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INEQUALITIES FOR FUNCTIONS OF EXPONENTIAL TYPE

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1. Introduction. Let $f(z)$ be an entire function of exponential type τ, with $|f(z)| \leqq M$ on the real axis. It is well known (for references see [1, p. 82]) that this implies

$$|f(x+iy)| \leqq M\, e^{\tau|y|} . \tag{1.1}$$

Duffin and Schaeffer [2] sharpened (1.1) for functions that are real on the real axis by showing that in this case

$$|f(x+iy)| \leqq M \cosh \tau y . \tag{1.2}$$

The object of this note is, first, to prove (1.2) by the method of interpolation series which deals so successfully with many other inequalities; and, second, to obtain some extensions of (1.2).

It turns out that, whether or not $f(z)$ is real on the real axis, (1.2) is true for at least one of $f(x+iy)$, $f(x-iy)$. (A. C. Schaeffer has pointed out that the validity of (1.2) for one of $f(x+iy)$, $f(x-iy)$ is also deducible from the Duffin-Schaeffer theorem for functions that are real on the real axis.) Consequently (1.2) is true for $y>0$ if $|f(x+iy)| \leqq |f(x-iy)|$ for $y>0$. (The latter inequality is true in particular if

$$\limsup_{y\to+\infty} y^{-1} \log|f(iy)| \leqq \limsup_{y\to-\infty} |y|^{-1} \log|f(iy)|$$

and $f(z)$ has no zeros in the lower half plane (B. Levin; see [1, p. 129])). That (1.2) holds, for a given x, either for y or for $-y$ follows from the general inequality (containing an arbitrary real ω)

$$|f(x+iy)e^{-i\omega} + f(x-iy)e^{i\omega}| \leqq 2M\,\{\cosh^2 \tau y - \sin^2 \omega\}^{1/2}; \tag{1.3}$$

for, given x and y, we can choose ω so that the left-hand side of (1.3) is $|f(x+iy)|+|f(x-iy)|$. For any particular ω (not an integral multiple of 2π) we get an inequality with a smaller bound. If $f(z)$ is real on the real axis, (1.3) becomes

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$$(1.4) \qquad |\mathrm{Re}\{f(x+iy)e^{-i\omega}\}| \leqq M\{\cosh^2 \tau y - \sin^2\omega\}^{1/2}.$$

For $\omega = \pi/2$ we obtain

$$(1.5) \qquad |\mathrm{Im} f(x+iy)| \leqq M \sinh \tau|y|,$$

which is included in a result of Hörmander's [3, Theorem 2]. By taking $\omega = \pm\pi/4$, we obtain

$$(1.6) \qquad |\mathrm{Re} f(x+iy)| + |\mathrm{Im} f(x+iy)| \leqq M\{\cosh 2\tau y\}^{1/2}.$$

All these inequalities can be specialized to give inequalities for polynomials; this was done by Duffin and Schaeffer [2] for (1.2).

An advantage of the proof by interpolation series is that it also yields L^p-results. Write $\|f(x)\|_p = \{\int_{-\infty}^{\infty} |f(x)|^p dx\}^{1/p}$. It is known (Plancherel and Pólya: see [1, p. 98]) that

$$\|f(x+iy)\|_p \leqq \|f(x)\|_p \, e^{\tau|y|}, \qquad p \geqq 1;$$

(1.1) is the limiting case $p=\infty$. Inequality (1.3) is also true if $| \ldots |$ is replaced by $\| \ldots \|_p$ and M is replaced by $\|f(x)\|_p$. In particular, for a function that is real on the real axis, we have

$$(1.7) \qquad \begin{aligned} \|\mathrm{Re} f(x+iy)\|_p &\leqq \|f(x)\|_p \cosh \tau y, \\ \|\mathrm{Im} f(x+iy)\|_p &\leqq \|f(x)\|_p \sinh \tau|y|. \end{aligned}$$

However, the immediate analogue of (1.2) is false. Indeed, for $p=2$ we have

$$(1.8) \qquad \|f(x+iy)\|_2 \leqq (\cosh 2\tau y)^{1/2} \|f(x)\|_2,$$

where the constant is the best possible.

An inequality for $\|f(x+iy)\|_p$, $p \neq 2$, can be deduced from (1.7) or (1.8), but presumably the best possible result cannot be obtained in this way. There are also analogous results for mean values.

2. The interpolation formula.

We establish the following lemma.

LEMMA. *If $f(z)$ is an entire function, of exponential type τ, which is bounded on the real axis, then for any real number ω we have*

$$(2.1) \qquad f(x+iy)e^{-i\omega} + f(x-iy)e^{i\omega} = 2\sum_{-\infty}^{\infty} (-1)^n c_n f(x-s+n\pi/\tau),$$

where

$$(2.2) \qquad c_n = \frac{\tau y \, \mathrm{Im}\{e^{-i\omega}\sin(s\tau + iy\tau)\}}{(n\pi - s)^2 + \tau^2 y^2}$$

and

$$(2.3) \qquad s\tau = \arg\{\cos(\omega + i\tau y)\}.$$

When $f(z)$ is real on the real axis, the left-hand side of (2.1) is

$$2\,\mathrm{Re}\{f(x+iy)e^{-i\omega}\} = 2\,\{\mathrm{Re}f(x+iy)\cos\omega + \mathrm{Im}f(x+iy)\sin\omega\}\,.$$

We need the explicit form (2.2) for c_n only to verify that c_n has the same sign for every n.

It is enough to prove the lemma when $f(z)$ has the special form

$$f(z) = \int_{-\tau}^{\tau} e^{-izt}\,d\alpha(t)\,, \tag{2.4}$$

with $\alpha(t)$ of bounded variation. For, in the general case

$$g(z) = f(z)(\delta z)^{-1}\sin\delta z$$

has this form with τ replaced by $\tau+\delta$ (by the Paley–Wiener Theorem, for which see [1, p. 103]). Since the expansion (2.1) for $g(z)$ converges uniformly in δ, we obtain (2.1) for $f(z)$ by letting $\delta \to 0$. When $f(z)$ has the form (2.4), the left-hand side of (2.1) is

$$2\int_{-\tau}^{\tau} e^{ixt}\cos(\omega - ity)\,d\alpha(t)\,,$$

and we obtain (2.1) by expanding $2e^{ist}\cos(\omega - ity)$ in a Fourier series and integrating term by term (cf. [1, pp. 207ff.]). The choice (2.3) for s makes $e^{is\tau}\cos(\omega - i\tau y) = e^{-is\tau}\cos(\omega + i\tau y)$, and this makes the Fourier series converge absolutely.

We need to evaluate $\sum|c_n|$; we could use the explicit formula (2.2); but it is simpler to apply (2.1) to the special case $f(z) = \cos\tau z$, with $x = s$, when we obtain

$$\sum |c_n| = \left|\mathrm{Re}\{e^{-i\omega}\cos\tau(s+iy)\}\right| = \{\cosh^2\tau y - \sin^2\omega\}^{1/2}\,, \tag{2.5}$$

if we use (2.3).

3. The inequalities. From (2.1) and (2.5) we have

$$|f(x+iy)e^{-i\omega} + f(x-iy)e^{i\omega}| \leqq 2 \sup_{-\infty<x<\infty} |f(x)|\,\{\cosh^2\tau y - \sin^2\omega\}^{1/2}\,. \tag{3.1}$$

Hence we have, by choosing ω suitably,

$$|f(x+iy)| + |f(x-iy)| \leqq 2\sup|f(x)|\cosh^2\tau y\,. \tag{3.2}$$

Similarly, by applying Jensen's inequality we have

$$\|f(x+iy)e^{-i\omega} + f(x-iy)e^{i\omega}\|_p \leqq 2\,\|f(x)\|_p\,\{\cosh^2\tau y - \sin^2\omega\}^{1/2}, \qquad p \geqq 1\,. \tag{3.3}$$

(Here we cannot deduce an analogue of (3.2), since the particular ω used in getting (3.2) depended on x.)

Now suppose that $f(z)$ is real on the real axis; then (3.1) becomes

$$|\mathrm{Re}\,\{f(x+iy)e^{-i\omega}\}| \leqq \sup|f(x)|\,\{\cosh^2\tau y - \sin^2\omega\}^{1/2} \tag{3.4}$$

and (3.3) becomes

$$\|\mathrm{Re}\,\{f(x+iy)e^{-i\omega}\}\|_p \leqq \|f(x)\|_p\,\{\cosh^2\tau y - \sin^2\omega\}^{1/2}. \tag{3.5}$$

By specializing ω we obtain (1.5), (1.6), and (1.7).

We could deduce (1.8) from (3.5), but it is simpler to proceed directly. If $f(z)$ belongs to L^2, we have by the Paley-Wiener Theorem

$$f(z) = \int_{-\tau}^{\tau} e^{izt}\varphi(t)\,dt, \qquad \varphi \in L^2,$$

and if $f(x)$ is also real for real x, we have $\varphi(-t) = \overline{\varphi(t)}$. Hence

$$\begin{aligned}
\int_{-\infty}^{\infty} |f(x+iy)|^2\,dx &= 2\pi\int_{-\tau}^{\tau} e^{-2yt}|\varphi(t)|^2\,dt \\
&= 2\pi\int_0^{\tau} e^{-2yt}\varphi(t)\varphi(-t)\,dt + 2\pi\int_0^{\tau} e^{2yt}\varphi(t)\varphi(-t)\,dt \\
&= 4\pi\int_0^{\tau} \cosh 2yt\,|\varphi(t)|^2\,dt \\
&\leqq 4\pi\cosh 2y\tau\int_0^{\tau} |\varphi(t)|^2\,dt \\
&= \cosh 2y\tau\int_{-\infty}^{\infty} |f(x)|^2\,dx,
\end{aligned}$$

and (1.8) follows. There is strict inequality unless $\varphi(t) = 0$ almost everywhere. If $c < \cosh 2\tau y$, we can contradict $\|f(x+iy)\|_2 \leqq c\|f(x)\|_2$ by taking $\varphi(t) = 0$ for $t > 0$ except in a small neighborhood of τ.

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