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A NOTE ON CONTRACTION SEMIGROUPS

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1. Let X be an abstract (L)-space, satisfying the conditions I–IX of Kakutani [5]. A positive linear operator P on X is called a *contraction operator* if

$$(1) \qquad \|Px\| \leqq \|x\| \quad \text{when} \quad x \geqq 0 .$$

Such an operator is necessarily bounded, with

$$(2) \qquad \|P\| \leqq 1 .$$

If equality holds in (1) for each $x \geqq 0$, and hence in (2) also, P is called a *transition operator.*

We shall be concerned with semigroups

$$\Sigma \equiv \{P_t \colon \ t \geqq 0\}$$

of contraction operators P_t which are such that

$$P_0 = I, \qquad P_{t+s} = P_t P_s \quad (t \geqq 0,\ s \geqq 0) ,$$
$$\|P_t x - x\| \to 0 \quad \text{as} \quad t \to +0, \quad \text{for each} \quad x \in X .$$

We call Σ a contraction (transition) semigroup when P_t, for each $t \geqq 0$, is a contraction (transition) operator. Finally, if $\Sigma' \equiv \{P_t'\}$ is another (contraction or transition) semigroup, we say that Σ' *dominates* Σ if

$$P_t' x \geqq P_t x \quad \text{when} \quad x \geqq 0 \quad \text{and} \quad t \geqq 0 .$$

In applications of semigroup theory (e.g. to the study of Markov processes) it is sometimes important to know whether a given semigroup Σ is dominated by any other semigroups Σ'. We shall show that the situation is very simple:

If Σ is a transition semigroup, no distinct contraction semigroup Σ' dominates Σ;

if Σ is a contraction, but not transition, semigroup, then there exist infinitely many distinct Σ' which dominate Σ, and amongst them are infinitely many transition semigroups.

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In the second case, we shall construct only some of the Σ' which dominate Σ; the problem of constructing all of them remains open.

2. We first recall some relevant facts from semigroup theory (see [3], [4]). The semigroup Σ has an infinitesimal generator Ω with dense domain $D(\Omega)$, and $\lambda I-\Omega$ has a bounded inverse J_λ (with domain X) for each $\lambda>0$. Thus the equation

$$\lambda y-\Omega y = x \qquad (\lambda > 0)\,, \tag{3}$$

for given $x\in X$, has the unique solution $y=J_\lambda x$. The resolvent operator J_λ can be calculated from the representation

$$J_\lambda x = \int_0^\infty e^{-\lambda t}\, P_t x\, dt; \tag{4}$$

conversely, P_t can be calculated from J_λ by either of the inversion formulae

$$\begin{cases} P_t x = \lim\limits_{n\to\infty} (nt^{-1}J_{n/t})^n x\,, \\ P_t x = \lim\limits_{\lambda\to\infty} \exp[t\lambda(\lambda J_\lambda - I)]x\,, \end{cases} \tag{5}$$

due to Hille [3] and Yosida [6] respectively. Finally, the fundamental Hille-Yosida theorem (see [4], [6], [7]) states that an operator Ω with dense domain generates a contraction (transition) semigroup if and only if, for each $\lambda>0$, Ω has a resolvent J_λ with domain X and such that λJ_λ is a contraction (transition) operator.

It is seen at once by using the identity

$$\|x+y\| = \|x\|+\|y\| \qquad (x \geqq 0,\ y \geqq 0)$$

that if a contraction operator P' dominates a transition operator P (in the sense that $P'x\geqq Px$ when $x\geqq 0$), then $P'\equiv P$. Thus, trivially, if a contraction semigroup Σ' dominates a transition semigroup Σ, then $\Sigma'\equiv\Sigma$. If Σ is not a transition semigroup, our construction of dominant semigroups Σ' will be motivated by a characterisation of operators Ω' which generate such a Σ' (Lemma 2 below).

For any $x\in X$, write

$$x^+ = \sup(x, 0), \qquad x^- = \sup(-x, 0)\,.$$

Then (see [5])

$$x = x^+-x^-, \qquad \|x\| = \|x^+\|+\|x^-\|\,.$$

An elementary argument shows that

$$(u, x) \equiv \|x^+\| - \|x^-\|$$

defines a positive linear functional u on X, and clearly $(u, x) = \|x\|$ if and only if $x \geqq 0$. Hence a positive linear operator is a contraction (transition) operator if and only if

$$(u, Px) \leqq (u, x) \qquad (= (u, x))$$

for all $x \geqq 0$. We now give a variant of the Hille-Yosida theorem which will be convenient for our purposes.

LEMMA 1. *A linear operator Ω with dense domain generates a contraction (transition) semigroup if and only if*

(i) $(u, \Omega x) \leqq 0 \ (=0)$ *whenever* $x \geqq 0$ *and* $x \in D(\Omega)$;

(ii) *for each* $\lambda > 0$ *and* $x \in X$, *the equation*

$$\lambda y - \Omega y = x$$

has a unique solution $y \equiv J_\lambda x \in D(\Omega)$, *and* $J_\lambda x \geqq 0$ *when* $x \geqq 0$.

PROOF. Since

$$(u, \Omega x) = \lim_{t \to 0} \big(u, t^{-1}(P_t x - x)\big) = \lim_{t \to 0} t^{-1}(\|P_t x\| - \|x\|),$$

condition (i) is clearly necessary; the necessity of (ii) is included in the Hille-Yosida theorem. Conversely, if (i) and (ii) hold, then for any $x \geqq 0$ and $\lambda > 0$ we have

$$(u, \lambda J_\lambda x) = (u, x) + (u, \Omega J_\lambda x) \leqq (u, x) \qquad (= (u, x)),$$

because $J_\lambda x \geqq 0$ and $J_\lambda x \in D(\Omega)$. Hence λJ_λ is a contraction (transition) operator, and the Hille-Yosida theorem shows that Ω generates a contraction (transition) semigroup.

LEMMA 2. *Let Ω generate a contraction semigroup Σ, and let Ω' be an operator with domain $D(\Omega') = D(\Omega)$. Then Ω' will generate a contraction semigroup Σ' which dominates Σ if and only if*

(i) $\Omega' x \geqq \Omega x$ *whenever* $x \geqq 0$ *and* $x \in D(\Omega)$;

(ii) $(u, \Omega' x) \leqq 0$ *whenever* $x \geqq 0$ *and* $x \in D(\Omega)$;

(iii) *For each* $\lambda > 0$, $\lambda I - \Omega'$ *has a positive inverse* J_λ' *with domain* X.

PROOF. The necessity of (ii) and (iii) follows from Lemma 1, and that of (i) follows at once from

$$\Omega' x - \Omega x = \lim_{t \to 0} t^{-1}(P_t' x - P_t x).$$

Conversely, if (i)–(iii) hold, then (ii) and (iii) together with Lemma 1 imply that Ω' generates a contraction semigroup Σ' with resolvent operator J_λ'. Also, if $x \geqq 0$ and $\lambda > 0$, then $J_\lambda' x \geqq 0$ and $J_\lambda' x \in D(\Omega)$, so that (i) gives

$$\Omega' J_\lambda' x \geqq \Omega J_\lambda' x ,$$

$$(\lambda I - \Omega) J_\lambda' x \geqq (\lambda I - \Omega') J_\lambda' x = x .$$

Operating with J_λ on the left, we obtain

$$J_\lambda' x \geqq J_\lambda x \qquad (\lambda > 0,\ x \geqq 0) ,$$

and now either of the inversion formulae (5) shows that

$$P_t' x \geqq P_t x \qquad (x \geqq 0,\ t \geqq 0)$$

so that Σ' dominates Σ.

3. We now suppose that Σ is a contraction but not transition semigroup, and look for semigroups Σ' which dominate Σ. We shall see that there exist such Σ' with generators Ω' such that $D(\Omega') = D(\Omega)$. Lemma 2 is a guide towards finding suitable operators Ω', and indeed it is easy to write down an Ω' which satisfies conditions (i) and (ii). To do this, choose a fixed element $c \in X$ such that $c \geqq 0$ and $0 < \|c\| \leqq 1$, and define Ω' with domain $D(\Omega') = D(\Omega)$ by

$$\Omega' x = \Omega x - (u, \Omega x) c . \tag{6}$$

Since $(u, \Omega x) \leqq 0$ when $x \geqq 0$ and $x \in D(\Omega)$, (i) clearly holds, and so does (ii) because

$$(u, \Omega' x) = (u, \Omega x)(1 - \|c\|) .$$

It so happens that Ω' also satisfies (iii), i.e. that the equation

$$\lambda y - \Omega' y = x \qquad (\lambda > 0) \tag{7}$$

has a unique solution $y \equiv J_\lambda' x$ in $D(\Omega')$, and that $J_\lambda' x \geqq 0$ when $x \geqq 0$. To prove this, write (7) as

$$(\lambda I - \Omega) y + (u, \Omega y) c = x .$$

Operating with J_λ (which is 1–1), this is equivalent to

$$y + (u, \Omega y) J_\lambda c = J_\lambda x ,$$

so that any solution of (7) necessarily has the form

$$y = J_\lambda x + \alpha J_\lambda c , \tag{8}$$

for some α (depending on x). Now y, as defined in (8), will satisfy (7) if and only if

$$x + \alpha c + \big(u, \Omega J_\lambda(x+\alpha c)\big)c = x ,$$

$$\alpha + \big(u, (\lambda J_\lambda - I)(x+\alpha c)\big) = 0 ,$$

$$\alpha[1+(u, \lambda J_\lambda c - c)] = (u, x - \lambda J_\lambda x) . \tag{9}$$

The coefficient of α in (9) is

$$1 - \|c\| + \|\lambda J_\lambda c\| > 0$$

since $1-\|c\| \geqq 0$ and $\|\lambda J_\lambda c\| > 0$; hence there is exactly one α such that (8) defines a solution of (7). Also, since

$$(u, x - \lambda J_\lambda x) = \|x\| - \|\lambda J_\lambda x\| \geqq 0 \text{ when } x \geqq 0 ,$$

we shall have $\alpha \geqq 0$ when $x \geqq 0$. We have now proved that (7) has a unique solution, $\geqq 0$ when $x \geqq 0$, and thus Ω' satisfies condition (iii) of Lemma 2. This concludes the proof that Ω' generates a contraction semigroup Σ' which dominates Σ.

Since Σ was assumed to be not a transition semigroup, Lemma 1 shows that

$$(u, \Omega x) < 0 \quad \text{for some } x \geqq 0 \text{ in } D(\Omega) . \tag{10}$$

Now $(u, \Omega' x) = (u, \Omega x)(1-\|c\|)$, so (10) implies that Σ' is a transition semigroup if and only if $\|c\| = 1$. It also implies that $\Omega' \neq \Omega$, and that distinct choices of c in (6) give rise to distinct Ω'; hence $\Sigma' \neq \Sigma$ and distinct c give rise to distinct Σ'. We are therefore able to summarise our conclusions in the following

THEOREM. *Let Σ be a contraction semigroup, generated by Ω. Then:*

(a) *If Σ is a transition semigroup, any contraction semigroup which dominates Σ coincides with Σ.*

(b) *If Σ is not a transition semigroup, the operator Ω_c defined by*

$$\Omega_c x = \Omega x - (u, \Omega x)c, \quad x \in D(\Omega) \quad (\text{with } c \geqq 0 \text{ and } 0 < \|c\| \leqq 1)$$

generates a contraction semigroup Σ_c dominating Σ. Also $\Sigma_c \neq \Sigma$, $\Sigma_{c_1} \neq \Sigma_{c_2}$ if $c_1 \neq c_2$, and Σ_c is a transition semigroup if and only if $\|c\| = 1$.

We remark that unless X is 1-dimensional, it will contain infinitely many distinct positive elements of norm 1, so that there will be infinitely many transition semigroups amongst the Σ_c.

4. The reader who is familiar with the theory of Markov processes may find it interesting to observe that the above construction of Σ_c from Σ is an analytical generalisation of a well known probabilistic construction due to Doob [1]. His construction converts a Markov process with a countable set of states, with transition probabilities $p_{ij}(t)$ such that $\Sigma_j p_{ij}(t) \leqq 1$, into a new process with transition probabilities $p_{ij}^*(t) \geqq p_{ij}(t)$ such that $\Sigma_j p_{ij}^*(t) = 1$. An analytical version of Doob's construction was found some time ago by David G. Kendall (unpublished); he also calculated the Laplace transform of $p_{ij}^*(t)$ from that of $p_{ij}(t)$, his result being equivalent to our solution of (7) in the special case when $X = (l)$, the space of absolutely convergent series. Recently W. Feller has also introduced an analytical version of Doob's construction for diffusion processes (cf. the "instantaneous return process" in [2]). Here X is the space of finite signed measures on the real line, but a direct comparison with our result is difficult because Feller considers a class of semigroups distinct from ours.

5. I am indebted to David G. Kendall for showing me his unpublished work on Markov processes, for suggesting the present more general investigation, and for some helpful comments during its progress.

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