MATH. SCAND. 3 (1955), 283–302

ON A THEOREM OF F. RIESZ

BENT FUGLEDE

Introduction.

The following characterization of the indefinite integral of functions from the class L^p corresponding to an interval (a, b), finite or infinite, was given in 1910 by F. Riesz [4, in particular § 5]:

Let $1 < p < \infty$. *In order that a function* $\varphi(x)$ *be an indefinite integral of some function* $f(x)$ *from the class* L^p, *it is necessary and sufficient that the sum*

$$\sum_{\nu=1}^{n} \frac{|\varphi(x_\nu) - \varphi(x_{\nu-1})|^p}{(x_\nu - x_{\nu-1})^{p-1}},$$

where $a \leqq x_0 < x_1 < \ldots < x_n \leqq b$, *is bounded by some constant* c *independent of the number and the location of the points of division* x_ν.—*Moreover, in the affirmative case, the smallest possible value of* c *is*

$$\int_a^b |f(x)|^p \, dx .$$

This result may be generalized in various directions, as it will be shown in the present paper. In the first place, Lebesgue measure on the interval (a, b) may be replaced by an arbitrary measure μ in an abstract space X. The differences $\varphi(x') - \varphi(x'')$, corresponding to subintervals (x', x''), should be replaced by the values of an additive set-function $\varphi(A)$ defined on a system $\mathfrak{A}$ of μ-measurable subsets of X. Under certain natural assumptions concerning the system $\mathfrak{A}$, one may prove Theorem I of Section 2.1 which is a direct generalization of Riesz' theorem[1]. Various definitions and well-known results from abstract measure theory, which will be used in the proof of Theorem I, are collected in Section 1.

Received August 30, 1955.

[1] In a recent article, R. E. Fullerton [1] has likewise treated the question of an abstract form of Riesz' theorem. His result in this direction may be described as the special case of our Theorem I (cf. 2.1) in which $\mathfrak{A}$ consists of *all* measurable subsets of X, i.e. $\mathfrak{A} = \mathfrak{M}$. Thus Fullerton's result does not contain the classical theorem of Riesz, in which $\mathfrak{A}$ consists merely of all (half-open) subintervals of (a, b), whereas $\mathfrak{M}$ is the system of Borel subsets of (a, b).

Secondly, it is easy to show that a quite similar result holds for systems of set-functions, or, in other words, set-functions whose values, instead of being real or complex numbers, are points (vectors) in a Euclidean space R^k of arbitrary dimension k (cf. Section 4.1). And, finally, the classes L^p may be replaced (cf. Section 4.2) by more general classes of functions (with values in R^k). Corresponding to a real-valued function $F(t)=F(t^1, \ldots, t^k)$ defined in R^k we may introduce the class L_F (with respect to a measure μ in X) of μ-measurable functions $f(x)$ defined in X (and with values in R^k), for which

$$\int_X F(f(x))\,\mu(dX) < \infty .$$

Under the additional assumptions that $F(t)$ is *convex* and that

$$F(t)/|t| \to +\infty \quad \text{as} \quad |t| \to \infty ,$$

we shall obtain a result (Theorem IV) quite analogous to Theorem I. (Here $|t|$ denotes the Euclidean norm $|t| = ((t^1)^2 + \ldots + (t^k)^2)^{1/2}$.)

For the sake of completeness, we have included (Section 3) a treatment of the corresponding problem for the extreme cases $p=\infty$ and $p=1$ of the L^p-classes. The case $p=\infty$ is similar to the case $1<p<\infty$, whereas the case $p=1$, as well known, is of an essentially different character even in the classical situation of Riesz' theorem: functions of a real variable with Lebesgue measure. The condition in Riesz' theorem expresses, for $p=1$, merely that $\varphi(x)$ is of bounded variation. Thus it becomes necessary to add, as a further condition, that φ is absolutely continuous in the sense that there exists, corresponding to every $\varepsilon > 0$, a $\delta > 0$ with the property that

$$\sum_{\nu=1}^{n} |\varphi(b_\nu) - \varphi(a_\nu)| < \varepsilon$$

for every finite system of mutually disjoint subintervals $(a_1, b_1), \ldots (a_n, b_n)$ of (a, b) for which

$$\sum_{\nu=1}^{n} (b_\nu - a_\nu) < \delta .$$

Theorem III is a generalization of this well-known result to the abstract situation. Closely related theorems have appeared in the literature (for example the theorem on p. 34 in B. Jessen [3]).

The results of the present paper were developed primarily in view of an application thereof in determining the structure of the functions which constitute the domain of the closure of a linear partial differential operator with constant coefficients. A separate publication concerning this problem is under preparation.

1. Auxiliary concepts and results from measure theory.

1.1. Operations with sets. We consider subsets of a fixed set X. The empty set is denoted by O. The union, intersection, and difference of sets will be denoted by $\cup$, $\cap$, and $-$, respectively. (The difference $A-B$ is to be defined even when B is not a subset of A). When $A, A_1, A_2, \dots$ are subsets of X, we write $A_n \to A$ if the characteristic function $a_n(x)$ of A_n converges to the characteristic function $a(x)$ of A at every point $x \in X$ when $n \to \infty$. If $A_n \to A$ and $B_n \to B$, then $A_n \cup B_n \to A \cup B$, $A_n \cap B_n \to A \cap B$, and $A_n - B_n \to A - B$. The symbols $A_n \uparrow A$ and $A_n \downarrow A$ shall denote that the sequence $\{A_n\}$ increases, respectively decreases, and that $\bigcup_n A_n = A$, resp. $\bigcap_n A_n = A$. If $A_n \uparrow A$ or if $A_n \downarrow A$, then $A_n \to A$.

1.2. Systems of sets. When $\mathfrak{A}$ is a system of sets (subsets of X), we shall denote by $\mathfrak{A}_\sigma$, resp. $\mathfrak{A}_\delta$, the system of all unions, resp. intersections, of countable systems of sets from $\mathfrak{A}$. Clearly, $\mathfrak{A}_\sigma$, resp. $\mathfrak{A}_\delta$, is closed under the formation of countable unions, resp. intersections, that is, $\mathfrak{A}_{\sigma\sigma} = \mathfrak{A}_\sigma$ and $\mathfrak{A}_{\delta\delta} = \mathfrak{A}_\delta$. If $\mathfrak{A}$ is closed under the formation of *finite* unions (intersections), then each set from $\mathfrak{A}_\sigma$ ($\mathfrak{A}_\delta$) may be represented as the union (intersection) of an increasing (a decreasing) sequence of sets from $\mathfrak{A}$. Using this remark, it is easy to show that if $\mathfrak{A}$ is closed under the formation of *finite unions and intersections*, then the same is true of $\mathfrak{A}_\sigma$ and of $\mathfrak{A}_\delta$, and hence also of $\mathfrak{A}_{\sigma\delta}$ and $\mathfrak{A}_{\delta\sigma}$, etc. This applies, in particular, to a *field* $\mathfrak{F}$, that is, a (non empty) system of subsets of X which is closed under the formation of differences and of finite unions (and hence also of finite intersections). If $A \in \mathfrak{F}_\sigma$ and $B \in \mathfrak{F}_\delta$, then $A - B \in \mathfrak{F}_\sigma$ and $B - A \in \mathfrak{F}_\delta$, as it may be easily shown. Similarly, if $A \in \mathfrak{F}_{\sigma\delta}$ and $B \in \mathfrak{F}_{\delta\sigma}$, then $A - B \in \mathfrak{F}_{\sigma\delta}$ and $B - A \in \mathfrak{F}_{\delta\sigma}$, etc.

A field $\mathfrak{F}$ which is closed under the formation of *countable* unions is called a *σ-field*; it is also closed under the formation of countable intersections. Thus $\mathfrak{F} = \mathfrak{F}_\sigma = \mathfrak{F}_\delta = \mathfrak{F}_{\sigma\delta}$, etc. For any system $\mathfrak{A}$, the intersection of all σ-fields that contain $\mathfrak{A}$ is itself a σ-field. We call it the σ-field *generated by* $\mathfrak{A}$.

1.3. Set-functions. We shall consider set-functions whose values are either finite complex (in particular finite real) numbers, or non-negative real numbers in which case the value $+\infty$ is admitted. The system of sets at which a set-function φ is defined is called the *domain* of φ. We call φ *bounded* if there is a finite constant c so that $|\varphi(A)| \leqq c$ for every A in the domain of φ. A set-function φ is called *additive* if $\varphi(A) = \sum_n \varphi(A_n)$ whenever A is the union of a finite system of mutually disjoint sets A_n, and A and each A_n belong to the domain of φ. If a similar statement holds even for countable unions of disjoint sets, then φ is called *countably additive.*

The real part φ_1 and the imaginary part φ_2 of a set-function φ are again set-functions with the same domain, and each of the properties: boundedness, additivity, or countable additivity applies to φ_1 and φ_2 if, and only if, it applies to φ. Consider now a bounded additive real-valued set-function φ defined on a field $\mathfrak{F}$, and define, for each $A \in \mathfrak{F}$, $\varphi^+(A) = \sup \varphi(B)$ and $\varphi^-(A) = \inf \varphi(B)$, where B ranges over all subsets of A which belong to $\mathfrak{F}$. It is easily proved that φ^+ and φ^- are likewise bounded additive set-functions on $\mathfrak{F}$ (Cf. B. Jessen [3, section 3.2]). Moreover, for every $A \in \mathfrak{F}$, we have $\varphi(A) = \varphi^+(A) + \varphi^-(A)$; $\varphi^+(A) \geqq 0$; $\varphi^-(A) \leqq 0$. If φ is countably additive, then so are φ^+ and φ^-.—More generally, every complex-valued bounded additive set-function φ defined on a field $\mathfrak{F}$ may be represented as $\varphi = \varphi_1{}^+ + \varphi_1{}^- + i\varphi_2{}^+ + i\varphi_2{}^-$, where $\varphi_1{}^+$, $-\varphi_1{}^-$, $\varphi_2{}^+$, and $-\varphi_2{}^-$ are non-negative bounded additive set-functions (i.e. bounded contents) on $\mathfrak{F}$. If φ is even countably additive, then so are the four "components", and if, in addition, $\mathfrak{F}$ is a σ-field, then the four components are bounded measures; (cf. the following definitions).

An additive set-function μ is called a *content* if the domain is a field $\mathfrak{F}$, if the values are non-negative (the value $+\infty$ being admitted), and if finally μ is *σ-finite,* whereby is meant that each set $A \in \mathfrak{F}$ for which $\mu(A) = +\infty$ (if any such set exists) may be covered by a countable system of sets from $\mathfrak{F}$ at each of which μ takes a finite value. A countably additive set-function μ is called a *measure* if the domain is a σ-field, if the values are non-negative (the value $+\infty$ being admitted), and if μ is σ-finite.

1.4. Extension theorems. A basic theorem in measure theory asserts that every countably additive content μ may be extended in just one way to a measure whose domain is the σ-field $\mathfrak{M}$ generated by the domain $\mathfrak{F}$ of the given content. (Cf. B. Jessen [3, section 1.5].) The extension is given by the formula

$$\mu(A) = \inf \sum_n \mu(A_n), \qquad A \in \mathfrak{M},$$

where the infimum is formed with respect to all sequences of sets $A_n \in \mathfrak{F}$

for which $A \subset \bigcup_n A_n$. It suffices, however, to consider coverings by mutually disjoint sets $A_n \in \mathfrak{F}$. Then

$$\sum_n \mu(A_n) = \lim_{n\to\infty} \sum_{\nu=1}^{n} \mu(A_\nu) = \lim_{n\to\infty} \mu\left(\bigcup_{\nu=1}^{n} A_\nu\right),$$

so that, in particular, the extension is bounded if the given content is bounded.

A similar extension theorem subsists for bounded complex-valued (in particular real-valued) set-functions: A bounded and countably additive set-function φ defined on a field $\mathfrak{F}$ may be extended in just one way to a bounded and countably additive set-function defined on the σ-field generated by $\mathfrak{F}$. In the proof of this theorem one may utilize the decomposition $\varphi = \varphi_1^+ + \varphi_1^- + i\varphi_2^+ + i\varphi_2^-$. (Cf. B. Jessen [3, section 3.3].)

1.5. Conditions for countable additivity. If φ is a finite (complex- or real-valued) additive set-function defined on a field $\mathfrak{F}$, then each of the following two conditions is necessary and sufficient in order that φ be countably additive:

a) $\varphi(A_n) \to \varphi(A)$ whenever $A_n \uparrow A$, and A and each A_n belong to $\mathfrak{F}$.

b) $\varphi(A_n) \to 0$ whenever $A_n \downarrow O$ and each $A_n \in \mathfrak{F}$.

If, in addition, φ is bounded, then the following condition is necessary (and sufficient) for φ to be countably additive:

c) $\varphi(A_n) \to \varphi(A)$ whenever $A_n \to A$, and A and each A_n belong to $\mathfrak{F}$.

As to the necessity of this last condition, the assumption that φ is countably additive implies that φ may be extended to a bounded, countably additive set-function defined on the σ-field $\mathfrak{M}$ generated by $\mathfrak{F}$. In view of the decomposition $\varphi = \varphi_1^+ + \varphi_1^- + i\varphi_2^+ + i\varphi_2^-$, it suffices to discuss the special case where φ on $\mathfrak{M}$ is non-negative, i.e. a bounded measure μ. Now, the characteristic function $a_n(x)$ of A_n converges to the characteristic function $a(x)$ of A at every point x of the set $E = \bigcup_n A_n \in \mathfrak{M}$, whose measure is finite. From Lebesgue's theorem on term by term integration (cf. B. Jessen [3, section 2.4, statement j]), it follows that

$$\int_E a_n(x)\, \mu(dX) \to \int_E a(x)\, \mu(dX),$$

that is, $\mu(A_n) \to \mu(A)$.

1.6. Hull and kernel. Let μ be a countably additive content with the domain $\mathfrak{F}$. The extension of μ to a measure on the σ-field $\mathfrak{M}$ generated by $\mathfrak{F}$ will likewise be denoted by μ. From the formula given in Section 1.4 for this extension it follows for every $A \in \mathfrak{M}$ that $\mu(A) = \inf \mu(B)$ where B ranges over those sets from $\mathfrak{F}_\sigma$ that contain A. In fact, $\mathfrak{F}_\sigma$ is the system of all countable unions of mutually disjoint sets from $\mathfrak{F}$. We infer now easily the existence of a "*hull*" for A (with respect to μ), that is, a set $\bar{A} \in \mathfrak{F}_{\sigma\delta}$ such that $\bar{A} \supset A$ and $\mu(\bar{A} - A) = 0$. (If $\mu(A) = +\infty$, the σ-finiteness of μ together with the fact that $\mathfrak{F}_{\sigma\sigma} = \mathfrak{F}_\sigma$ must be taken into account in order to determine sets $B \in \mathfrak{F}_\sigma$ for which $B \supset A$ and $\mu(B - A)$ is arbitrarily small.)—Similarly we may determine a "*kernel*" for A, that is, a set $\underline{A} \in \mathfrak{F}_{\delta\sigma}$ for which $\underline{A} \subset A$ and $\mu(A - \underline{A}) = 0$. In fact, let $B \in \mathfrak{F}_\sigma$, $B \supset A$, and $\mu(B - A) < \infty$. Let $H \in \mathfrak{F}_{\sigma\delta}$ be a hull for $B - A$. Then $B - H \in \mathfrak{F}_{\delta\sigma}$ is easily shown to be a kernel for A.

More generally, let a finite (or denumerably infinite) system of countably additive contents $\mu_1, \mu_2, \ldots$ be given, each with the domain $\mathfrak{F}$. Let $\bar{A}_1, \bar{A}_2, \ldots \in \mathfrak{F}_{\sigma\delta}$ be hulls and $\underline{A}_1, \underline{A}_2, \ldots \in \mathfrak{F}_{\delta\sigma}$ be kernels for a set $A \in \mathfrak{M}$ with respect to the extensions of $\mu_1, \mu_2, \ldots$ from $\mathfrak{F}$ to measures on $\mathfrak{M}$ (the σ-field generated by $\mathfrak{F}$). Then $\bar{A} = \bigcap_n \bar{A}_n \in \mathfrak{F}_{\sigma\delta\delta} = \mathfrak{F}_{\sigma\delta}$ is a hull and $\underline{A} = \bigcup_n \underline{A}_n \in \mathfrak{F}_{\delta\sigma\sigma} = \mathfrak{F}_{\delta\sigma}$ is a kernel for A with respect to all the measures $\mu_1, \mu_2, \ldots$ simultaneously.—If φ is a bounded, countably additive set-function defined on the field $\mathfrak{F}$, it follows at once from this remark, in view of the decomposition $\varphi = \varphi_1^+ + \varphi_1^- + i\varphi_2^+ + i\varphi_2^-$, that there exist for every set $A \in \mathfrak{M}$ a hull $\bar{A} \in \mathfrak{F}_{\sigma\delta}$ and a kernel $\underline{A} \in \mathfrak{F}_{\delta\sigma}$ for A (with respect to the extension of φ to $\mathfrak{M}$), in the sense that $\underline{A} \subset A \subset \bar{A}$ and $\varphi(\underline{A}) = \varphi(A) = \varphi(\bar{A})$. And if there is given, besides φ, a content μ as above, then $\bar{A}$ and $\underline{A}$ may be chosen so as to be a hull and a kernel for A with respect to the extensions of φ and μ simultaneously.

1.7. The L^p-classes. Let μ be a measure in X defined on a σ-field $\mathfrak{M}$, and let $1 \leq p < \infty$. If $E \in \mathfrak{M}$, we denote by $L^p(E)$ or $L^p(E, \mathfrak{M}, \mu)$ the class of μ-measurable functions $f(x)$, defined in E, for which

$$\int_E |f(x)|^p \, \mu(dX) < \infty .$$

By $L^\infty(E)$ we denote the class of μ-measurable functions that are defined and essentially bounded in E. In both cases, it suffices that the function be defined almost everywhere (a.e.) in E (instead of everywhere in E). By *almost everywhere* is meant: everywhere except in some μ-measurable set whose measure μ is zero. A μ-measurable function $f(x)$ is called *essentially bounded* in E if there is a finite constant c with the property that the inequality $|f(x)| \leq c$ holds a.e. in E. The smallest such constant is denoted by $\operatorname{ess\,sup}_{x \in E} |f(x)|$ (*essential supremum*).—A measurable function is said to be *essentially uniquely determined* when subjected to certain conditions, if any two measurable functions that satisfy the conditions are equal a.e.

1.8. The Radon-Nikodym theorem. Given a measure μ in X defined on a σ-field $\mathfrak{M}$, and a bounded, countably additive set-function φ, likewise with the domain $\mathfrak{M}$. Assume that $X \in \mathfrak{M}$. We say that φ is continuous with respect to μ (or simply: *μ-continuous*) if $\varphi(A) = 0$ when $A \in \mathfrak{M}$ and $\mu(A) = 0$. A function $f \in L^1(X, \mathfrak{M}, \mu)$ is called a *derivative* of φ with respect to μ if

$$\varphi(A) = \int_A f(x)\,\mu(dX) \qquad \text{for every} \quad A \in \mathfrak{M}\,.$$

The Radon-Nikodym theorem (cf. B. Jessen [3, the theorem on p. 36]) asserts that φ possesses a derivative with respect to μ if, and only if, φ is μ-continuous; the derivative is then essentially uniquely determined in X.

2. An abstract form of F. Riesz' theorem.

2.1. Let μ be a measure (defined on a σ-field $\mathfrak{M}$ of subsets of a fixed set X), and assume that $X \in \mathfrak{M}$. Let φ be a complex-valued additive set-function defined on such a system $\mathfrak{A} \subset \mathfrak{M}$ that all finite unions of disjoint sets from $\mathfrak{A}$, together with the empty set O, form a field[2] $\mathfrak{F}$ which generates $\mathfrak{M}$. Finally, it is assumed that $0 < \mu(A) < \infty$ for every $A \in \mathfrak{A}$. Then Riesz' theorem may be generalized as follows:

THEOREM I. *Let $1 < p < \infty$. In order that there exist a function $f(x) \in L^p(X, \mathfrak{M}, \mu)$ with the property that*

$$\varphi(A) = \int_A f(x)\mu(dX) \qquad \textit{for every} \quad A \in \mathfrak{A}\,,$$

it is necessary and sufficient that there is a finite constant c so that the inequality

$$(*) \qquad \sum_{\nu=1}^{n} \frac{|\varphi(A_\nu)|^p}{\mu(A_\nu)^{p-1}} \leqq c$$

holds for every finite system of disjoint sets $A_1, A_2, \ldots, A_n$ from $\mathfrak{A}$.—The function f is then essentially uniquely determined, and the smallest possible value of c is $\int_X |f(x)|^p \mu(dX)$.

[2] In order that the system $\mathfrak{F}$ consisting of O and all finite unions of disjoint sets from $\mathfrak{A}$ be a field, it is necessary and sufficient that $A \cap B \in \mathfrak{F}$ and $A - B \in \mathfrak{F}$ whenever $A \in \mathfrak{A}$ and $B \in \mathfrak{A}$, as it is easily shown. Clearly, this holds, in particular, if $\mathfrak{A} \cup \{O\}$ is a field.

2.2. As to the necessity of the condition, let $f \in L^p(X, \mathfrak{M}, \mu)$, and let

$$\varphi(A) = \int_A f(x)\,\mu(dX) \qquad \text{for every} \quad A \in \mathfrak{A}\,.$$

Since $\mu(A) < \infty$, it follows from Hölder's inequality (cf. Hardy-Littlewood-Pólya [2], Theorem 189, p. 140) that $f \in L^1(A)$ and that

$$|\varphi(A)|^p \leqq \mu(A)^{p-1} \int_A |f(x)|^p\,\mu(dX)\,.$$

This implies, with the notations employed in the formulation of the theorem, that

$$\sum_{\nu=1}^{n} \frac{|\varphi(A_\nu)|^p}{\mu(A_\nu)^{p-1}} \leqq \sum_{\nu=1}^{n} \int_{A_\nu} |f(x)|^p\,\mu(dX) \leqq \int_X |f(x)|^p\,\mu(dX)\,.$$

The condition is thus necessary, and the last integral may serve as the constant c in the inequality (*).

2.3. Concerning the sufficiency of the condition (*), we observe that φ may be extended to an additive set-function on the field $\mathfrak{F}$ as follows. If $A \in \mathfrak{F}$, then either $A = O$, or A is the union of a finite system of disjoint sets $A_1, A_2, \ldots, A_r$ from $\mathfrak{A}$. In the first case we define $\varphi(A) = 0$; in the second case we define $\varphi(A) = \sum_\varrho \varphi(A_\varrho)$, but it must then be verified that this sum is independent of the particular way in which A is composed of sets from $\mathfrak{A}$. Let $B_1, \ldots, B_s$ likewise be disjoint sets from $\mathfrak{A}$ with the union A. Then $A = \bigcup_{\varrho,\sigma}(A_\varrho \cap B_\sigma)$, where $A_\varrho \cap B_\sigma$ belongs to $\mathfrak{F}$ and hence, if not empty, is the union of some disjoint sets $E_{\varrho,\sigma,1}, \ldots, E_{\varrho,\sigma,t}$ from $\mathfrak{A}$. Thus $A_\varrho = \bigcup_\sigma(A_\varrho \cap B_\sigma) = \bigcup_{\sigma,\tau} E_{\varrho,\sigma,\tau}$ and $B_\sigma = \bigcup_\varrho(A_\varrho \cap B_\sigma) = \bigcup_{\varrho,\tau} E_{\varrho,\sigma,\tau}$. By virtue of the additivity of φ (on $\mathfrak{A}$), it follows that $\sum_\varrho \varphi(A_\varrho) = \sum_{\varrho,\sigma,\tau} \varphi(E_{\varrho,\sigma,\tau}) = \sum_\sigma \varphi(B_\sigma)$.

We shall now estimate $\varphi(A)$ when $A \in \mathfrak{F}$ (and $A \neq O$). Let A be the union of disjoint sets $A_1, \ldots, A_r$ from $\mathfrak{A}$. For abbreviation, put $q_\varrho = \mu(A_\varrho)/\mu(A)$ and $t_\varrho = \varphi(A_\varrho)/\mu(A_\varrho)$, $\varrho = 1, 2, \ldots, r$. Then $q_\varrho > 0$, $\sum_\varrho q_\varrho = 1$, and $\varphi(A)/\mu(A) = \sum_\varrho \varphi(A_\varrho)/\mu(A) = \sum_\varrho q_\varrho t_\varrho$ is the mean of order 1 of the numbers t_ϱ with the weights q_ϱ. From Hölder's inequality (cf. Hardy-Littlewood-Pólya [2, Theorem 13, p. 24]), or, more directly, Schlömilch' inequality between means of different orders ([2, Theorem 16, p. 26]), it follows that

$$\frac{|\varphi(A)|^p}{\mu(A)^{p-1}} = \mu(A)\left|\sum_{\varrho=1}^{r} q_\varrho t_\varrho\right|^p \leqq \mu(A) \sum_{\varrho=1}^{r} q_\varrho |t_\varrho|^p = \sum_{\varrho=1}^{r} \frac{|\varphi(A_\varrho)|^p}{\mu(A_\varrho)^{p-1}}\,.$$

It is now apparent that the inequality (*) remains valid (with the same constant c) for disjoint non-empty sets $A_1, \ldots, A_n$ from $\mathfrak{F}$. In fact, let A_ν be the union of disjoint sets $A_{\nu,1}, \ldots, A_{\nu,r_\nu}$ from $\mathfrak{A}$. Then

$$\sum_{\nu=1}^{n} \frac{|\varphi(A_\nu)|^p}{\mu(A_\nu)^{p-1}} \leqq \sum_{\nu=1}^{n} \sum_{\varrho=1}^{r_\nu} \frac{|\varphi(A_{\nu,\varrho})|^p}{\mu(A_{\nu,\varrho})^{p-1}} \leqq c\,,$$

the $r_1 + \ldots + r_n$ sets $A_{\nu,\varrho}$ from $\mathfrak{A}$ being mutually disjoint.

The additive set-function φ on the field $\mathfrak{F}$ is even countably additive. Let $A_n \in \mathfrak{F}$ and $A_n \downarrow O$ (cf. Section 1.5, condition b); then $\mu(A_n) \to 0$, since μ is finite on $\mathfrak{A}$ and hence on $\mathfrak{F}$. Applying the inequality (*) to one single set $A \in \mathfrak{F}$, we obtain $|\varphi(A)|^p \leqq c\mu(A)^{p-1}$; this holds even if $A = O$. Taking $A = A_n$ and making $n \to \infty$, we conclude that $\varphi(A_n) \to 0$; q.e.d.

In the rest of the proof of Theorem I, we distinguish the cases $\mu(X) < \infty$ and $\mu(X) = \infty$.

2.4. The case $\mu(X) < \infty$. In this case, φ is bounded. For every $A \in \mathfrak{F}$ we have $|\varphi(A)| \leqq c^{1/p}\mu(A)^{1-1/p} \leqq c^{1/p}\mu(X)^{1-1/p}$. The unique extension (cf. 1.4) of φ to a bounded, countably additive set-function on $\mathfrak{M}$ will likewise be denoted by φ. We propose to verify that the inequality (*) remains valid (with the same constant c) when the disjoint sets $A_1, \ldots, A_n$ belong to $\mathfrak{M}$ and have positive measures. Instead of passing directly from $\mathfrak{F}$ to $\mathfrak{M}$, we begin by assuming that $A_1, \ldots, A_n$ belong to $\mathfrak{F}_\delta$, (cf. 1.2). Let $A_{\nu,q} \downarrow A_\nu$ as $q \to \infty$, $A_{\nu,q} \in \mathfrak{F}$, $\nu = 1, 2, \ldots, n$. The slight difficulty that the n sets $A_{\nu,q}$, for given q, cannot be expected to be mutually disjoint, is easily overcome by replacing them by new sets $B_{\nu,q} \in \mathfrak{F}$ as follows: $B_{\nu,q} = A_{\nu,q} - \bigcup_{\lambda \neq \nu} A_{\lambda,q}$. For fixed q, these n sets are mutually disjoint; for fixed ν, $B_{\nu,q} \to A_\nu - \bigcup_{\lambda \neq \nu} A_\lambda = A_\nu$ as $q \to \infty$. Hence $\varphi(B_{\nu,q}) \to \varphi(A_\nu)$ and $\mu(B_{\nu,q}) \to \mu(A_\nu)$, by virtue of 1.5, condition c, applied to φ and to μ. Consequently, $\mu(B_{\nu,q}) > 0$ for every ν and all sufficiently large values of q, and the desired inequality (*) for the sets $A_1, \ldots, A_n$ follows from the corresponding inequality for the sets $B_{1,q}, \ldots, B_{n,q}$ from $\mathfrak{F}$ by letting $q \to \infty$. In a similar manner, it may be proved that $\varphi(A) = 0$ when $A \in \mathfrak{F}_\delta$ and $\mu(A) = 0$.—Next, let $A_1, \ldots, A_n$ be disjoint sets from $\mathfrak{M}$, each of positive measure. For each $\nu = 1, 2, \ldots, n$, choose a kernel $\underline{A}_\nu \in \mathfrak{F}_{\delta\sigma}$ for A with respect to φ and μ simultaneously (cf. 1.6), and let $A_{\nu,q} \uparrow \underline{A}_\nu$, whereby each $A_{\nu,q}$ should belong to $\mathfrak{F}_\delta$. For each q, the n sets $A_{\nu,q}$ are mutually disjoint, and if q is sufficiently large, they have positive measures since $\mu(A_{\nu,q}) \to \mu(\underline{A}_\nu) = \mu(A_\nu) > 0$. The desired inequality (*) for the sets $A_1, \ldots, A_n$ from $\mathfrak{M}$ follows from the corresponding inequality for the sets $A_{1,q}, \ldots, A_{n,q}$ by letting $q \to \infty$. Likewise, it is shown that φ is continuous with respect to μ, that is, $\varphi(A)=0$ when $A \in \mathfrak{M}$ and $\mu(A)=0$.

To the bounded, countably additive, and μ-continuous set-function φ defined on the σ-field $\mathfrak{M}$, there corresponds, by the Radon-Nikodym theorem (cf. Section 1.8), a complex-valued function $f(x) \in L^1(X, \mathfrak{M}, \mu)$ with the property that

$$\varphi(A) = \int_A f(x)\,\mu(dX) \qquad \text{for every} \quad A \in \mathfrak{M}$$

(in particular for $A \in \mathfrak{A}$). It remains to be proved that $f \in L^p(X, \mathfrak{M}, \mu)$, and that

$$\int_X |f(x)|^p\,\mu(dX) \leqq c\,.$$

For a given natural number n, divide the complex plane into equal disjoint (half-open) squares S_k, $k=1, 2, \ldots$, each with the diagonal $1/n$. Denote by A_k the set of points $x \in X$ for which $f(x) \in S_k$. Each $x \in X$ belongs to just one of these sets A_k; if $\mu(A_k)>0$, define

$$f_n(x) = \mu(A_k)^{-1}\varphi(A_k) = \mu(A_k)^{-1}\int_{A_k} f(x)\,\mu(dX)\,.$$

This mean-value of $f(x)$ over A_k belongs to S_k since a square is convex. The function $f_n(x)$ is thus defined a.e. in X, and the inequality $|f_n(x)-f(x)| \leqq 1/n$ subsists a.e. in X. Hence, $|f_n(x)|^p \to |f(x)|^p$ a.e. in X as $n \to \infty$. Now,

$$\int_X |f_n(x)|^p\,\mu(dX) = \Sigma'_k\,|\varphi(A_k)|^p/\mu(A_k)^{p-1}\,,$$

the prime indicating that the summation is to be restricted to those values of k for which $\mu(A_k)>0$. As each partial sum on the right is bounded by c, we conclude that

$$\int_X |f_n(x)|^p\,\mu(dX) \leqq c\,.$$

Letting $n \to \infty$, it follows from Fatou's theorem that

$$\int_X |f(x)|^p\,\mu(dX) \leqq \liminf_n \int_X |f_n(x)|^p\,\mu(dX) \leqq c\,.$$

We have now, for the case $\mu(X)<\infty$, established the existence of a function f with the desired properties. Suppose that g is another function possessing these properties. Then $g \in L^1(X)$ since $g \in L^p(X)$ and $\mu(X) < \infty$. Further,

$$\int_A f(x)\, \mu(dX) = \int_A g(x)\, \mu(dX) \qquad \text{for every} \quad A \in \mathfrak{A} ,$$

both integrals being equal to $\varphi(A)$. Clearly, the two integrals are equal also when $A \in \mathfrak{F}$. They even coincide when $A \in \mathfrak{M}$, since they both determine the unique extension of the bounded, countably additive set-function φ from the field $\mathfrak{F}$ to the σ-field $\mathfrak{M}$ generated by $\mathfrak{F}$. Hence f and g are both derivatives of φ with respect to μ, so that $f(x) = g(x)$ a.e. in X.

2.5. The case $\mu(X) = \infty$. As to the sufficiency of the condition of Theorem I, it was proved in Section 2.3 that the set-function φ on $\mathfrak{A}$, satisfying the condition of the theorem, may be extended to a countably additive set-function φ on the field $\mathfrak{F}$, and the inequality (*) remains valid. Since X belongs to $\mathfrak{M}$, it may be covered by a sequence of sets $X_n \in \mathfrak{F}$. (In fact, the system of those subsets of X that have this covering property, is a σ-field which contains $\mathfrak{F}$ and hence contains $\mathfrak{M}$.) We may even choose a sequence of sets $X_n \in \mathfrak{F}$ so that $X_n \to X$ as $n \to \infty$. Denote by $\mathfrak{F}_n$ and $\mathfrak{M}_n$ the systems of subsets of X_n which belong to $\mathfrak{F}$ and to $\mathfrak{M}$, respectively. Then φ and μ, considered on $\mathfrak{F}_n - \{O\}$ and $\mathfrak{M}_n$ (instead of $\mathfrak{A}$ and $\mathfrak{M}$), respectively, satisfy the condition of the theorem as well as the assumptions preceding it. Since $\mu(X_n) < \infty$, it follows from 2.4 that there exists a function $f_n \in L^p(X_n, \mathfrak{M}, \mu)$ with the properties that

$$\varphi(A) = \int_A f_n(x)\, \mu(dX) \qquad \text{for every} \quad A \in \mathfrak{F}_n ,$$

and that

$$\int_{X_n} |f_n(x)|^p\, \mu(dX) \leqq c .$$

In view of the uniqueness of f in the case $\mu(X) < \infty$, it follows for $m > n$ that $f_m(x) = f_n(x)$ a.e. in X_n. Hence, there exists a μ-measurable function f, defined in X, for which $f_n(x) = f(x)$ a.e. in X_n. Consequently,

$$\int_{X_n} |f(x)|^p\, \mu(dX) = \int_{X_n} |f_n(x)|^p\, \mu(dX) \leqq c ,$$

from which it follows, by letting $n \to \infty$, that

$$f \in L^p(X, \mathfrak{M}, \mu) \qquad \text{and that} \qquad \int_X |f(x)|^p\, \mu(dX) \leqq c .$$

In particular, if $A \in \mathfrak{A}$, then $f \in L^1(A, \mathfrak{M}, \mu)$ since $\mu(A) < \infty$. Thus

$$\int_A f(x)\,\mu(dX) = \lim_{n\to\infty} \int_{A \cap X_n} f(x)\,\mu(dX) = \lim_{n\to\infty} \varphi(A \cap X_n) = \varphi(A)$$

since φ is countably additive, as it was shown in 2.3.

It remains to be proved (in the case $\mu(X) = \infty$) that f is essentially uniquely determined. With the above notations, it follows, however, easily from the uniqueness result at the end of Section 2.4 that f is essentially uniquely determined in each X_n, and hence also in X.

2.6. Remark 1 (to Theorem I). The requirement that $\mu(A) > 0$ for every $A \in \mathfrak{A}$ (and hence for every $A \in \mathfrak{F} - \{O\}$) may be abolished if, at the same time, *the inequality* (*) *of p.* 289 *is maintained for all finite systems of disjoint sets from* $\mathfrak{A}$ *of positive measure, and if it is added, moreover, that* $\varphi(A) = 0$ *when* $A \in \mathfrak{A}$ *and* $\mu(A) = 0$. In fact, with this interpretation the condition (*) remains necessary, since the proof in Section 2.2 involving Hölder's inequality works as before when the measures of $A_1, \ldots, A_n$ are positive; and clearly

$$\int_A f(x)\,\mu(dX) = 0 \qquad \text{when} \qquad \mu(A) = 0\,.$$

As to the sufficiency, there is no change to be made in the proof (from Section 2.3) that the additive set-function φ may be extended from $\mathfrak{A}$ to $\mathfrak{F}$, whereas the proof that the condition (*) in its new interpretation remains valid for φ on $\mathfrak{F}$ requires only little modification: Evidently, $\varphi(A) = 0$ if $A \in \mathfrak{F}$ and $\mu(A) = 0$. And if $A_1, \ldots, A_n$ are disjoint sets from $\mathfrak{F}$, each of positive measure, then we divide again each A_ν into mutually disjoint sets $A_{\nu,\varrho}$ from $\mathfrak{A}$. In estimating $\varphi(A_\nu)$, we may then neglect such sets $A_{\nu,\varrho}$ for which $\mu(A_{\nu,\varrho}) = 0$, since $\varphi(A_{\nu,\varrho}) = 0$, too.—It follows, as before, that φ on $\mathfrak{F}$ is countably additive. In the case $\mu(X) < \infty$ it is proved just like in Section 2.4 that the condition (*) (now in its new interpretation) remains fulfilled for the extension of φ from $\mathfrak{F}$ to $\mathfrak{M}$. From this point, the proof continues precisely as before, also in the case $\mu(X) = \infty$.

Without this additional requirement that $\varphi(A) = 0$ when $\mu(A) = 0$ ($A \in \mathfrak{A}$), the condition (*) would no longer be sufficient in general (when we admit sets of measure zero in $\mathfrak{A}$), even if $\mathfrak{A}$ together with O is a σ-field $\mathfrak{M}$. (This fact seems to have been overlooked in Fullerton [1].) As an example, let X consist of just two points x_1 and x_2, let $\mathfrak{A}$ consist of the sets $\{x_1\}$, $\{x_2\}$, and X, so that $\mathfrak{M}$ = all four subsets of X. Let μ and φ be given by $\mu(\{x_1\}) = 0$, $\varphi(\{x_1\}) = 1$; $\mu(\{x_2\}) = 1$, $\varphi(\{x_2\}) = 0$; $\mu(X) = \varphi(X) = 1$.

Then the inequality (*) subsists with $c=1$, for any p, $1<p<\infty$. Nevertheless $\varphi(\{x_1\})=1$, although $\mu(\{x_1\})=0$.

2.7. Remark 2 (to Theorem I). A special consequence of the condition (*) of Theorem I is that $|\varphi(A)| \leqq c^{1/p} \mu(A)^{1-1/p}$ for every $A \in \mathfrak{A}$. Even if $\mathfrak{A}$ together with O is a field $(=\mathfrak{F})$, *this special inequality is*, however, *not a sufficient condition.* As an example, let X be the interval $0<x<1$; let $\mathfrak{F}$ be all finite unions of intervals of the form $a \leqq x<b$ contained in X, together with the empty set; let μ be the Borel measure in X; and let

$$\varphi(A) = \int_A x^{-1/p} dx$$

for every A from the field $\mathfrak{F}$. Since $x^{-1/p}$ is a decreasing positive function, we have

$$0 \leqq \varphi(A) \leqq \int_0^{\mu(A)} x^{-1/p} dx = [p/(p-1)] \mu(A)^{1-1/p} .$$

Nevertheless, the derivative $x^{-1/p}$ of φ with respect to μ is not in the class $L^p(X)$.

3. The extreme cases $p=\infty$ and $p=1$.

3.1. The case $p=\infty$. Under the same assumptions as for the case $1<p<\infty$ (see Section 2.1), except that we now, like in Remark 1, admit sets of measure zero in $\mathfrak{A}$, the following result may be obtained:

THEOREM II. *In order that there exist a bounded, μ-measurable function $f(x)$, defined in X, with the property that*

$$\varphi(A) = \int_A f(x)\, \mu(dX) \quad \textit{for every} \quad A \in \mathfrak{A} ,$$

it is necessary and sufficient that there is a finite constant c so that the inequality

$$|\varphi(A)| \leqq c\mu(A)$$

holds for every set $A \in \mathfrak{A}$. — The function f is then essentially uniquely determined, and the smallest possible value of c is $\operatorname{ess\,sup}_{x \in X} |f(x)|$.

A proof of this theorem may be obtained from the proof of Theorem I in the version described in Remark 1 (see 2.6), by obvious modifications.

3.2. The case $p=1$. This second limiting case is essentially different from the case $1<p<\infty$. Again, we do not assume that $\mu(A)>0$ for

$A \in \mathfrak{A}$, but otherwise the assumptions are the same as in Theorem I (see Section 2.1). The result is:

THEOREM III. *In order that there exist a function $f(x) \in L^1(X, \mathfrak{M}, \mu)$ with the property that*

$$\varphi(A) = \int_A f(x)\mu(dX) \qquad \textit{for every} \quad A \in \mathfrak{A},$$

the following two conditions are necessary and, when combined, sufficient:

1° To every number $\varepsilon > 0$ there corresponds a number $\delta > 0$ with the property that $\sum_{\nu=1}^n |\varphi(A_\nu)| \leqq \varepsilon$ for every finite system of disjoint sets $A_1, \ldots, A_n$ from $\mathfrak{A}$ for which $\sum_{\nu=1}^n \mu(A_\nu) < \delta$.

2° There is a finite constant c with the property that $\sum_{\nu=1}^n |\varphi(A_\nu)| \leqq c$ for every finite system of disjoint sets $A_1, \ldots, A_n$ from $\mathfrak{A}$.

The function f is then essentially uniquely determined, and the smallest possible value of c in condition 2° is $\int_X |f(x)|\mu(dX)$.

Under the additional assumption that $\mu(X) < \infty$, condition 2° is a consequence of condition 1°.

3.3. The necessity of condition 2° is immediately verified together with the fact that $\int_X |f(x)|\mu(dX)$ may serve as the constant c. That condition 1° is necessary may be shown as follows. Introduce the set E_m of points $x \in X$ at which $|f(x)| > m$, $m = 1, 2, \ldots$. Then $E_m \in \mathfrak{M}$ and $\mu(E_m) < \infty$ since $f \in L^1(X, \mathfrak{M}, \mu)$. Moreover, as $m \to \infty$, we have $E_m \downarrow O$; hence, by Lebesgue's theorem on term by term integration,

$$\int_{E_m} |f(x)|\mu(dX) \to 0 .$$

To any given $\varepsilon > 0$, we may, therefore, choose m so large that

$$\int_{E_m} |f(x)|\, \mu(dX) < \varepsilon/2 .$$

Now, let $A_1, \ldots, A_n$ be disjoint sets from $\mathfrak{A}$, and assume that

$$\sum_{\nu=1}^n \mu(A_\nu) < \varepsilon/(2m) \quad (=\delta) .$$

The union $A = \bigcup_{\nu=1}^n A_\nu$ belongs to $\mathfrak{M}$, and $\mu(A) < \delta$. Hence,

$$\sum_{\nu=1}^n |\varphi(A_\nu)| \leqq \sum_{\nu=1}^n \int_{A_\nu} |f(x)|\, \mu(dX) = \int_A |f(x)|\, \mu(dX)$$

$$= \int_{A \cap E_m} + \int_{A - E_m} < \frac{\varepsilon}{2} + m\mu(A) < \varepsilon .$$

3.4. The sufficiency of the two combined conditions 1° and 2° may be proved in the following way. First, the additive set-function φ may be extended from $\mathfrak{A}$ to $\mathfrak{F}$, and the two conditions remain valid for φ on $\mathfrak{F}$. This is shown in a straightforward way. Condition 2° implies that φ is bounded on $\mathfrak{F}$, whereas condition 1 implies that φ is countably additive on $\mathfrak{F}$ (in view of condition b from Section 1.5). By the method from the proof of Theorem I (see Section 2.4), it is proved, next, that the extension of φ from $\mathfrak{F}$ to $\mathfrak{M}$ satisfies conditions 1° and 2° with the same constant c (in 2°) and the same δ corresponding to a given ε (in 1°). From condition 1° for φ on $\mathfrak{M}$ it follows, in particular, that φ is μ-continuous. By the Radon-Nikodym theorem (see Section 1.8), φ possesses a derivative f with respect to μ. Like in Section 2.4, it is shown that

$$\int_X |f(x)|\,\mu(dX) \leqq c$$

and that f is essentially uniquely determined.

3.5. Let $\mu(X) < \infty$, and let φ satisfy condition 1° on $\mathfrak{A}$. Again, φ may be extended to an additive set-function defined on the field $\mathfrak{F}$, and condition 1° remains valid. From Remark 3 (see below), it follows that condition 2° will be satisfied (with some value of c) for φ on $\mathfrak{F}$ if we can prove that φ is a bounded set-function on $\mathfrak{F}$. Assume that, on the contrary, φ is unbounded on $\mathfrak{F}$. Corresponding to any natural number n, we shall determine a system of n disjoint sets $A_1, \ldots, A_n$ from $\mathfrak{F}$ with the property that $|\varphi(A_\nu)| > 1$, $\nu = 1, 2, \ldots, n$. When n is sufficiently large, the measure $\mu(A_\nu)$ will, for some value of ν, be smaller than the number δ which corresponds to $\varepsilon = 1$ in condition 1°; and thus we arrive at a contradiction. It remains to be proved, under the hypothesis that φ be unbounded on $\mathfrak{F}$, that a system of n sets of the desired kind exists for every natural number n. Proceeding by induction, we choose such a system of n sets $A_1, \ldots, A_n$ and also a set $E \in \mathfrak{F}$ for which

$$|\varphi(E)| > 1 + \sum_{\nu=1}^{n} \big(|\varphi(A_\nu)| + 1\big) .$$

With the abbreviation $E \cap A_\nu = E_\nu$, $\nu = 1, 2, \ldots, n$, we have $E = \bigcup_{\nu=0}^{n} E_\nu$, where $E_0 \subset X - \bigcup_{\nu=1}^{n} A_\nu$. It follows that either $|\varphi(E_0)| > 1$ or else $|\varphi(E_\nu)| > |\varphi(A_\nu)| + 1$ for at least one value of $\nu \geqq 1$. In the first case, we obtain a system of $n+1$ sets of the desired kind by adding E_0 to the system $A_1, \ldots, A_n$. In the second case, we replace A_ν by the two sets E_ν and $A_\nu - E_\nu$; in fact

$$|\varphi(A_\nu - E_\nu)| = |\varphi(A_\nu) - \varphi(E_\nu)| \geqq |\varphi(E_\nu)| - |\varphi(A_\nu)| > 1 .$$

3.6. Remark 3 (to Theorem III). If $\mathfrak{A}$ (or just $\mathfrak{A}$ together with O) is a field ($=\mathfrak{F}$), then conditions 1° and 2° may be replaced by the following two simpler conditions, in which only single sets $A \in \mathfrak{F}$ enter:

1′. To every $\varepsilon > 0$ corresponds a $\delta' > 0$ so that $|\varphi(A)| \leqq \varepsilon$ when $\mu(A) < \delta'$.
2′. The set-function φ is bounded, say $|\varphi(A)| \leqq c'$.

The smallest possible value of c' in condition 2′ is $\leqq \int_X |f(x)| \mu(dX)$, but the sign of equality applies only if φ is of constant argument (when sets for which φ is zero are neglected). If φ is real-valued, the smallest possible value of c' is

$$\max\left\{ \int_X f^+(x)\mu(dX), \int_X -f^-(x)\,\mu(dX) \right\}, \qquad \text{where} \quad f(x) = f^+(x) + f^-(x)$$

is a decomposition of f (which may be chosen as real-valued) into a non-negative and a non-positive part.

In the proof that conditions 1′ and 2′ imply conditions 1 and 2, respectively, when $\mathfrak{A}$ (or $\mathfrak{A} \cup \{O\}$) is a field $\mathfrak{F}$, it suffices to consider the case where φ is real-valued. If $A_1, \ldots, A_n$ are disjoint sets from $\mathfrak{F}$, then the union A^+, respectively A^-, of those sets A_ν for which $\varphi(A_\nu) \geqq 0$, resp. < 0, is likewise a set from $\mathfrak{F}$, and

$$\mu(A^+) + \mu(A^-) = \sum_{\nu=1}^{n} \mu(A_\nu); \qquad \varphi(A^+) - \varphi(A^-) = \sum_{\nu=1}^{n} |\varphi(A_\nu)|\,.$$

Applying conditions 1′ and 2′ to A^+ and A^-, we infer that, first, $\Sigma_{\nu=1}^{n} |\varphi(A_\nu)| \leqq 2\varepsilon$ if $\mu(A^+) < \delta'$ and $\mu(A^-) < \delta'$, in particular if $\Sigma_{\nu=1}^{n} \mu(A_\nu) < \delta'$; and, secondly, $\Sigma_{\nu=1}^{n} |\varphi(A_\nu)| \leqq 2c'$.—The statements concerning the smallest possible value of c' are easily verified.

4. Extension to more general classes of functions.

4.1. The only property of the complex numbers which is relevant in the preceding theorems is that they represent the points of the Euclidean plane. It is straightforward to verify that instead of complex-valued set-functions $\varphi(A)$ and point-functions $f(x)$ one might just as well consider set-functions and point-functions whose values are points $t = (t^1, \ldots, t^k)$ in Euclidean k-dimensional space R^k; $k = 1, 2, \ldots$. (In other words, the values are vectors in the k-dimensional real vector space with the norm $|t| = ((t^1)^2 + \ldots + (t^k)^2)^{1/2}$. This norm replaces, then, the absolute value of a complex number.) The properties: boundedness, additivity, or countable additivity, are defined for such set-functions in obvious analogy to the complex-valued case (cf. Section 1.3), and the extension theorem remains valid, as well as the conditions for countable additivity and the results about hull and kernel (cf. Sections 1.4, 1.5, and 1.6). A function $f(x)$ with values in R^k is called μ-measurable if its domain of definition E belongs to $\mathfrak{M}$ and if each co-ordinate $f^{(\varkappa)}$, $(\varkappa = 1, 2, \ldots, k)$, is μ-measurable. The class $L^p(E, \mathfrak{M}, \mu)$ is characterized by the additional requirement that

$$\int_E |f(x)|^p \, \mu(dX) < \infty .$$

Theorems I, II, and III, and the remarks 1, 2, and 3, subsist with unchanged proofs.

4.2. A somewhat less obvious, further generalization arises when the classes L^p are replaced by more general classes of functions. Consider a real-valued continuous convex function $F(t)$, defined in R^k, with the property that $F(t) \to +\infty$ as $|t| \to \infty$. By $L_F(E) = L_F(E, \mathfrak{M}, \mu)$ we understand the class of μ-measurable functions (with values in R^k), defined a.e. in E and subjected to the condition that

$$\int_E F(f(x)) \, \mu(dX) < \infty .$$

(Thus the classes L^p, $1 \leqq p < \infty$, correspond to the special functions $F(t) = |t|^p$.) It follows from the properties of $F(t)$ that

$$\liminf_{|t| \to \infty} F(t)/|t| > 0 ,$$

from which we shall infer presently that $L_F(E) \subset L^1(E)$ when $\mu(E) < \infty$. Moreover, the convexity of $F(t)$ will replace the special cases of Hölder's inequality which were used in the previous proofs. Under the further assumption that $F(t)/|t| \to +\infty$ as $|t| \to \infty$, we shall obtain a result quite analogous to Theorem I (and containing this theorem as a special case corresponding to the function $F(t) = |t|^p$, $1 < p < \infty$). With the assumptions and notations introduced in connection with Theorem I (cf. Section 2.1), except that the complex values are replaced, as described in Section 4.1, by points in R^k, the result is the following theorem.

THEOREM IV. *Let $F(t)$ be a real-valued continuous convex function, defined in R^k, for which $F(t)/|t| \to +\infty$ as $|t| \to \infty$. In order that there exist a function $f(x)$ (with values in R^k) belonging to the class $L_F(X, \mathfrak{M}, \mu)$ and possessing the property that*

$$\varphi(A) = \int_A f(x)\,\mu(dX) \qquad \text{for every} \quad A \in \mathfrak{A}\,,$$

it is necessary and sufficient that there is a finite constant c so that the inequality

$$(**) \qquad \sum_{\nu=1}^{n} F\big(\varphi(A_\nu)/\mu(A_\nu)\big)\mu(A_\nu) \leqq c$$

holds for every finite system of disjoint sets $A_1, A_2, \ldots, A_n$ from $\mathfrak{A}$.—The function f is then essentially uniquely determined, and the smallest possible value of c is $\int_X F(f(x))\mu(dX)$.

Remark 1 from Section 2.6 remains likewise valid: If sets of measure 0 are admitted in $\mathfrak{A}$, then Theorem IV subsists only if we add to (**) the extra condition that $\varphi(A)=0$ for every $A \in \mathfrak{A}$ for which $\mu(A)=0$.

4.3. The proof of Theorem IV may be conducted exactly like that of Theorem I. Aside from the obvious modifications, mentioned in 4.1 in connection with the replacement of complex numbers by points of R^k, the only new modifications are the following:

A. The inequality

$$\left|\sum_{\varrho=1}^{r} q_\varrho t_\varrho\right|^p \leqq \sum_{\varrho=1}^{r} q_\varrho |t_\varrho|^p$$

is replaced by the following analogous inequality which expresses the convexity of $F(t)$:

$$F\left(\sum_{\varrho=1}^{r} q_\varrho t_\varrho\right) \leqq \sum_{\varrho=1}^{r} q_\varrho F(t_\varrho)\,,$$

the assumptions being: $q_\varrho \geqq 0$, $\sum_\varrho q_\varrho = 1$, and now: $t_\varrho \in R^k$. (Cf. Hardy-Littlewood-Pólya [2, Theorem 98, p. 80], where the inequality is formulated for the two-dimensional case $k=2$).

B. In the proof of the necessity of the condition (**) we apply (cf. Section 2.2) the continuous analogue of the convexity property described under A:

$$F\left(\mu(A)^{-1}\int_A f(x)\mu(dX)\right) \leqq \mu(A)^{-1}\int_A F\big(f(x)\big)\mu(dX)$$

in generalization of the inequality

$$\left|\int_A f(x)\mu(dX)\right|^p \leqq \mu(A)^{p-1}\int_A |f(x)|^p\,\mu(dX)\,.$$

C. If $E \in \mathfrak{M}$, $\mu(E) < \infty$, and $f \in L_F(E, \mathfrak{M}, \mu)$, then $f \in L^1(E, \mathfrak{M}, \mu)$. In fact, since $\liminf_{|t| \to \infty} F(t)/|t| > 0$, there exist finite constants a and b so that $|t| \leqq bF(t)$ whenever $|t| > a$. In other words, the inequality

$$|t| \leqq \max[a, bF(t)]$$

holds for every $t \in R^k$. Inserting $f(x)$ for t, we conclude that $f \in L^1(E)$.

D. The proof of the countable additivity of the extension of φ from $\mathfrak{A}$ to $\mathfrak{F}$ (cf. 2.3) depends again on the fact that $\varphi(A_n) \to 0$ whenever $\mu(A_n) \to 0$, $A_n \in \mathfrak{F}$. This may now be proved by contradiction as follows. Let $A_1, A_2, \ldots$ be a sequence of sets from $\mathfrak{F}$ for which $\mu(A_n) \to 0$, but $|\varphi(A_n)| > \eta > 0$, η being fixed. Then $\mu(A_n) > 0$. Putting $t_n = \varphi(A_n)/\mu(A_n)$, we infer that $|t_n| \to \infty$, and hence $F(t_n)/|t_n| \to +\infty$ as $n \to \infty$. Multiplying by $|\varphi(A_n)|$, which exceeds η, we conclude that

$$F\big(\varphi(A_n)/\mu(A_n)\big)\,\mu(A_n) \to +\infty,$$

in contradiction with the inequality (**) for φ on $\mathfrak{F}$.

E. In the case $\mu(X) < \infty$, the extension of φ from $\mathfrak{A}$ to the field $\mathfrak{F}$ is bounded (cf. Section 2.4). In fact, the inequality (**), applied to a single set, states that $F\big(\varphi(A)/\mu(A)\big)\,\mu(A) \leqq c$ when $A \in \mathfrak{F}$ and $A \neq O$. Replacing t in the inequality $|t| \leqq \max[a, bF(t)]$ (see under C) by $\varphi(A)/\mu(A)$ and multiplying by $\mu(A)$, we get $|\varphi(A)| \leqq \max[a\mu(X), bc]$.

It is not difficult to show that all the assumptions made about the function $F(t)$ are indeed indispensable for the validity of Theorem IV.

4.4. To obtain a generalization of Theorem III, one may replace the assumption that $F(t)/|t| \to +\infty$ by the original weaker assumption that $F(t) \to +\infty$ as $|t| \to \infty$. Then it becomes necessary, like in Theorem III, to add to (**) a further condition (to ensure that $\varphi(A) \to 0$ whenever $\mu(A) \to 0$), namely condition 1° of Theorem III. This modified version of Theorem IV holds, like Theorem III, even if sets of measure 0 are admitted in $\mathfrak{A}$. Under the further assumptions that

$$\limsup_{|t| \to \infty} F(t)/|t| < +\infty \quad \text{and} \quad \mu(X) < \infty,$$

condition (**) becomes a consequence of the extra condition (cond. 1° of Theorem III).—As no new features are involved in these remarks, we may refrain from presenting the proofs.

4.5. Examples. A simple type of functions $F(t)$, $t \in R^k$, are those which (like the functions $|t|^p$) depend only on $|t|$. Thus $F(t) = \Phi(|t|)$, where $\Phi(r)$ is a real-valued function defined for $0 \leqq r < \infty$. In order that $F(t)$

be continuous and convex, it is necessary and sufficient that $\Phi(r)$ is non-decreasing and convex (and hence continuous). In fact, these two conditions, taken together, are sufficient to ensure that $F(t)$ is convex, since

$$F\left(\sum_\nu q_\nu t_\nu\right) = \Phi\left(\left|\sum_\nu q_\nu t_\nu\right|\right) \leqq \Phi\left(\sum_\nu q_\nu |t_\nu|\right) \leqq \sum_\nu q_\nu \Phi(|t_\nu|) = \sum_\nu q_\nu F(t_\nu)$$

when $q_\nu \geqq 0$, $\Sigma_\nu q_\nu = 1$, and $t_\nu \in R^k$. And conversely, the two conditions are necessary, as we may see by considering the values of $F(t)$ on a line through the origin given by the parametric representation $t = r \cdot e$, where e is a fixed vector of length $|e| = 1$, and the parameter r ranges over the real axis $-\infty < r < \infty$. A convex, even function defined over the entire real axis is obviously non-decreasing over the positive semi-axis.

Next, in order that $F(t) = \Phi(|t|)$ satisfy the condition $\lim_{|t| \to \infty} F(t) = +\infty$, it is necessary and sufficient that $\lim_{r \to \infty} \Phi(r) = +\infty$. And similarly, the condition $\lim_{|t| \to \infty} F(t)/|t| = +\infty$, which enters in Theorem IV, is fulfilled if, and only if, $\lim_{r \to \infty} \Phi(r)/r = +\infty$.

As a specific example, apart from the functions $|t|^p$, $1 < p < \infty$, it may be noted that Theorem IV holds with the function $\Phi(r) = r \log^+ r$, that is,

$$F(t) = |t| \log^+ |t| = \begin{cases} |t| \log |t| & \text{for } |t| \geqq 1 . \\ 0 & \text{for } 0 \leqq |t| \leqq 1 . \end{cases}$$

The corresponding class $L_F = Z$ was considered by A. Zygmund in connection with conjugate trigonometric series and the theory of strong differentiation. Thus, under the assumptions of Theorem I (cf. Section 2.1), a necessary and sufficient condition that an additive set-function φ be the indefinite integral of some function f from the class Z, is that there is a constant c so that

$$\sum_{\nu=1}^{n} |\varphi(A_\nu)| \log^+ \frac{|\varphi(A_\nu)|}{\mu(A_\nu)} \leqq c$$

for all finite systems of disjoint sets $A_1, \ldots, A_n$ from the domain $\mathfrak{A}$ of φ.

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