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ON MAXIMAL SYMMETRIC ORDINARY DIFFERENTIAL OPERATORS

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1. Introduction. The purpose of this note is to show how the results of the immediately preceding paper (in the sequel referred to as (A)) can be extended to the case of maximal symmetric ordinary differential operators. We assume the reader is familiar with the notation and results of (A).

The simplest example of a formally self-adjoint ordinary differential operator L which gives rise to a maximal symmetric operator which is not self-adjoint is $Lu = idu/dx$ on $0 \leqq x < \infty$. Let $\mathfrak{D}$ be the set of all functions in $\mathfrak{L}^2(0, \infty)$ which are absolutely continuous on every interval of the form $0 \leqq x \leqq c$, $0 < c < \infty$, and for which $Lu \in \mathfrak{L}^2(0, \infty)$. For these L and $\mathfrak{D}$ let the operators S and T in $\mathfrak{L}^2(0, \infty)$ be defined as in the Introduction of (A). Since $e^{-x} \in \mathfrak{L}^2(0, \infty)$, and $e^{x} \bar{\in} \mathfrak{L}^2(0, \infty)$, the deficiency index of S is $(0, 1)$. This implies S has no self-adjoint extensions, and that the closure $\tilde{S}$ of S is maximal symmetric. In fact, $\tilde{S}$ is the operator having a domain consisting of those $u \in \mathfrak{D}$ satisfying $u(0) = 0$, and for these u, $\tilde{S}u = idu/dx$.

Let L be the formally self-adjoint differential operator on the interval (a, b) which was defined in the Introduction of (A), and let S and T be the corresponding operators in $\mathfrak{H} = \mathfrak{L}^2(a, b)$. The operator S always has maximal symmetric extensions, and we show in section 2 below that the domains of such extensions, and their adjoints, may be described by certain homogeneous boundary conditions.

Every maximal symmetric extension H of S has a unique generalized resolvent R_l, in the sense of M. A. Neumark (see, for example [1], p. 278). We prove that R_l is an integral operator whose kernel is a limit of Green's functions G_δ associated with appropriate self-adjoint boundary value problems on closed bounded subintervals δ of (a, b).

Associated with the generalized resolvent is a unique generalized resolution of the identity [1, p. 277]. This is a one-parameter family $F(\lambda)$, $-\infty < \lambda < \infty$, of operators satisfying

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(a) $F(\lambda)-F(\mu)$ is positive bounded if $\lambda>\mu$,
(b) $F(-\infty)=0$, $F(\infty)=E$ (the identity operator),
(c) $F(\lambda+0)=F(\lambda)$.

(In [1, p. 263], the condition $F(\lambda-0)=F(\lambda)$ replaces the normalization (c), but this is of no importance). A generalized resolution of the identity becomes an ordinary resolution of the identity if the further restriction

$$F(\lambda)F(\mu) = F(\nu), \qquad \nu = \min(\lambda, \mu),$$

is imposed. Thus the operators $F(\lambda)$ need not be projections. We show how $F(\lambda)-F(\mu)$ can be expressed in terms of an essentially unique spectral matrix ϱ and a fundamental set of solutions of $Lu=lu$. The Parseval equality and expansion theorem, corresponding to a maximal symmetric operator H, is then a direct consequence of this representation.

With non-essential modifications the results remain valid for L operating on vector-valued functions. We intend, in a future paper, to show how some of our results carry over to the case of elliptic partial differential operators.

2. Boundary conditions and domains of maximal symmetric extensions. Let S have the deficiency index (ω^+, ω^-). The two integers ω^+, ω^- satisfy $0\leqq\omega^+, \omega^-\leqq n$. If $\mathfrak{E}(i)$ and $\mathfrak{E}(-i)$ are the eigenspaces of T for i and $-i$ respectively then

$$\omega^+ = \dim \mathfrak{E}(i), \qquad \omega^- = \dim \mathfrak{E}(-i).$$

We assume, without any real restriction, that $\omega^+\leqq\omega^-$; the case $\omega^+=\omega^-$ was treated in (A).

It was shown in [2, Theorem 2], that the closure of S is a symmetric operator T_0 whose domain $\mathfrak{D}_0$ is the set of all $u\in\mathfrak{D}$ satisfying $\langle uv\rangle=0$ for all $v\in\mathfrak{D}$. Since $S\subseteqq T$, $T_0\subseteqq T$, and therefore $T_0u=Lu$ for $u\in\mathfrak{D}_0$. From the general theory of Hilbert space [3, p. 38] it follows that

$$\mathfrak{D} = \mathfrak{D}_0+\mathfrak{E}(i)+\mathfrak{E}(-i),$$

and every $u\in\mathfrak{D}$ may be expressed uniquely as a sum

$$u = u_0+u^++u^- \qquad \big(u_0\in\mathfrak{D}_0,\ u^+\in\mathfrak{E}(i),\ u^-\in\mathfrak{E}(-i)\big).$$

The set of all maximal symmetric extensions of S (and hence of T_0) are in a one-to-one correspondence with the set of all isometric operators V of $\mathfrak{E}(i)$ into $\mathfrak{E}(-i)$. If H is a maximal symmetric extension of S, there exists a unique isometry V of $\mathfrak{E}(i)$ into $\mathfrak{E}(-i)$ such that the domain $\mathfrak{D}_H$ of H is the set of all $u\in\mathfrak{D}$ of the form

$$u = u_0 + (I - V)u^+ \qquad (u_0 \in \mathfrak{D}_0,\ u^+ \in \mathfrak{E}(i)), \tag{2.1}$$

where I is the identity operator on $\mathfrak{E}(i)$. For such $u \in \mathfrak{D}_H$, $Hu = Lu$. Conversely every isometry V of $\mathfrak{E}(i)$ into $\mathfrak{E}(-i)$ generates a maximal symmetric extension in this way.

The domain $\mathfrak{D}_{H^*}$ of the adjoint H^* of H is given by

$$\mathfrak{D}_{H^*} = \mathfrak{D}_H + (\mathfrak{E}(-i) \ominus V\mathfrak{E}(i)), \tag{2.2}$$

where $\mathfrak{E}(-i) \ominus V\mathfrak{E}(i)$ is the orthogonal complement of $V\mathfrak{E}(i)$ in $\mathfrak{E}(-i)$.

Let $\varphi_1, \ldots, \varphi_{\omega+}$ be an orthonormal basis for $\mathfrak{E}(i)$. If V is any isometry of $\mathfrak{E}(i)$ into $\mathfrak{E}(-i)$, the set of functions of the form (2.1) can also be described as the set of all $u \in \mathfrak{D}$ of the form

$$u = u_0 + \sum_{j=1}^{\omega+} a_j v_j \qquad (u_0 \in \mathfrak{D}_0), \tag{2.3}$$

where

$$v_j = \varphi_j - V\varphi_j \qquad (j = 1, \ldots, \omega^+), \tag{2.4}$$

and the a_j are complex constants. For any such V let $V\varphi_j = \psi_j$ for $j = 1, \ldots, \omega^+$. The ψ_j are orthonormal in $\mathfrak{E}(-i)$. Let $\psi_{\omega^+ +1}, \ldots, \psi_{\omega-}$ be an orthonormal basis for the space $\mathfrak{E}(-i) \ominus V\mathfrak{E}(i)$. Then $\psi_1, \ldots, \psi_{\omega-}$ is an orthonormal basis for $\mathfrak{E}(-i)$.

We now show that the domains $\mathfrak{D}_H$ and $\mathfrak{D}_{H^*}$ can be characterized by functions satisfying certain homogeneous boundary conditions. We recall [2, p. 198] that the boundary conditions $\langle u\alpha_j \rangle = 0$, $j = 1, \ldots, m$, for functions $u \in \mathfrak{D}$ (α_j are fixed in $\mathfrak{D}$) are linearly independent if and only if the α_j are linearly independent mod $\mathfrak{D}_0$. A linearly independent set $\langle u\alpha_j \rangle = 0$, $j = 1, \ldots, m$, is said to be self-adjoint if $\langle \alpha_j \alpha_k \rangle = 0$ for $j, k = 1, \ldots, m$.

THEOREM 1. *Let H be a maximal symmetric extension of S with domain the set of all $u \in \mathfrak{D}$ of the form* (2.1). *Then $\mathfrak{D}_{H^*}$ is the set of all $u \in \mathfrak{D}$ satisfying the self-adjoint set of boundary conditions*

$$\langle uv_j \rangle = 0 \qquad (j = 1, \ldots, \omega^+), \tag{2.5}$$

and $\mathfrak{D}_H$ is the set of all $u \in \mathfrak{D}_{H^*}$ satisfying the (*non-self-adjoint*) *boundary conditions*

$$\langle u\psi_j \rangle = 0 \qquad (j = \omega^+ + 1, \ldots, \omega^-). \tag{2.6}$$

PROOF. Let H be a maximal symmetric extension of S with domain (2.1). The boundary conditions (2.5) are linearly independent. For suppose $\sum \gamma_j v_j \in \mathfrak{D}_0$ for some complex constants γ_j. Since

$$\Sigma \gamma_j v_j \in \mathfrak{E}(i) + \mathfrak{E}(-i) ,$$

it follows that $\Sigma \gamma_j v_j = 0$, and this implies that $\Sigma \gamma_j \varphi_j = \Sigma \gamma_j V \varphi_j = 0$. The linear independence of the φ_j implies that $\gamma_1 = \ldots = \gamma_{\omega+} = 0$.

An easy calculation, using Green's formula, yields

$$(2.7) \qquad \langle \varphi_j \varphi_k \rangle = 2i\varepsilon_{jk}, \qquad \langle \psi_j \psi_k \rangle = -2i\varepsilon_{jk}, \qquad \langle \varphi_j \psi_k \rangle = 0$$

for $j, k = 1, \ldots, \omega^+$. Therefore

$$(2.8) \qquad \langle v_j v_k \rangle = \langle \varphi_j \varphi_k \rangle - \langle \psi_j \varphi_k \rangle - \langle \varphi_j \psi_k \rangle + \langle \psi_j \psi_k \rangle = 0$$

for $j, k = 1, \ldots, \omega^+$, thus proving that the set (2.5) is self-adjoint.

From (2.2) and (2.3) any $u \in \mathfrak{D}_{H^*}$ has the form

$$u = u_0 + \sum_{p=1}^{\omega+} a_p v_p + \sum_{p=\omega^++1}^{\omega-} b_p \psi_p ,$$

where a_p, b_p are complex constants. Thus, if $j = 1, \ldots, \omega^+$

$$\langle u v_j \rangle = \langle u_0 v_j \rangle + \sum_{p=1}^{\omega+} a_p \langle v_p v_j \rangle + \sum_{p=\omega^++1}^{\omega-} b_p \langle \psi_p v_j \rangle = 0 ,$$

for by the definition of $\mathfrak{D}_0$, $\langle u_0 v \rangle = 0$ for all $v \in \mathfrak{D}$, $\langle v_p v_j \rangle = 0$ by (2.8), and

$$\langle \psi_p v_j \rangle = \langle \psi_p \varphi_j \rangle - \langle \psi_p \psi_j \rangle = 0 \qquad (p = \omega^+ + 1, \ldots, \omega^-; j = 1, \ldots, \omega^+) .$$

This shows that every $u \in \mathfrak{D}_{H^*}$ satisfies the conditions (2.5). Conversely, let $u \in \mathfrak{D}$ and satisfy (2.5). Since $u \in \mathfrak{D} = \mathfrak{D}_0 + \mathfrak{E}(i) + \mathfrak{E}(-i)$,

$$u = u_0 + \sum_{p=1}^{\omega+} a_p \varphi_p + \sum_{p=1}^{\omega+} \tilde{a}_p \psi_p + \sum_{p=\omega^++1}^{\omega-} b_p \psi_p$$

for some constants a_p, $\tilde{a}_p$, b_p. Now using the relations (2.7) in the equalities $\langle u v_j \rangle = 0$ it follows that $\tilde{a}_p = -a_p$, and hence

$$u = u_0 + \sum_{p=1}^{\omega+} a_p v_p + \sum_{p=\omega^++1}^{\omega-} b_p \psi_p ,$$

which means $u \in \mathfrak{D}_{H^*}$. Thus $\mathfrak{D}_{H^*}$ is precisely the set of all $u \in \mathfrak{D}$ satisfying (2.5).

If $u \in \mathfrak{D}_H$ then u has the form (2.3), and for $j = \omega^+ + 1, \ldots, \omega^-$,

$$\langle u \psi_j \rangle = \langle u_0 \psi_j \rangle + \sum_{p=1}^{\omega+} a_p \langle v_p \psi_j \rangle = 0 .$$

Conversely, let $u \in \mathfrak{D}_{H^*}$, and satisfy the conditions

$$\langle u \psi_j \rangle = 0 \qquad (j = \omega^+ + 1, \ldots, \omega^-) .$$

Then

$$u = u_1 + \sum_{p=\omega^++1}^{\omega^-} b_p \psi_p \qquad (u_1 \in \mathfrak{D}_H),$$

for some complex numbers b_p, and for $j = \omega^+ + 1, \ldots, \omega^-$,

$$0 = \langle u\psi_j \rangle = \langle u_1 \psi_j \rangle + \sum_{p=\omega^++1}^{\omega^-} b_p \langle \psi_p \psi_j \rangle = 0 - 2ib_j .$$

Thus $\mathfrak{D}_H$ consists of just those $u \in \mathfrak{D}_{H^*}$ satisfying the conditions (2.6). Since $\langle \psi_j \psi_k \rangle = -2i\varepsilon_{jk}$ for $j, k = \omega^+ + 1, \ldots, \omega^-$, this set of boundary conditions is not self-adjoint.

This completes the proof of Theorem 1.

3. The generalized resolvent of a maximal symmetric extension. Let H be a fixed maximal symmetric extension of S whose adjoint H^* has a domain consisting of those $u \in \mathfrak{D}$ satisfying (2.5), whereas $\mathfrak{D}_H$ is the set of $u \in \mathfrak{D}_{H^*}$ satisfying (2.6). The generalized resolvent of H is the operator R_l defined for $\operatorname{Im} l \neq 0$ by

$$R_l = (H^* - l)^{-1} \qquad (\operatorname{Im} l > 0),$$
$$R_l = (H - l)^{-1} \qquad (\operatorname{Im} l < 0);$$

see [1, p. 278]. We shall show that R_l is an integral operator $G(l)$ with a kernel which is a limit of Green's functions for self-adjoint problems on closed bounded subintervals δ of (a, b).

Indeed, for any such δ let $\varphi_{1\delta}, \ldots, \varphi_{\omega^+\delta}$ be $\varphi_1, \ldots, \varphi_{\omega^+}$ orthonormalized to $\mathfrak{L}^2(\delta)$, and let

$$v_{j\delta} = \varphi_{j\delta} - V\varphi_{j\delta} \qquad (j = 1, \ldots, \omega^+),$$

where V is the isometry associated with H. Let $\varphi_{\omega^++1\delta}, \ldots, \varphi_{n\delta}$ be functions such that $\varphi_{1\delta}, \ldots, \varphi_{n\delta}$ is an orthonormal basis in $\mathfrak{L}^2(\delta)$ for the solutions of $Lu = iu$, and let $\psi_{\omega^++1\delta}, \ldots, \psi_{n\delta}$ be such that $V\varphi_{1\delta}, \ldots, V\varphi_{\omega^+\delta}$, $\psi_{\omega^++1\delta}, \ldots, \psi_{n\delta}$ is an orthonormal basis in $\mathfrak{L}^2(\delta)$ for the solutions of $Lu = -iu$. Further define

$$v_{j\delta} = \varphi_{j\delta} - \psi_{j\delta} \qquad (j = \omega^+ + 1, \ldots, n).$$

Then it is clear that the problem

$$Lu = lu, \quad \langle u v_{j\delta} \rangle_\delta = 0 \qquad (j = 1, \ldots, n),$$

is self-adjoint in $\mathfrak{L}^2(\delta)$, and

$$v_{j\delta} \to v_j \qquad (\delta \to (a, b);\ j = 1, \ldots, \omega^+)$$

in the pointwise sense, as well as in $\mathfrak{L}^2(a, b)$. For $\operatorname{Im} l \neq 0$, let $G_\delta = G_\delta(x, y, l)$

be Green's function for this problem. The same arguments which led to the proof of Theorem 1 of (A) may now be applied. The set $\{G_\delta\}$ is uniformly bounded and equicontinuous on every compact (x, y, l)-region where $\mathrm{Im}\, l \neq 0$. If G is the limit of any convergent (uniform on any compact subset of $a<x, y<b$, $\mathrm{Im}\, l \neq 0$) sequence $G_m = G_{\delta_m}$, then $G(l)f$, where

$$G(l)f(x) = \int_a^b G(x, y, l) f(y)\, dy \qquad (f \in \mathfrak{H},\ \mathrm{Im}\, l \neq 0)$$

satisfies the boundary conditions

$$\langle G(l)f\, v_j \rangle = 0 \qquad (j = 1, \ldots, \omega^+)$$

if $\mathrm{Im}\, l \neq 0$. This implies the following

Theorem 2. *Every convergent sequence $\{G_m\}$ of $\{G_\delta\}$ tends to the same limit G, and hence*

$$G_\delta \to G \qquad \big(\delta \to (a, b)\big)\,,$$

uniformly on any compact (x, y, l)-region where $\mathrm{Im}\, l \neq 0$. *If $G(l)$ is the integral operator defined above, then*

$$G(l) = (H^* - l)^{-1} \qquad (\mathrm{Im}\, l > 0)\,,$$
$$G(l) = (H - l)^{-1} \qquad (\mathrm{Im}\, l < 0)\,,$$

that is, $G(l) = R_l$, the generalized resolvent of H.

Proof. Let G be the limit of any convergent sequence $\{G_m\}$. The argument of the proof of the Corollary to Theorem 1 of (A) shows that $G(l) = (H^* - l)^{-1}$ for $\mathrm{Im}\, l > 0$. Indeed, if $f \in \mathfrak{H}$, $G(l)f \in \mathfrak{D}_{H^*}$ for any $\mathrm{Im}\, l \neq 0$, and $(H^* - l)G(l)f = (L - l)G(l)f = f$. Conversely, let $u \in \mathfrak{D}_{H^*}$ and put $(H^* - l)u = (L - l)u = f$. Then $w = u - G(l)f$ is in $\mathfrak{D}_{H^*}$, and $(H^* - l)w = 0$. For $\mathrm{Im}\, l > 0$ the equation $(H^* - l)w = 0$ has only the solution $w = 0$ which is in $\mathfrak{D}_{H^*}$. Thus $u = G(l)f$, or $G(l)(H^* - l)u = u$, proving that $G(l) = (H^* - l)^{-1}$ for $\mathrm{Im}\, l > 0$.

The symmetry $G(x, y, l) = \overline{G}(y, x, \bar{l})$ implies that $G(l) = \big(G(\bar{l})\big)^*$. But if $\mathrm{Im}\, l < 0$, $G(\bar{l}) = (H^* - \bar{l})^{-1}$, and hence $G(l) = \big(G(\bar{l})\big)^* = \big((H^* - \bar{l})^{-1}\big)^* = (H - l)^{-1}$. Thus $G(l)$ is the generalized resolvent of H, and this easily implies $G_\delta \to G$ uniformly on any compact (x, y, l)-region where $\mathrm{Im}\, l \neq 0$.

4. The generalized resolution of the identity associated with a maximal symmetric extension. It is now clear that Theorems 2 and 3 of (A) are valid provided we understand by G there the limit function which is the kernel of the generalized resolvent $G(l)$ of H. We call ϱ the spectral matrix associated with H.

It is known [1, pp. 277, 278] that corresponding to each maximal symmetric extension H of S there exists a unique generalized resolution of the identity $F(\lambda)$ which is related to the generalized resolvent $G(l)$ via the formula

$$\big(G(l)f, g\big) = \int_{-\infty}^{\infty} (\lambda - l)^{-1}\, d\big(F(\lambda)f, g\big) \qquad (f, g \in \mathfrak{H}, \operatorname{Im} l \neq 0)\,.$$

This implies inversely, that if $\Delta = (\mu, \lambda]$ is any finite interval whose end points are continuity points of $F(\lambda)$, and $F(\Delta) = F(\lambda) - F(\mu)$, then

$$\big(F(\Delta)f, g\big) = \lim_{\varepsilon \to +0} \frac{1}{2\pi i} \int_{\mu}^{\lambda} \big(K(\nu + i\varepsilon)f, g\big)\, d\nu\,,$$

where $K(l) = G(l) - G(\bar{l})$, $\operatorname{Im} l \neq 0$. Using this formula it is now obvious that the analogue of Theorem 4 of (A) is valid.

THEOREM 3. *If $f \in C^n$ vanishes outside a closed bounded subinterval of (a, b), and λ, μ are continuity points of $F(\lambda)$ and ϱ, then*

$$F(\Delta)f(x) = \int_{\Delta} \sum_{j,k=1}^{n} s_j(x, \lambda) \hat{f}_k(\lambda)\, d\varrho_{jk}(\lambda)\,.$$

This result is sufficient to imply the Parseval equality and expansion theorem, as in Theorem 5 of (A). Thus we see that all results of (A) can be generalized to the case of an arbitrary maximal symmetric extension H of S.

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