arising in a natural way, which can be proved infinite, yet have a distribution which does not exceed $Ax/\log x$ for any $A > 0$. If $\{p_n\}$ is the sequence of the primes, then $\{p_{n^2}\}$ meets all the requirements except for "arising in a natural way". Similarly for $\{p_{n^3}\}$, $\{p_{n!}\}$, etc. Clearly one criterion for "naturalness" should be that there is no explicit reference to subscripts in the sequence $\{p_n\}$. However, such sets as $\{p_{n!}\}$ serve to demonstrate that there can be no doubt as to the *existence* of infinite sets of primes with arbitrarily sparse distributions.

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In this article, a large family of sets of primes will be exhibited, wherein each set is proved to be infinite, yet at the same time is shown to have a distribution which does not exceed $Ax/\log x$ for any $A > 0$. (Such sets of primes will be said to have *intermediate density*, or simply to be *intermediate sets*). Two independent motivations for the *naturalness* of such sets will be given, and some important connections between these sets and the existing literature will be established. Another intermediate set, arising in a somewhat different manner, will also be studied.

2. First motivation. The usual test for whether or not a polynomial $f(x)$, with coefficients in a field F, has any *repeated* factors, is to determine (if necessary, by the Euclidean algorithm) whether or not $f(x)$ and its derivative $f'(x)$ have any *common* factors. If the field F has characteristic 0, then

i) every common factor of f and f' is a repeated factor of f, and conversely.

If $\mathrm{char}(F) \neq 0$, the test remains *partly* valid, in that

ii) every repeated factor of f is a common factor of f and f' — although now the converse no longer follows. Nonetheless, when looking for repeated factors of f, it suffices to confine attention to common factors of f and f'; and unless f' vanishes identically, this is some simplification, at least.

The intention here is to examine the corresponding situation for integers. It has long been recognized that whether or not an integer is "square-free" (*i.e.* free of repeated factors) is one of its basic arithmetic properties. For example, the Möbius function $\mu(n)$ vanishes if and only if n has repeated factors. In view of the situation with polynomials, the idea here is to relate the question of repeated factors to the analog of *derivative* for integers. The first question then is how to *define* the derivative of an integer.

It seems decidedly unfair to say:

$$\text{if} \quad n = \prod_{i=1}^{k} p_i^{\alpha_i}, \quad \alpha_i \geqq 1, \qquad \text{then let} \qquad n' = \prod_{i=1}^{k} p_i^{\alpha_i - 1};$$

for although this has the property that (n, n') gives precisely the *repeated part* of n, it is moreover true that $(n, n') = n'$, a situation which does not hold for polynomials.

If

$$f(n) = \sum_{d|n} g(d),$$

Wintner [9, p. 1] calls $g(n)$ the "arithmetical derivative" of $f(n)$, and claims that this idea is really due to Euler, at least in some special cases. Selberg [7] is probably thinking along the same lines when he defines

$$f'(d) = f(d) \prod_{p|d} (1 - 1/f(p))$$

for the case that $f(d)$ is multiplicative. This agrees with Wintner's definition, at least when d is square-free.

From either point of view, $n' = \varphi(n)$, where $\varphi(n)$ is Euler's function, since it is well-known that

$$n = \sum_{d|n} \varphi(d), \qquad \text{and} \qquad \varphi(n) = n \prod_{p|n} (1 - 1/p).$$

Then, since $\varphi(n) = \Pi p_i^{\alpha_i - 1}(p_i - 1)$ when $n = \Pi p_i^{\alpha_i}$, it is at least true that

ii) every repeated factor of n is a common factor of n and $\varphi(n)$, though not, in general, conversely.

The first illustration of the failure of the converse occurs with $n = 21$ and $\varphi(n) = 12$; then $(n, \varphi(n)) = 3$ even though n is square-free. It is natural to ask about *systems* of integers in which $(n, \varphi(n))$ gives *precisely* the repeated part of n. However, before analyzing this question further, another and perhaps more compelling instance of the spontaneous occurrence of $(n, \varphi(n))$ in mathematics will be mentioned.

3. Second motivation. There is an elegant if little-publicized result in the theory of finite groups which may be worded as follows:

"There is one and only one group of order n, if and only if $(n, \varphi(n)) = 1$".

In particular, if n is prime, $(n, n-1) = 1$, and there is only the cyclic group. But even when $n = 15$, $\varphi(n) = 8$, and there is only one group. On the other hand, when $n = 21$, $(n, \varphi(n)) = 3$, so that there is more than one group of order 21. Burnside [2, p. 48] proves that if $n = pq$ is the product of two primes, $p < q$, then there is a non-cyclic group of order n if and only if $q \equiv 1 \pmod{p}$. From this proof and the fact that there are always two groups of order p^2, it is not difficult to prove the assertion that $(n, \varphi(n)) = 1$ is necessary and sufficient for the existence of only one group of order n.

It is noteworthy that Erdös [3] has obtained an asymptotic expression involving the numbers n which satisfy $(n, \varphi(n)) = 1$. If $A(x)$ denotes the number of such positive integers n which do not exceed x, Erdös's result may be stated

$$A(x) \sim \frac{x e^{-\gamma}}{\log \log \log x}, \tag{1}$$

where "$\backsim$" means "asymptotic, as $x \to \infty$", and where γ, as one might suspect, is Euler's constant.

The appearance of $x/\log\log\log x$ in this result is both surprising and reasonable—surprising in that it certainly does not arise in very many problems; but reasonable in that $\pi(x) < A(x) < S(x)$ expresses not merely an inequality, but in fact an ordering by inclusion, where $\pi(x)$ counts the primes, and $S(x)$ counts the square-free integers. Since $\pi(x) \backsim x/\log x$, and $S(x) \backsim 6x/\pi^2$, and since $A(x)$ is distinctly intermediate, it is very reasonable that $A(x)$ has the order of x divided by a suitably iterated logarithm.

4. Arithmetical semi-groups. Following the usage (if not the precise definition) of Wintner [9, p. 18], a set G of positive integers will be called an *arithmetical semi-group* if it consists of the number 1, a fixed set P of primes, and all products of powers of the primes in P.

Let $X(n)$ be the characteristic function of G. Defining $\mu_G(n) = \mu(n)X(n)$, where $\mu(n)$ is the Möbius function, Wintner [8, p. 70][9, p. 19] observes that

$$\zeta_G(s) = \sum_{n \in G} \frac{1}{n^s} = \prod_{p \in P} (1 - p^{-s})^{-1}, \qquad \text{Re}\, s > 1\,, \tag{2}$$

and the related identity

$$1/\zeta_G(s) = \sum_{n \in G} \frac{\mu_G(n)}{n^s} = \prod_{p \in P} (1 - p^{-s}), \qquad \text{Re}\, s > 1\,. \tag{3}$$

Wintner is concerned primarily with the generalization of the Prime Number Theorem to arithmetical semi-groups, and his results, while certainly interesting, will not be required here.

In the spirit of the preceding sections, suppose that one wishes to construct an arithmetical semi-group G of integers for which, if $n \in G$, and n is square-free, then $(n, \varphi(n)) = 1$. (Equivalently, for *any* n in G, $(n, \varphi(n))$ must yield precisely the repeated part of n). Moreover, suppose that G is required to be *maximal* with respect to this property. Call such a set G a *regular semi-group*. Then the following theorem applies.

THEOREM 1. *If G is a regular semi-group, it has either the single prime generator 2 (so that $G = \{1, 2, 4, 8, 16, 32, \dots\}$); or else, G is generated by infinitely many odd primes.*

PROOF. The condition $(n, \varphi(n)) = 1$ for square-free $n \in G$ is equivalent to the statement that if p_1 and p_2 are any two prime generators of G, then $p_1 \not\equiv 1 \pmod{p_2}$. (That is, if $n = \prod p_i$, then $\varphi(n) = \prod(p_i - 1)$, and

$(n, \varphi(n)) = 1$ will always hold if and only if $p_i - 1$ is always relatively prime to p_j, for any primes p_i and p_j in G).

If $2 \in G$, it is impossible for any *odd* prime p to satisfy $p \not\equiv 1 \pmod 2$. Thus $G_0 = \{1, 2, 4, 8, \ldots\}$ is already maximal. On the other hand, if G has only a *finite* set of *odd* prime generators $p_1, p_2, \ldots, p_k$, let $A = p_1 p_2 \ldots p_k$, and take any prime q in the arithmetic progression $An - 1$. (By Dirichlet's Theorem, there are infinitely many such q). Then $q = An_0 - 1$; and $q - 1 = An_0 - 2$ is divisible by none of the primes $p_1, p_2, \ldots, p_k$, all of which divide A but not 2. Hence q could be added to G, contradicting maximality. This completes the proof.

There is a "most natural" regular semi-group G_1, obtained by going consecutively through the odd primes, and allowing each prime as a generator of G_1 so long as it does not conflict with any previous generator. This set G_1 contains the following 46 primes up to 1000.

3	113	317	479	653	857
5	149	353	503	659	863
17	173	359	509	677	887
23	197	383	557	683	947
29	257	389	563	773	977
53	263	419	569	797	983
83	269	449	593	809	
89	293	467	617	827	

TABLE 1. Primes $\leqq 1000$ belonging to G_1.

It is interesting to note that all the Fermat primes necessarily belong to G_1.

Any integer n which satisfies $(n, \varphi(n)) = 1$ belongs to at least one regular semi-group G, obtained by starting with the prime factors of n as generators of G, and then including enough other primes to make G maximal with respect to the condition $p_i \not\equiv 1 \pmod{p_j}$ for all primes p_i, p_j in G. Conversely, defining $\overline{G}$ to be the set of *square-free* integers in G, the union of the $\overline{G}$'s, extended over all regular semi-groups, is precisely the set of integers n which satisfy $(n, \varphi(n)) = 1$. (If P is the set of primes which generate G, then $\overline{G}$ consists of 1, and all products of *distinct* members of P).

Using Dirichlet's Theorem once more, it is easy to establish the following result:

THEOREM 2. *The number of regular semi-groups is infinite.*

PROOF. In the process described for forming G_1, one may deviate at the i^{th} stage by rejecting q_i, and using a prime of the form

$$q_i' = (q_1 q_2 \ldots q_i)n - (q_i - 1)$$

instead. The number q_i' exists by Dirichlet's Theorem, which applies because $(q_1 q_2 \ldots q_i, q_i - 1) = 1$. Note that $q_i' \not\equiv 1 \pmod{q_j}$ for $j = 1, 2, \ldots, i-1$. Completing this set (containing $q_1, \ldots, q_{i-1}, q_i'$) in any fashion to form a regular semi-group (call it H_i), q_i will not be included, because it conflicts with q_i', in that $q_i' \equiv 1 \pmod{q_i}$. Then for every i, H_i agrees with G_1 in its first $i-1$ prime generators, and differs on the i^{th}. Thus, all the sets H_i are distinct from each other and from G_1. Hence they furnish an infinity of examples of regular semi-groups.

Actually, the number of regular semi-groups is non-denumerable. This can be shown by supposing a denumeration exists, selecting a prime from each semi-group (these representatives need not be all distinct), and using Dirichlet's Theorem to find a consistent set of primes conflicting with each prime of this representative set.

5. The upper bound. It is of interest to compare the density of the prime generators of G_1 with the twin primes. Denoting by $g(x)$ and $T(x)$ the number of primes in G_1, and of pairs of twin primes, respectively, which are $\leq x$, the following table shows the comparison, by hundreds, up to $x = 1000$.

x	$g(x)$	$T(x)$
100	8	8
200	12	15
300	16	19
400	21	21
500	25	24
600	31	27
700	36	29
800	38	29
900	43	34
1000	46	34

TABLE 2.

Even on the basis of such inadequate evidence, there is the distinct impression that the growth of $g(x)$ is more rapid than the growth of $T(x)$.

This has a simple explanation (non-rigorous, to be sure) from the *sieve* point of view. The twin primes result from a *double* application of the Sieve of Eratosthenes, whereas the primes of G_1 are determined by a "one-and-one-half-fold" application of the sieve.

More precisely, to determine the twin primes between $x^{\frac{1}{2}}$ and x, one lists all the primes $p_i \leqq x^{\frac{1}{2}}$, and eliminates from the list of integers between $x^{\frac{1}{2}}$ and x any n which satisfies either

i) $n \equiv 0 \pmod{p_i}$ for some p_i, or

ii) $n \equiv -2 \pmod{p_i}$ for some p_i.

This is a *double* elimination process, and the numbers which remain are the primes p between $x^{\frac{1}{2}}$ and x such that $p+2$ is also prime.

On the other hand, to determine the prime generators of G_1 between $x^{\frac{1}{2}}$ and x, one lists all the primes $p_i \leqq x^{\frac{1}{2}}$, and all the prime generators $q_i \leqq x^{\frac{1}{2}}$ of G_1, and then eliminates those integers n from the listing of integers between $x^{\frac{1}{2}}$ and $2x^{\frac{1}{2}}$ which satisfy either

i)′ $n \equiv 0 \pmod{p_i}$ for some p_i, or

ii)′ $n \equiv 1 \pmod{q_i}$ for some q_i.

What remains is the set of prime generators of G_1 between $x^{\frac{1}{2}}$ and $2x^{\frac{1}{2}}$. Since the q_i are a proper subset of *all* the primes, the elimination ii)′ is only a *partial* application of the sieve; hence the (imprecise) term "one-and-one-half-fold" application. (Enlarging the set of q_i's to extend up to $2x^{\frac{1}{2}}$, one then proceeds to "sift" the integers between $2x^{\frac{1}{2}}$ and $4x^{\frac{1}{2}}$; and this process can be continued until x is reached).

If ii)′ is strengthened to

ii)″ $n \equiv 1 \pmod{p_i}$ for some odd prime p_i,

the corresponding *two*-fold application of the sieve determines precisely the Fermat primes $p = 2^{2^n}+1$ between $x^{\frac{1}{2}}$ and x. (Thus the assertion in section 4 that all the Fermat primes belong to G_1). It seems to have been generally overlooked in the literature that the Fermat primes can thus be obtained by a double application of the sieve. Similarly the Mersenne primes, using the condition:

iii) $n \equiv -1 \pmod{p_i}$ for some odd prime p_i,

instead of the condition ii)″. Thus, not all two-fold applications of the sieve are likely to yield sets of primes with the same asymptotic distribution.

The purpose of the next section is to show that the "one-and-one-half-fold" application of the sieve can be mechanized, along the lines of Brun [1] or Selberg [7], to prove that $g(x) > Ax/\log x$ does *not* hold for any $A > 0$, and all $x > x_0$. Since $g(x)$ appears to be bigger than $T(x)$, and the best result [1] on $T(x)$ is $T(x) = O(x/\log^2 x)$, this result on $g(x)$ cannot be considered trivial. Furthermore, since $g(x) \to \infty$ as $x \to \infty$ (by Theorem

1), the prime generators of G_1 form a set which can be *proved* to have intermediate density—infinite, yet not greater than $Ax/\log x$ for any $A > 0$.

6. Applying the sieve.

THEOREM 3. *The set of primes which generate G_1 has intermediate density.*

PROOF. Let $a(p) = 2$ if p is a prime in G_1, and $a(p) = 1$ for all other primes. Extend the definition of $a(n)$ to all the integers by making it completely multiplicative. (Actually, $a(n)$ will only be needed for square-free n. Its rôle will be to count residue classes modulo n—how many are "eliminated" in the sieve process by the various prime divisors of n).

Using almost any form of Brun's sieve method, the crucial step is to obtain an upper bound for the product

$$\prod_{p<y}(1 - a(p)/p) \leqq \prod_{p<y}(1-1/p)^{a(p)} = \prod_{p<y}(1-1/p)\prod_{q<y}(1-1/q), \tag{4}$$

where p runs through all the primes, and q runs through the prime generators of G_1. ($y > 3$ is a real number). The estimate

$$\prod_{p<y}(1-1/p) < 1/\log y \tag{5}$$

is classical; so the problem is to estimate

$$Q(y) = \prod_{q<y}(1-1/q)\,. \tag{6}$$

To begin with,

$$\log Q(y) = \sum_{q<y}\log(1-1/q) = -\sum_{q<y}\sum_{k=1}^{\infty}1/kq^k\,. \tag{7}$$

Since

$$1/q < \sum_{k=1}^{\infty}1/kq^k < \sum_{k=1}^{\infty}1/q^k = 1/(q-1)\,,$$

$$-\sum_{q<y}1/(q-1) < \log Q(y) < -\sum_{q<y}1/q\,, \tag{8}$$

and the problem is now to estimate $\Sigma_{q<y}1/q$.

At first glance, it appears reasonable to protest "but if there is already a good estimate for this sum, why bother with the sieve method?" The answer is simple enough: Using the machinery of the sieve, a *large* estimate for $\Sigma_{q<y}1/q$ will lead to a *low* density for the primes in G_1.

Suppose $g(x) > Ax/\log x$ for some fixed $A > 0$, and all large x. Then

$$n = g(q_n) > Aq_n/\log q_n \qquad \text{and} \qquad q_n < A^{-1}n\log q_n\,,$$

for all sufficiently large n. Moreover, $q_n < n^2$ for all sufficiently large n (otherwise, $g(x) < x^{\frac{1}{2}}$ would occur for arbitrarily large values of x), so that

$$q_n < cn \log n , \tag{9}$$

for some constant $c > 0$, and all $n > n_0$. Thus

$$\sum_{q_n < y} 1/q_n > \sum_{q_{n_0} < q_n < y} 1/q_n > \sum_{n_0 < n < y/(c \log y)} 1/cn \log n > c_1 \log\log y , \tag{10}$$

for all y sufficiently large. Combining this inequality with (8),

$$Q(y) < e^{-c_1 \log\log y} = e^{-\log(\log y)^{c_1}} = 1/(\log y)^{c_1}; \tag{11}$$

and recalling (4) and (5), this yields the important estimate

$$\prod_{p < y} \left(1 - a(p)/p\right) < 1/(\log y)^{1+c_1} . \tag{12}$$

Using Landau's $\Omega(d)$ notation [6, pp. 71–78], $|\mu(d)\, a(d)| \leqq |\mu(d)\, 2^{\Omega(d)}|$ for all d, so that the remainder-term approximations which Landau makes in the course of *his* application of the sieve method surely hold for the present problem. Then, setting $y = x^{1/c_2 \log\log x}$,

$$(\log y)^{1+c_1} = \left(\frac{\log x}{c_2 \log\log x}\right)^{1+c_1} > (\log x)^{1+c_3} \tag{13}$$

for all sufficiently large x, where again $c_3 > 0$.

Finally, still following Landau's pattern [6],

$$g(x) < \frac{c_4 x}{(\log y)^{1+c_1}} < \frac{c_4 x}{(\log x)^{1+c_3}} , \tag{14}$$

in direct contradiction to the assumption $g(x) > Ax/\log x$ for all large x. This assumption is therefore false.

In view of Theorem 1, the set of primes which generate G_1 has now been proved *intermediate* in the sense of section 1. Moreover, the proofs of Theorems 1 and 3 apply to all regular semi-groups other than G_0. Since it was shown (Theorem 2) that the number of regular semi-groups is infinite, an infinite number of examples of intermediate sets of primes is thereby furnished, though it is indeed a moot point whether *all* of them arise "naturally".

7. Other intermediate sets. It is no doubt possible to discover many other intermediate sets of primes, proceeding along completely different lines. One such example is the "Fermat-Pólya" set described herewith.

Pólya bases a proof (see [5, p. 14]) that the number of primes is infinite on the Fermat numbers $F_n = 2^{2^n} + 1$, no two of which have a common divisor greater than 1. Let P_n be the largest prime divisor of F_n. Since $(P_i, P_j) = 1$ if $i \neq j$, the set $\{P_n\}$ is infinite. Moreover, it can be shown that $\{P_n\}$ is intermediate, and that $\Sigma 1/P_n$ converges. The following lemma is useful.

Lemma. *Every prime factor of F_n has the form*

$$p = 2^{n+1}K + 1 ,$$

with K a positive integer.

Proof. If p divides F_n, then

$$2^{2^n} \equiv -1 \pmod{p} ,$$

so that

$$2^{2^{n+1}} \equiv (-1)^2 \equiv 1 \pmod{p} .$$

Let $(p-1)/k$ be the index of 2 modulo p. Then $2^{n+1} = h(p-1)/k$, where h is an integer. Suppose $h = 2^a m$. Then $2^{n+1-a} = m(p-1)/k$; and

$$2^{2^{n+1-a}} \equiv 1 \pmod{p} .$$

But

$$\left(2^{2^{n+1-a}}\right)^{2^{a-1}} = 2^{2^n} \equiv -1 \pmod{p} ,$$

which means that $a = 0$. Hence h is odd. But since 2^{n+1} has no odd factors, h must cancel completely with the factors of k. Let $K = k/h$. Then $2^{n+1} = (p-1)/K$, and the assertion follows.

Theorem 4. *The Fermat-Pólya set $\{P_n\}$ has intermediate density. Also, $\Sigma 1/P_n$ converges.*

Proof. By the lemma, $P_n > 2^n$, $\Sigma 1/P_n < \Sigma 1/2^n = 1$; and $P_n < x$ implies $2^n < x$, $n = O(\log x)$.

Theorem 4 can also be made to depend on a result due to Erdös [4], which states that

$$\sum_{1 < d | F_n} 1/d = o(1) \quad \text{as} \quad n \to \infty , \tag{15}$$

where the sum is extended over *all* the divisors of $F_n = 2^{2^n} + 1$ except 1.

In view of the lemma, the set of *smallest* prime divisors of the Fermat numbers also satisfies Theorem 4. It would be interesting to discover whether or not the set of *all* prime divisors of the Fermat numbers is intermediate.

Had Fermat's conjecture been correct, and every F_n a prime, they would not in themselves generate a regular semi-group, but would at least be included among the generators of G_1, as well as of other regular semi-groups. However, the Fermat-Pólya set of Theorem 4 is not likely to satisfy the $p_i \not\equiv 1 \pmod{p_j}$ condition for regular semi-groups. Thus,

$$F_5 = 2^{32}+1 = 641 \cdot 6700417 ,$$

with

$$641 \equiv 1 \pmod{F_1} \quad \text{and} \quad 6700417 \equiv 1 \pmod{F_0} .$$

If $P_0 = F_0 = 3$ is included in the Fermat-Pólya set $\{P_n\}$, then a violation of $p_i \not\equiv 1 \pmod{p_j}$ is already illustrated: $P_5 \equiv 1 \pmod{P_0}$. The other congruence, $641 \equiv 1 \pmod 5$, is an adequate counter-example for the set of *smallest* prime factors of F_n.

For those who are not content with but one intermediate set of primes constructed along the lines just presented, it is possible to consider $F_n' = a^{2^n}+1$, where a is any even number, as well as $F_n'' = \frac{1}{2}(b^{2^n}+1)$, where $b > 1$ is odd. The analogous definitions and theorems should be obvious.

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