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ON SELF-ADJOINT ORDINARY DIFFERENTIAL OPERATORS

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1. Introduction. Let L denote the formal ordinary differential operator

$$L = p_0 D^n + p_1 D^{n-1} + \ldots + p_n ,$$

where $D = d/dx$, the p_k are complex-valued functions having $n-k$ continuous derivatives on an open real interval $a < x < b$, and $p_0(x) \neq 0$ on (a, b); $a = -\infty$ or $b = +\infty$, or both, are allowed. We further assume L is formally self-adjoint, i.e., L coincides with its Lagrange adjoint

$$L^+ = (-1)^n D^n \bar{p}_0 + (-1)^{n-1} D^{n-1} \bar{p}_1 + \ldots + \bar{p}_n .$$

Let $\mathfrak{H}$ be the Hilbert space of all complex-valued functions on (a, b) whose magnitudes are square summable on (a, b), i.e., $\mathfrak{H} = \mathfrak{L}^2(a, b)$. We denote by $\mathfrak{D}$ the set of all $u \in \mathfrak{H}$ which have continuous derivatives up to order $n-1$ on (a, b), $u^{(n-1)}$ is absolutely continuous on every closed subinterval of (a, b), and $Lu \in \mathfrak{H}$. Let $\mathfrak{D}_S$ be the set of all $u \in \mathfrak{D}$ such that u vanishes outside some closed bounded subinterval of (a, b) (the interval may depend on u), and define the operator S in $\mathfrak{H}$ to have the domain $\mathfrak{D}_S$, and

$$Su = Lu \qquad (u \in \mathfrak{D}_S) .$$

Then S is a symmetric operator whose adjoint is the operator T, with domain $\mathfrak{D}$, defined by

$$Tu = Lu \qquad (u \in \mathfrak{D});$$

see [2].

Under the assumption that S has a self-adjoint extension H, we show how to define self-adjoint boundary value problems on finite closed subintervals δ of (a, b) in such a way as to produce, in the limit $\delta \to (a, b)$, the unique spectral matrix associated with the expansion theorem and Parseval equality for H. This spectral matrix is related to the Green's function for $H - l$, $\operatorname{Im} l \neq 0$, which we prove is a limit of Green's functions for the problems defined on the subintervals of (a, b). Finally we show

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how the spectral family of projections $E(\lambda)$ associated with H can be represented in terms of the spectral matrix and solutions of $Lu=\lambda u$. This representation implies the uniqueness of the spectral matrix, the expansion theorem, and the Parseval equality.

In [1] we obtained the unique spectral matrix and Green's function for the cases (I) when $H=T$ is self-adjoint (and hence no boundary conditions are required to specify the domain of H), and (II) when the point a is finite and (a, b) can be replaced by $[a, b)$, and the domain of H results by imposing boundary conditions on $u \in \mathfrak{D}$ at a alone. Here we show how the method of [1] can be adapted to the case of an arbitrary self-adjoint extension H. Use will be made of the characterization which we gave in [2], of such an H by homogeneous boundary conditions.

With minor changes our results remain valid for differential operators defined for vector-valued functions.

2. The resolvent of a self-adjoint extension. Let H be a self-adjoint extension of S. It satisfies $S \subseteqq H \subseteqq T$, and its domain consists of those $u \in \mathfrak{D}$ satisfying certain boundary conditions, which we now describe.

If $a<y<x<b$ and u, v are in $\mathfrak{D}$, then Green's formula is

$$\int_y^x (\bar{v}Lu - u\overline{Lv}) = [uv](x) - [uv](y),$$

where $[uv](x)$ is the form

$$[uv](x) = \sum_{m=1}^{n} \sum_{j+k=m-1} (-1)^j u^{(k)}(x)(p_{n-m}\bar{v})^{(j)}(x).$$

From Green's formula it follows that the limits

$$[uv](a) = \lim_{x\to a}[uv](x), \quad [uv](b) = \lim_{x\to b}[uv](x)$$

exist for all $u, v \in \mathfrak{D}$. Let $\langle uv\rangle = [uv](b) - [uv](a)$.

Since we assume S has a self-adjoint extension, there exist ω, $0 \leqq \omega \leqq n$, linearly independent solutions of $Lu=iu$, and of $Lu=-iu$, which are in $\mathfrak{D}$. Let $\varphi_1, \ldots, \varphi_\omega$ be an orthonormal basis for the solutions of $Lu=iu$ in $\mathfrak{D}$, and let $\psi_1, \ldots, \psi_\omega$ be a corresponding orthonormal basis for the solutions of $Lu=-iu$ in $\mathfrak{D}$. Corresponding to H there exists a unique unitary matrix $U=(u_{jk})$, $j, k=1, \ldots, \omega$, such that the domain $\mathfrak{D}_H$ of H is the set of all $u \in \mathfrak{D}$ satisfying

$$\langle uv_j\rangle = 0 \quad (j=1, \ldots, \omega),$$

where

$$v_j = \varphi_j - \sum_{k=1}^{\omega} u_{jk}\psi_k \qquad (j = 1, \ldots, \omega);$$

see [2, Theorem 3]. Moreover, every ω by ω unitary matrix determines a self-adjoint extension of S in this way.

We now define self-adjoint boundary value problems on closed bounded subintervals $\delta=[\tilde{a}, \tilde{b}]$ of (a, b), and show that the resolvent $(H-l)^{-1}$, $\operatorname{Im} l \neq 0$, is an integral operator whose kernel is the limit of Green's functions for the problems defined on the subintervals.

The inner product and norm in $\mathfrak{L}^2(\delta)$ will be denoted by $(\ ,\ )_\delta$ and $\|\ \|_\delta$ respectively, whereas in $\mathfrak{L}^2(a, b)$ these will be denoted by $(\ ,\ )$ and $\|\ \|$. Further we define $\langle uv\rangle_\delta$ by

$$\langle uv\rangle_\delta = [uv](\tilde{b})-[uv](\tilde{a}) .$$

Using the Gram-Schmidt process let $\varphi_{1\delta}, \ldots, \varphi_{\omega\delta}$ be $\varphi_1, \ldots, \varphi_\omega$ orthonormalized to $\mathfrak{L}^2(\delta)$; similarly let $\psi_{1\delta}, \ldots, \psi_{\omega\delta}$ denote $\psi_1, \ldots, \psi_\omega$ orthonormalized to $\mathfrak{L}^2(\delta)$. Then, for $j=1, \ldots, \omega$,

$$\varphi_{j\delta} = \sum_{k=1}^{j} a_{jk}(\delta)\varphi_k, \quad \psi_{j\delta} = \sum_{k=1}^{j} b_{jk}(\delta)\psi_k ,$$

where $A(\delta)=(a_{jk}(\delta))$, and $B(\delta)=(b_{jk}(\delta))$, are certain matrices having the property that

$$A(\delta) \to E, \qquad B(\delta) \to E \qquad (\delta \to (a, b)) ,$$

where E is the ω by ω unit matrix. Let $\varphi_{\omega+1\delta}, \ldots, \varphi_{n\delta}$ be functions such that $\varphi_{1\delta}, \ldots, \varphi_{n\delta}$ is a basis for the solutions of $Lu=iu$, orthonormalized in $\mathfrak{L}^2(\delta)$; similarly adjoin $\psi_{\omega+1\delta}, \ldots, \psi_{n\delta}$ to the set $\psi_{1\delta}, \ldots, \psi_{\omega\delta}$. We define the functions $v_{j\delta}$ by

$$\begin{aligned} v_{j\delta} &= \varphi_{j\delta} - \sum_{k=1}^{\omega} u_{jk}\psi_{k\delta} && (j = 1, \ldots, \omega) , \\ v_{j\delta} &= \varphi_{j\delta} - \psi_{j\delta} && (j = \omega+1, \ldots, n) . \end{aligned} \tag{2.1}$$

Here $U=(u_{jk})$ is the unique unitary matrix, mentioned above, which corresponds to the self-adjoint extension H. Clearly the matrix $U(\delta)=(u_{jk}(\delta))$, $(j, k=1, \ldots, n)$, where

$$\begin{aligned} u_{jk}(\delta) &= u_{jk} && (j, k = 1, \ldots, \omega) , \\ u_{jj}(\delta) &= 1 && (j = \omega+1, \ldots, n) , \\ u_{jk}(\delta) &= 0 && \text{all other } j, k , \end{aligned}$$

is unitary, and (2.1) may be written as

$$v_{j\delta} = \varphi_{j\delta} - \sum_{k=1}^{n} u_{jk}(\delta)\psi_{k\delta} \qquad (j = 1, \ldots, n).$$

From [2, Theorem 3], applied to the interval δ, it follows that the problem

$$(2.2) \qquad Lu = lu, \qquad \langle uv_{j\delta}\rangle_\delta = 0 \qquad (j = 1, \ldots, n)$$

(l a complex parameter) is a self-adjoint boundary value problem in $\mathfrak{L}^2(\delta)$.

More precisely, let $\mathfrak{D}_\delta$ be the set of all $u \in \mathfrak{L}^2(\delta)$ for which $u^{(n-1)}$ is absolutely continuous on δ, $Lu \in \mathfrak{L}^2(\delta)$, and $\langle uv_{j\delta}\rangle_\delta = 0$, $j = 1, \ldots, n$. Then the operator L_δ defined by $L_\delta u = Lu$ for $u \in \mathfrak{D}_\delta$ is a self-adjoint operator in $\mathfrak{L}^2(\delta)$. For $\operatorname{Im} l \neq 0$ the resolvent $(L_\delta - l)^{-1}$ is an integral operator $G_\delta(l)$, with a kernel called Green's function $G_\delta = G_\delta(x, y, l)$, which is defined for all $f \in \mathfrak{L}^2(\delta)$ by

$$G_\delta(l)f(x) = \int_\delta G_\delta(x, y, l)f(y)\,dy.$$

It was shown in [1, Lemma 4], that the set of functions $\{G_\delta\}$ is uniformly bounded and equicontinuous on every compact (x, y, l)-region where $\operatorname{Im} l \neq 0$. From this it follows that there exists a sequence of intervals $\delta_m \subset (a, b)$, $m = 1, 2, \ldots, \delta_m \to (a, b)$, such that the corresponding Green's functions $G_m = G_{\delta_m}$ tend uniformly, on any compact subset of $a < x$, $y < b$, $\operatorname{Im} l > 0$ (or $\operatorname{Im} l < 0$), to a continuous limit function G. From Theorem 1 in [1], any such limit G is in $\mathfrak{H}$ as a function of y for each fixed x, and if $G(l)$ is defined by

$$(2.3) \qquad G(l)f(x) = \int_a^b G(x, y, l)f(y)\,dy \qquad (f \in \mathfrak{H},\ \operatorname{Im} l \neq 0),$$

then $\|G(l)f\| \leq |\operatorname{Im} l|^{-1}\|f\|$, $G(l)f \in \mathfrak{D}$, and $(L - l)G(l)f = f$.

THEOREM 1. *Let G be the limit of any convergent sequence $\{G_m\}$ of the set $\{G_\delta\}$ of Green's functions associated with the self-adjoint boundary value problems* (2.2). *If $f \in \mathfrak{H}$, then $G(l)f$, defined by* (2.3), *satisfies the boundary conditions*

$$(2.4) \qquad \langle G(l)f\, v_j\rangle = 0 \qquad (j = 1, \ldots, \omega).$$

We remark that the theorem remains valid if $U(\delta)$ is replaced by any matrix of the form

$$\begin{pmatrix} U_1(\delta) & 0 \\ 0 & U_2(\delta) \end{pmatrix},$$

where $U_1(\delta)$ is an ω by ω matrix tending to U as $\delta \to (a, b)$, and $U_2(\delta)$ is an arbitrary $n-\omega$ by $n-\omega$ unitary matrix.

A direct consequence of Theorem 1 is the following

COROLLARY. *Every convergent sequence $\{G_m\}$ of $\{G_\delta\}$ tends to the same limit G, and hence*

$$G_\delta \to G \qquad (\delta \to (a, b)), \tag{2.5}$$

uniformly on any compact (x, y, l)-region where $\operatorname{Im} l \neq 0$. If $G(l)$ is defined by (2.3), then

$$G(l) = (H-l)^{-1} \qquad (\operatorname{Im} l \neq 0). \tag{2.6}$$

PROOF OF THE COROLLARY. Let G be the limit of any convergent sequence $\{G_m\}$, and for $\operatorname{Im} l \neq 0$ let $G(l)$ be the corresponding integral operator defined by (2.3). For any $f \in \mathfrak{H}$, $G(l)f \in \mathfrak{D}$ and (2.4) is valid, thus showing that $G(l)f \in \mathfrak{D}_H$. Moreover $(H-l)G(l)f = (L-l)G(l)f = f$ for every $f \in \mathfrak{H}$. Conversely, let $u \in \mathfrak{D}_H$ and put $(H-l)u = f$. Then $w = u - G(l)f$ is in $\mathfrak{D}_H$, and $(H-l)w = 0$, implying $w = 0$, for the spectrum of H is real. Thus $u = G(l)f$, or $G(l)(H-l)u = u$ for every $u \in \mathfrak{D}_H$. This proves (2.6), and this readily implies (2.5).

Because of (2.6) we call G the *Green's function* for $H-l$, $\operatorname{Im} l \neq 0$.

PROOF OF THEOREM 1. Let G be a limit function of some convergent sequence $\{G_m\}$, and let $G(l)$ be the integral operator given by (2.3) for this G. The theorem will first be proved for the case when $f \in \mathfrak{H}$ vanishes outside some closed bounded subinterval $\delta_0 = [a_0, b_0]$ of (a, b). In the following let j be a fixed integer from the set $1, \ldots, \omega$, and $\operatorname{Im} l \neq 0$. Since

$$\langle G(l)f\, v_j \rangle = \lim \langle G(l)f\, v_j \rangle_\delta \qquad (\delta \to (a, b)),$$

we have to show that, given any $\varepsilon > 0$, there exists a subinterval $\delta(\varepsilon)$ such that

$$|\langle G(l)f\, v_j \rangle_\delta| < \varepsilon \tag{2.7}$$

is valid for all δ satisfying $\delta(\varepsilon) \subset \delta \subset (a, b)$.

Since $G_\delta(l) = (L_\delta - l)^{-1}$,

$$\langle G_\delta(l)f\, v_{j\delta} \rangle_\delta = 0 \qquad (\delta \supset \delta_0). \tag{2.8}$$

For $\delta = \delta_m$ let $v_{j\delta} = v_{jm}$, $\varphi_{j\delta} = \varphi_{jm}$, $\psi_{j\delta} = \psi_{jm}$, $G_\delta(l) = G_m(l)$, $(u, v)_\delta = (u, v)_m$, $\|u\|_\delta = \|u\|_m$, and $\langle uv \rangle_\delta = \langle uv \rangle_m$. We prove that, given any $\varepsilon > 0$, there exists a $\delta(\varepsilon)$ having the property that for any $\delta \supset \delta(\varepsilon)$ there exists a $\tilde{\delta} \supset \delta_0$, depending on δ and ε, such that

$$|\langle G_m(l)f\, v_{jm} \rangle_m - \langle G(l)f\, v_j \rangle_\delta| < \varepsilon \tag{2.9}$$

is valid for all $\delta_m \supset \tilde{\delta}$. However, since $\langle G_m(l)f\, v_{jm}\rangle_m = 0$ by (2.8), it follows that (2.7) is true for all $\delta \supset \delta(\varepsilon)$.

Let δ be fixed, $\delta \supset \delta_0$, and $\delta_m \supset \delta$. Then by Green's formula

$$(2.10)\quad \begin{aligned}&\langle G_m(l)f\, v_{jm}\rangle_m - \langle G(l)f\, v_j\rangle_\delta \\ &= \langle G_m(l)f\, v_{jm}\rangle_\delta - \langle G(l)f\, v_j\rangle_\delta + \big(LG_m(l)f,\, v_{jm}\big)_{m-\delta} - \big(G_m(l)f,\, Lv_{jm}\big)_{m-\delta}\,,\end{aligned}$$

where $m-\delta$ stands for $\delta_m - \delta$. We estimate separately the difference between the first two terms, and the difference between the last two terms.

Using Green's formula

$$(2.11)\quad \begin{aligned}&\langle G_m(l)f\, v_{jm}\rangle_\delta - \langle G(l)f\, v_j\rangle_\delta \\ &= \big(LG_m(l)f,\, v_{jm}\big)_\delta - \big(G_m(l)f,\, Lv_{jm}\big)_\delta - \big(LG(l)f,\, v_j\big)_\delta + \big(G(l)f,\, Lv_j\big)_\delta\,.\end{aligned}$$

We shall show that for the fixed δ, as $m \to \infty$,

$$(2.12)\quad \begin{aligned}&\text{(a)}\quad \|G_m(l)f - G(l)f\|_\delta \to 0\,,\\ &\text{(b)}\quad \|LG_m(l)f - LG(l)f\|_\delta \to 0\,,\\ &\text{(c)}\quad \|v_{jm} - v_j\|_\delta \to 0\,,\\ &\text{(d)}\quad \|Lv_{jm} - Lv_j\|_\delta \to 0\,.\end{aligned}$$

From (2.11) it is then clear that for given $\varepsilon > 0$, $\delta \supset \delta_0$, there exists a $\tilde{\delta} \supset \delta$, such that

$$(2.13)\quad |\langle G_m(l)f\, v_{jm}\rangle_\delta - \langle G(l)f\, v_j\rangle_\delta| < \varepsilon/2 \qquad \big(\delta_m \supset \tilde{\delta}\big)\,.$$

As to (2.12) (a) we have, since f vanishes outside δ_0, and $\delta_0 \subset \delta \subset \delta_m$,

$$\|G_m(l)f - G(l)f\|_\delta^2 = \int_\delta \left| \int_{\delta_0} \big(G_m(x,y,l) - G(x,y,l)\big) f(y)\, dy \right|^2 dx\,,$$

and this tends to zero as $m \to \infty$ because $G_m \to G$ uniformly for $x \in \delta$, $y \in \delta_0$. Relation (2.12) (b) follows from (2.12) (a) and the fact that

$$(L-l)G_m(l)f = (L-l)G(l)f = f\,.$$

Turning to (2.12) (c) we have

$$(2.14)\quad \begin{aligned}\|v_{jm} - v_j\|_\delta &= \|\varphi_{jm} - \varphi_j - \sum_{k=1}^{\infty} u_{jk}(\psi_{km} - \psi_k)\|_\delta \\ &\leqq \|\varphi_{jm} - \varphi_j\|_\delta + \sum_{k=1}^{\infty} |u_{jk}|\, \|\psi_{km} - \psi_k\|_\delta\,.\end{aligned}$$

If $\varepsilon_{jk}=1$ or 0 according as $j=k$ or $j\neq k$,

$$\|\varphi_{jm}-\varphi_j\|_\delta = \left\| \sum_{k=1}^{\omega} \left(a_{jk}(\delta_m)-\varepsilon_{jk}\right)\varphi_k \right\|_\delta$$

$$\leqq \sum_{k=1}^{\omega} |a_{jk}(\delta_m)-\varepsilon_{jk}|\,\|\varphi_k\|\,,$$

and, since $\|\varphi_k\|_\delta \leqq \|\varphi_k\|=1$, this is less than or equal to

$$\sum_{k=1}^{\omega} |a_{jk}(\delta_m)-\varepsilon_{jk}|\,,$$

which tends to zero as $m\to\infty$. Similarly $\|\psi_{km}-\psi_k\|_\delta \to 0$ as $m\to\infty$. From (2.14) we now see that (2.12) (c) results. Finally, since

$$(2.15)\qquad Lv_{jm} = L\left(\varphi_{jm}-\sum_{k=1}^{\omega} u_{jk}\psi_{km}\right) = i\left(\varphi_{jm}+\sum_{k=1}^{\omega} u_{jk}\psi_{km}\right),$$

and similarly

$$Lv_j = i\left(\varphi_j+\sum_{k=1}^{\omega} u_{jk}\psi_k\right),$$

we see (2.12) (d) follows from (2.12) (c).

Now we estimate the difference between the last two terms in (2.10). We let

$$\Lambda = \left|\left(LG_m(l)f,\, v_{jm}\right)_{m-\delta}-\left(G_m(l)f,\, Lv_{jm}\right)_{m-\delta}\right|.$$

Then

$$\Lambda \leqq \|LG_m(l)f\|_{m-\delta}\,\|v_{jm}\|_{m-\delta}+\|G_m(l)f\|_{m-\delta}\,\|Lv_{jm}\|_{m-\delta}$$

$$\leqq \|LG_m(l)f\|_m\,\|v_{jm}\|_{(a,\,b)-\delta}+\|G_m(l)f\|_m\,\|Lv_{jm}\|_{(a,\,b)-\delta}\,.$$

Since $LG_m(l)f=lG_m(l)f+f$, and $\|G_m(l)f\|_m \leqq |\operatorname{Im} l|^{-1}\|f\|_m=|\operatorname{Im} l|^{-1}\|f\|$, we have

$$(2.16)\quad \Lambda \leqq (1+|l|\,|\operatorname{Im} l|^{-1})\,\|f\|\,\|v_{jm}\|_{(a,\,b)-\delta}+|\operatorname{Im} l|^{-1}\,\|f\|\,\|Lv_{jm}\|_{(a,\,b)-\delta}\,.$$

Now

$$\|v_{jm}\|_{(a,\,b)-\delta} = \|\varphi_{jm}-\sum_{k=1}^{\omega} u_{jk}\psi_{km}\|_{(a,\,b)-\delta} \leqq \|\varphi_{jm}\|_{(a,\,b)-\delta}+\sum_{k=1}^{\omega} |u_{jk}|\,\|\psi_{km}\|_{(a,\,b)-\delta}\,,$$

and

$$\|\varphi_{jm}\|_{(a,\,b)-\delta} = \left\| \sum_{p=1}^{\omega} a_{jp}(\delta_m)\varphi_p \right\|_{(a,\,b)-\delta} \leqq \sum_{p=1}^{\omega} |a_{jp}(\delta_m)|\,\|\varphi_p\|_{(a,\,b)-\delta}\,.$$

Since $A(\delta_m)=\left(a_{jp}(\delta_m)\right)$ tends to $E=(\varepsilon_{jp})$ as $m\to\infty$, there exists a δ^0 such that

$$|a_{jp}(\delta_m)| < 2 \qquad (j,\, p = 1,\, \ldots,\, \omega;\; \delta_m \supset \delta^0)\,.$$

Thus

$$\|\varphi_{jm}\|_{(a,\,b)-\delta} \leqq 2\sum_{p=1}^{\omega} \|\varphi_p\|_{(a,\,b)-\delta} \qquad (\delta_m \supset \delta^0)\,,$$

and a similar estimate is valid for $\|\psi_{km}\|_{(a,b)-\delta}$, resulting in

$$(2.17)\quad \|v_{jm}\|_{(a,b)-\delta} \leqq 2\sum_{p=1}^{\omega}\|\varphi_p\|_{(a,b)-\delta}+2\sum_{k=1}^{\omega}|u_{jk}|\sum_{p=1}^{\omega}\|\psi_p\|_{(a,b)-\delta} \qquad (\delta_m\supset\delta^0)\,.$$

By virtue of (2.15) we see $\|Lv_{jm}\|_{(a,b)-\delta}$ is majorized by the same quantity for $\delta_m\supset\delta^0$. Since $\varphi_p, \psi_p\in\mathfrak{H}$ it follows that, as $\delta\to(a,b)$,

$$\sum_{p=1}^{\omega}\|\varphi_p\|_{(a,b)-\delta}\to 0, \qquad \sum_{p=1}^{\omega}\|\psi_p\|_{(a,b)-\delta}\to 0\,.$$

Therefore, from (2.17) and (2.16) we see that, given any $\varepsilon>0$, there exists a $\delta(\varepsilon)\supset\delta^0$ such that $\Lambda<\varepsilon/2$ provided that $\delta\supset\delta(\varepsilon)$, and $\delta_m\supset\delta$. This, combined with (2.13) and (2.10), proves (2.9). The proof is thus complete in case f vanishes outside a closed bounded subinterval of (a,b).

Now let f be an arbitrary element of $\mathfrak{H}$, and let f_n, $n=1,2,\ldots$, be functions in $\mathfrak{H}$ vanishing outside closed bounded subintervals of (a,b) such that $\|f_n-f\|\to 0$ as $n\to\infty$. Since $\langle G(l)f_n\, v_j\rangle=0$ for $j=1,\ldots,\omega$, and $n=1,2,\ldots$, we have

$$\begin{aligned}|\langle G(l)f\, v_j\rangle| &= |\langle G(l)f\, v_j\rangle-\langle G(l)f_n\, v_j\rangle|\\ &= \left|\left(L(G(l)f-G(l)f_n),\, v_j\right)-\left(G(l)f-G(l)f_n,\, Lv_j\right)\right|\\ &\leqq \|LG(l)(f-f_n)\|\,\|v_j\|+\|G(l)(f-f_n)\|\,\|Lv_j\|\,.\end{aligned}$$

But

$$\|G(l)(f-f_n)\| \leqq |\operatorname{Im} l|^{-1}\|f-f_n\|\,,$$

and

$$\|LG(l)(f-f_n)\| = \|lG(l)(f-f_n)+(f-f_n)\| \leqq (|l|\,|\operatorname{Im} l|^{-1}+1)\,\|f-f_n\|\,.$$

Thus, letting $n\to\infty$, we see that $\langle G(l)f\, v_j\rangle=0$ for $j=1,\ldots,\omega$, completing the proof of Theorem 1.

3. The spectral matrix associated with a self-adjoint extension. Let $\varrho_\delta=(\varrho_{\delta jk})$ be the spectral matrix associated with the self-adjoint problem (2.2) on δ. It is hermitian, non-decreasing (i.e., $\varrho_\delta(\lambda)-\varrho_\delta(\mu)$ is positive semi-definite if $\lambda>\mu$), the total variation of $\varrho_{\delta jk}$ is finite on every finite λ-interval, and $\varrho_\delta(\lambda+0)=\varrho_\delta(\lambda)$, $\varrho_\delta(0)=0$. In terms of ϱ_δ the Parseval equality

$$\|u\|_\delta^2 = \int_{-\infty}^{\infty}\sum_{j,k=1}^{n}\overline{\hat{u}_{\delta j}(\lambda)}\,\hat{u}_{\delta k}(\lambda)\,d\varrho_{\delta jk}(\lambda)$$

is valid for $u\in\mathfrak{L}^2(\delta)$. Here

$$\hat{u}_{\delta j}(\lambda) = \big(u, s_j(\lambda)\big)_\delta ,$$

where the $s_j(l)$, $j=1, \ldots, n$, are n linearly independent solutions of $Lu=lu$ satisfying

$$s_j^{(k-1)}(c, l) = \varepsilon_{jk} \qquad (j, k = 1, \ldots, n) ,$$

for some fixed c, $\tilde{a}<c<\tilde{b}$. The following theorem is a direct consequence of Theorem 4 in [1], and Theorem 1 of the previous section.

THEOREM 2. *There exists an hermitian, non-decreasing matrix $\varrho=(\varrho_{jk})$ whose elements are of bounded variation on every finite λ-interval, and such that, if $\Delta=(\mu, \lambda]$,*

$$\varrho_{\delta jk}(\Delta) \to \varrho_{jk}(\Delta) \qquad \big(\delta \to (a, b)\big) ,$$

provided the end points of Δ are continuity points for ϱ_{jk}. Further

$$\varrho_{jk}(\Delta) = \frac{1}{2\pi i} \lim_{\varepsilon \to +0} \int_\mu^\lambda P_{jk}(\sigma+i\varepsilon)\, d\sigma ,$$

where

$$P_{jk}(l) = \frac{\partial^{j+k-2} K}{\partial x^{j-1}\, \partial y^{k-1}}(c, c, l), \qquad K(x, y, l) = G(x, y, l)-G(x, y, \bar{l}) ,$$

and G is the Green's function for $H-l$.

The matrix ϱ is called the *spectral matrix* associated with H (and the fundamental set $s_1, \ldots, s_n$).

4. The spectral family of projections associated with a self-adjoint extension. Let $E(\lambda)$ be the spectral family of projections associated with the self-adjoint operator H via the spectral theorem, i.e.,

$$H = \int_{-\infty}^{\infty} \lambda\, dE(\lambda) .$$

We show how the $E(\lambda)$ may be expressed in terms of the spectral matrix ϱ and the fundamental set $s_1, \ldots, s_n$, thus connecting more intimately ϱ with H. If

$$K(l) = G(l)-G(\bar{l}) \qquad (\operatorname{Im} l \neq 0) ,$$

we first prove

THEOREM 3. *If $f, g \in C^n$, and vanish outside closed bounded subintervals of (a, b), then*

$$(K(l)f, g) = 2i \operatorname{Im} l \int_{-\infty}^{\infty} \sum_{j,k=1}^{n} \overline{\hat{g}_j(\lambda)}\, \hat{f}_k(\lambda)\, |\lambda - l|^{-2}\, d\varrho_{jk}(\lambda),$$

where

$$\hat{f}_k(\lambda) = \int_a^b f(x)\overline{s_k(x, \lambda)}\, dx, \qquad \hat{g}_j(\lambda) = \int_a^b g(x)\overline{s_j(x, \lambda)}\, dx.$$

PROOF. Let f and g vanish outside δ_0, and in the following let $\delta \supset \delta_0$. Suppose $\{\chi_{\delta m}\}$ is a complete orthonormal set of eigenfunctions for the problem (2.2) on δ, and let $\{\lambda_{\delta m}\}$ be the corresponding eigenvalues. If

$$K_\delta(l) = G_\delta(l) - G_\delta(\bar{l}) = 2i \operatorname{Im} l\, G_\delta(l) G_\delta(\bar{l}),$$

we have from the Parseval equality

$$\begin{aligned}(K_\delta(l)f, g)_\delta &= 2i \operatorname{Im} l \big(G_\delta(l) G_\delta(\bar{l})f, g\big)_\delta \\ &= 2i \operatorname{Im} l \big(G_\delta(\bar{l})f, G_\delta(\bar{l})g\big)_\delta \\ &= 2i \operatorname{Im} l \sum_m \big(G_\delta(\bar{l})f, \chi_{\delta m}\big)_\delta \overline{\big(G_\delta(\bar{l})g, \chi_{\delta m}\big)_\delta}.\end{aligned}$$

But

$$\big(G_\delta(\bar{l})f, \chi_{\delta m}\big)_\delta = \big(f, G_\delta(l)\chi_{\delta m}\big)_\delta = (\lambda_{\delta m} - \bar{l})^{-1} (f, \chi_{\delta m})_\delta.$$

Therefore, using the definition of the matrix ϱ_δ, we have

$$(4.2) \qquad (K_\delta(l)f, g)_\delta = 2i \operatorname{Im} l \int_{-\infty}^{\infty} \sum_{j,k=1}^{n} \overline{\hat{g}_j(\lambda)}\, \hat{f}_k(\lambda)\, |\lambda - l|^{-2}\, d\varrho_{\delta jk}(\lambda).$$

We show that by letting $\delta \to (a, b)$ the equality (4.2) leads to (4.1).

If $K_\delta(x, y, l) = G_\delta(x, y, l) - G_\delta(x, y, \bar{l})$, then

$$(K_\delta(l)f, g)_\delta = \int_{\delta_0} \Big(\int_{\delta_0} K_\delta(x, y, l) f(y)\, dy\Big) \overline{g(x)}\, dx,$$

and, since $G_\delta \to G$ uniformly for $x, y \in \delta_0$, we have

$$(K_\delta(l)f, g)_\delta \to (K(l)f, g) \qquad (\delta \to (a, b)).$$

It remains to show that the right side of (4.2) tends to the right side of (4.1). Let

$$d\tau_\delta(\lambda; f, g) = \sum_{j,k=1}^{n} \overline{\hat{g}_j(\lambda)}\hat{f}_k(\lambda)\, d\varrho_{\delta jk}(\lambda),$$

and

$$d\tau(\lambda; f, g) = \sum_{j,k=1}^{n} \overline{\hat{g}_j(\lambda)}\hat{f}_k(\lambda)\, d\varrho_{jk}(\lambda).$$

If $\mu > 0$ we have

$$\int_{-\mu}^{\mu} \frac{|d\tau_\delta(\lambda; f, g)|}{|\lambda - l|^2} \leqq \left(\int_{-\mu}^{\mu} \frac{d\tau_\delta(\lambda; f, f)}{|\lambda - l|^2}\right)^{\frac{1}{2}} \left(\int_{-\mu}^{\mu} \frac{d\tau_\delta(\lambda; g, g)}{|\lambda - l|^2}\right)^{\frac{1}{2}}$$
$$\leqq \left(\int_{-\infty}^{\infty} \frac{d\tau_\delta(\lambda; f, f)}{|\lambda - l|^2}\right)^{\frac{1}{2}} \left(\int_{-\infty}^{\infty} \frac{d\tau_\delta(\lambda; g, g)}{|\lambda - l|^2}\right)^{\frac{1}{2}}$$
$$= \|G_\delta(\bar{l})f\|_\delta \, \|G_\delta(\bar{l})g\|_\delta$$
$$\leqq |\mathrm{Im}\, l|^{-2} \, \|f\| \, \|g\| \, .$$

Letting $\delta \to (a, b)$, and then $\mu \to \infty$, we see that

$$\int_{-\infty}^{\infty} \frac{|d\tau(\lambda; f, g)|}{|\lambda - l|^2} \leqq |\mathrm{Im}\, l|^{-2} \, \|f\| \, \|g\| \, , \tag{4.3}$$

thus showing the convergence of the integral on the right side of (4.1). Now let $\mu > 1 + |l|$. Then if $|\lambda| \geqq \mu$, $|\lambda - l| \geqq |\lambda| - |l| > 1$, or $|\lambda - l|^{-2} < 1$. Therefore

$$\int_{|\lambda| \geqq \mu} \frac{|d\tau_\delta(\lambda; f, g)|}{|\lambda - l|^2} < \int_{|\lambda| \geqq \mu} |d\tau_\delta(\lambda; f, g)|$$
$$\leqq \mu^{-2} \int_{|\lambda| \geqq \mu} \lambda^2 \, |d\tau_\delta(\lambda; f, g)| \leqq \mu^{-2} \int_{-\infty}^{\infty} \lambda^2 \, |d\tau_\delta(\lambda; f, g)|$$
$$\leqq \mu^{-2} \left(\int_{-\infty}^{\infty} \lambda^2 d\tau_\delta(\lambda; f, f)\right)^{\frac{1}{2}} \left(\int_{-\infty}^{\infty} \lambda^2 d\tau_\delta(\lambda; g, g)\right)^{\frac{1}{2}}$$
$$= \mu^{-2} \, \|Lf\| \, \|Lg\| \, .$$

We then have

$$\left| \int_{-\infty}^{\infty} \frac{d\tau_\delta(\lambda; f, g)}{|\lambda - l|^2} - \int_{-\infty}^{\infty} \frac{d\tau(\lambda; f, g)}{|\lambda - l|^2} \right|$$
$$\leqq \left| \int_{-\mu}^{\mu} \frac{d\tau_\delta(\lambda; f, g)}{|\lambda - l|^2} - \int_{-\infty}^{\infty} \frac{d\tau(\lambda; f, g)}{|\lambda - l|^2} \right| + \mu^{-2} \, \|Lf\| \, \|Lg\| \, ,$$

and letting first $\delta \to (a, b)$, and then $\mu \to \infty$, we see that the right side of (4.2) tends to the right side of (4.1). This completes the proof of Theorem 3.

If $\Delta = (\mu, \lambda]$ is any finite interval let $E(\Delta) = E(\lambda) - E(\mu)$, where $E(\lambda)$ is the spectral family of projections associated with H.

THEOREM 4. *If $f \in C^n$ vanishes outside a closed bounded subinterval of (a, b), and λ, μ are continuity points of $E(\lambda)$ and ϱ, then*

$$E(\Delta)f(x) = \int_\Delta \sum_{j,k=1}^{n} s_j(x, \sigma)\hat{f}_k(\sigma)\, d\varrho_{jk}(\sigma)\,. \tag{4.4}$$

PROOF. We apply the known formula

$$(E(\Delta)f, g) = \lim_{\varepsilon \to +0} \frac{1}{2\pi i} \int_\mu^\lambda (K(\nu + i\varepsilon)f, g)\, d\nu$$

to functions $f, g \in C^n$ which vanish outside closed bounded subintervals of (a, b). From (4.1) we have

$$\frac{1}{2\pi i} \int_\mu^\lambda (K(\nu + i\varepsilon)f, g)\, d\nu = \frac{1}{\pi} \int_\mu^\lambda \left(\int_{-\infty}^{\infty} \frac{\varepsilon}{(\sigma - \nu)^2 + \varepsilon^2} d\tau(\sigma; f, g) \right) d\nu\,.$$

If $\xi > 2|\lambda| + 2|\mu|$, and $|\sigma| \geqq \xi$, then

$$\frac{1}{(\sigma - \nu)^2 + \varepsilon^2} \leqq \frac{\alpha(\lambda, \mu)}{1 + \sigma^2}, \quad \alpha(\lambda, \mu) = \frac{1}{(|\lambda| + |\mu|)^2} + 4\,.$$

Thus

$$\begin{aligned} \int_{|\sigma| \geqq \xi} \frac{|d\tau(\sigma; f, g)|}{(\sigma - \nu)^2 + \varepsilon^2} &\leqq \alpha(\lambda, \mu) \int_{|\sigma| \geqq \xi} \frac{|d\tau(\sigma; f, g)|}{1 + \sigma^2} \\ &\leqq \alpha(\lambda, \mu) \int_{-\infty}^{\infty} \frac{|d\tau(\sigma; f, g)|}{1 + \sigma^2} \\ &\leqq \alpha(\lambda, \mu)\, \|f\|\, \|g\|\,, \end{aligned}$$

where the last inequality follows from (4.3) for $l = i$. Therefore

$$\begin{aligned} &\lim_{\varepsilon \to +0} \frac{1}{2\pi i} \int_\mu^\lambda (K(\nu + i\varepsilon)f, g)\, d\nu \\ &\qquad = \frac{1}{\pi} \int_{|\sigma| \leqq \xi} \lim_{\varepsilon \to +0} \left[\arctan\left(\frac{\lambda - \sigma}{\varepsilon}\right) - \arctan\left(\frac{\mu - \sigma}{\varepsilon}\right) \right] d\tau(\sigma; f, g) \\ &\qquad = \int_\mu^\lambda d\tau(\sigma; f, g)\,, \end{aligned}$$

proving that

$$\begin{aligned}(E(\Delta)f, g) &= \int_{\Delta} d\tau(\sigma; f, g) \\ &= \int_{\delta_0} \left(\int_{\Delta} \sum_{j,k=1}^{n} s_j(x, \sigma)\hat{f}_k(\sigma)\, d\varrho_{jk}(\sigma) \right) \overline{g(x)}\, dx ,\end{aligned}$$

where g vanishes outside δ_0. This readily implies (4.4).

Let $\zeta = (\zeta_1, \ldots, \zeta_n)$, $\eta = (\eta_1, \ldots, \eta_n)$ be vector functions of λ, and introduce the inner product

$$(\zeta, \eta) = \int_{-\infty}^{\infty} \sum_{j,k=1}^{n} \overline{\eta_j(\lambda)}\, \zeta_k(\lambda)\, d\varrho_{jk}(\lambda)$$

and norm $\|\zeta\| = (\zeta, \zeta)^{\frac{1}{2}}$. Let $\mathfrak{L}^2(\varrho)$ be the Hilbert space of all ζ, measurable with respect to ϱ, such that $\|\zeta\| < \infty$.

THEOREM 5. *If $f \in \mathfrak{L}^2(a, b)$ the vector $\hat{f} = (\hat{f}_j)$, where*

$$\hat{f}_j(\lambda) = \int_a^b f(x) \overline{s_j(x, \lambda)}\, dx ,$$

converges in norm in $\mathfrak{L}^2(\varrho)$, and

$$\|f\| = \|\hat{f}\| \qquad (\textit{Parseval equality}) .$$

In terms of this $\hat{f}$,

$$f(x) = \int_{-\infty}^{\infty} \sum_{j,k=1}^{n} s_j(x, \lambda)\hat{f}_k(\lambda)\, d\varrho_{jk}(\lambda) \qquad (\textit{Expansion theorem}) ,$$

where the integral converges in norm in $\mathfrak{L}^2(a, b)$.

PROOF. Let $f \in C^n$ and vanish outside a closed bounded subinterval of (a, b). The Parseval equality for f results from (4.4) and the fact that $(E(\Delta)f, f) \to \|f\|^2$ as $\Delta \to (-\infty, \infty)$. The expansion theorem for f results since $\|f - E(\Delta)f\| \to 0$ as $\Delta \to (-\infty, \infty)$. The denseness of these f in $\mathfrak{L}^2(a, b)$ allows one to extend these results to all of $\mathfrak{L}^2(a, b)$.

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