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THE STRUCTURE OF SEMISPACES

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1. Introduction. When L is a real linear space and p a point of L, a *semispace* at p (in L) is a maximal convex subset of $L \sim \{p\}$. To obtain an intuitive idea of this notion, the reader may read the first paragraph of § 2 and the statements of (2.1), (2.2), and (2.5). The notion was recently introduced by Hammer [2], who showed that the class of all semispaces in L is the smallest intersection-base for the class of all convex proper subsets of L. (See also Motzkin [7]). The present paper studies (in § 2) the structure of semispaces and (in § 3) of sets which are the intersection of countably many semispaces. In § 4, some results of § 3 are extended to separable (F) spaces, under certain topological restrictions on the convex sets involved.

It is proved that every semispace in L can be generated in a simple way by an ordered family of linear functionals on L, and that L is of countable dimension if and only if every semispace can be generated by the family of "coördinate" functionals associated with some basis for L. The number of different isomorphism-types represented by the semispaces in L is determined in terms of the dimension of L. It is proved that if L is of countable dimension and C is a convex proper subset of L, then C is the intersection of a countable family of semispaces if and only if every family of convex sets whose intersection is C contains a countable subfamily whose intersection is C.

By a *variety* V in L is meant a translate of a linear subspace of L. A *hyperplane* H in V is a set which is maximal among the varieties properly contained in V; H *bounds* in V two *open halfspaces*, the maximal convex subsets of $V \sim H$. For points x and y of L,

$$[x, y] = \{tx+(1-t)y\colon 0 \leqq t \leqq 1\}, \ ]x, y[= \{tx+(1-t)y\colon 0 < t < 1\},$$

etc. The neutral element of L will be denoted by Φ, the empty set by Λ, the closure of a set A by $\mathrm{cl}\,A$, the part of A not in B by $A \sim B$, the union and intersection of a family $\mathfrak{X}$ of sets by $\sigma\mathfrak{X}$ and $\pi\mathfrak{X}$ respectively. For

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sets in a finite-dimensional linear space, words of a topological nature refer to the usual "Euclidean" topology. The remaining notation and terminology should require no explanation.

2. The structure of semispaces. If S is a semispace at p in L, it is easy to check that S is a convex cone with vertex p, that the set $L \sim (S \cup \{p\})$ is also a semispace at p and is identical with the set $2p - S$. As is proved in [2], the class of all semispaces at p is the smallest intersection-base for the class of all convex cones with vertex p in $L \sim \{p\}$, and the class of all semispaces in L is the smallest intersection-base for the class of all convex proper subsets of L. Since each semispace at p is of the form $S + p$ for some semispace S at Φ, in studying the structure of semispaces we may as well consider only those at Φ. Henceforth, *in this section only*, "semispace" will mean "semispace at Φ". Verification of the following remark is left to the reader.

(2.1) *Let* $\mathfrak{F}$ *be a set of linear functionals on* L *and* r *a linear order on* $\mathfrak{F}$ *such that for each* $x \in L \sim \{\Phi\}$ *there is a first* $f_x \in \mathfrak{F}$ *(in the order* r*) having* $f_x x \neq 0$, *that is,* $f_x r g$ *whenever* $g \in \mathfrak{F} \sim \{f_x\}$ *and* $gx \neq 0$. *Then the set* $\{x : f_x x > 0\}$ *is a semispace in* L.

The semispace described in (2.1) will be denoted by $S(F, r)$. A case of special interest arises when B is a basis for L, t a linear order on B, F_B the set of "coördinate" functionals associated with B in the usual way, and r the order on $\mathfrak{F}_B$ naturally induced by t. The semispace $S(\mathfrak{F}_B, r)$ will then be denoted also by $S[B, t]$, and those semispaces obtainable from a basis in this way will be called *basic semispaces*. We shall see below that every semispace can be represented in the form (2.1) but that not all semispaces are basic.

The following proposition collects some useful properties of semispaces, some of which are given in [2].

(2.2) *A convex subset* X *of* $L \sim \{\Phi\}$ *is a semispace (at* Φ*) if and only if* $X \cup -X = L \sim \{\Phi\}$. *If* M *is a linear subspace of* L *and* $T \subset M$, *then* T *is a semispace in* M *if and only if* $T = S \cap M$ *for some semispace* S *in* L. *If* H *is a hyperplane in* L, Q *is one of the open halfspaces bounded by* H, *and* T *is a semispace in* H, *then* $T \cup Q$ *is a semispace in* L; *conversely, if* L *is finite-dimensional then every semispace in* L *has the form* $T \cup Q$ *in this way.*

Proof. The first assertion is easy to check, and leads at once to the "if" part of the second assertion and the first part of the third assertion. Now consider a semispace T in M; let M' be a subspace complementary to M in L and T' a semispace in M'. Then the set $S = (T + M') \cup T'$ is a semispace in L with $S \cap M = T$, whence the proof of the second assertion is complete. Finally, consider a semispace S in a finite-dimensional L. By the basic separation theorem for convex sets, there are in L a hyperplane H through Φ and an open halfspace Q bounded by H such that $S \subset H \cup Q$. But $S \cup -S = L \sim \{\Phi\}$, so $Q \subset S$ and $S = (S \cap H) \cup Q$, completing the proof of (2.2).

We must now introduce a certain partial order on convex sets (discussed also in [8]). Let C be a convex set having more than one point and $x, y \in C$. We write $x \prec y$ provided $[x, y] \subset [x, z]$ for some $z \in C \sim \{x\}$, $y \succ x$ provided $x \prec y$, $x \prec' y$ provided $x \prec y$ and not $y \prec x$, $y' \succ x$ provided $x \prec' y$, and $x \approx y$ provided $x \prec y$ and $y \prec x$. It is easily verified that $\prec$ is transitive, whence $\approx$ is an equivalence relation. When $\prec [\prec']$ is transferred in the natural way to the set of equivalence classes, it becomes a proper partial order which is reflexive [antireflexive]. Clearly each equivalence class is convex.

(2.3) *Suppose S is a semispace in L and X is a representative set from S (i.e., $X \subset S$ and X includes exactly one element from each of the equivalence classes into which S is decomposed by $\approx$). Then X is linearly independent and is linearly ordered by $\prec$. If L is finite-dimensional, then X is a basis for L and $S = S[X, \succ]$.*

PROOF. It suffices, even for the first assertion, to consider the case in which L is finite-dimensional. When $\dim L = 1$, the assertions are obvious. Suppose they have been proved for $\dim L = n$, and consider an $(n+1)$-dimensional L, a semispace S in L, X as described. By (2.2), there are an open halfspace Q in L and bounding hyperplane H such that $S = (S \cap H) \cup Q$. Clearly Q is an equivalence class in S and $x \prec' y$ for each $x \in S \cap H$, $y \in Q$, so $X \cap Q$ consists of a single point z and $x \prec' z$ for each $x \in X \sim \{z\}$. The desired conclusions then follow at once from the inductive hypothesis as applied to H.

(2.4) *Suppose S is a semispace in L and $\mathfrak{C}$ is the set of all equivalence classes in S under $\approx$. For $\xi, \eta \in \mathfrak{C}$, write $\xi \prec' \eta$ provided $x \prec' y$ whenever $x \in \xi$, $y \in \eta$. Then if $\eta \in \mathfrak{C}$ and L_η is the linear extension of η, η is an open halfspace in L_η whose bounding hyperplane contains all $\xi \prec' \eta$.*

PROOF. Let A be the union of all $\xi \prec' \eta$. We shall see below that A is convex, $S \cap L_\eta = A \cup \eta$, and $\eta = \operatorname{core} \eta$ in L (i.e., for each $y \in \eta$ and $p \in L_\eta \sim \{\Phi\}$ there exists $t > 0$ such that $[y, y + tp] \subset \eta$). Suppose for the moment that this has been done, and let us see how to complete the proof. Since η is convex and $A \cap \eta = A$, it follows by the basic separation theorem for convex sets (using the fact that the A is convex and

$\eta = \operatorname{core}\eta$ in L_η) that there is in L_η an open halfspace Q which contains η and misses A. Since clearly $]x, y] \subset \eta$ whenever $x \in A$ and $y \in \eta$, it follows that A is contained in the bounding hyperplane H of Q. Now the set $T = (S \cap H) \cup Q$ is a semispace in L_η, as is the set $S \cap L_\eta$. But $S \cap L_\eta = A \cup \eta \subset T$, and since both are semispaces at Φ (hence maximal convex subsets of $L_\eta \sim \{\Phi\}$) it follows that $S \cap L_\eta = T$, whence $\eta = Q$ and the proof is complete. It remains to establish the facts assumed above.

To show that A is convex it suffices to check that if $y \in \eta$, $u_i \in S$, $u_i \prec' y$, and $p \in]u_1, u_2[$, then $p \prec' y$, and this follows easily by application of (2.3) to the linear extension M of $\{u_1, u_2, y\}$, since $S \cap M$ is a semispace in M and (2.3) shows that every semispace in M is basic. To see that $S \cap L_\eta = A \cup \eta$, observe first that η is a convex cone with vertex Φ, whence $L_\eta = \eta - \eta$. Thus if $p \in S \cap L_\eta$, we have $p = p_1 - p_2$ with $p_i \in \eta$, and application of (2.3) to the linear extension of $\{p_1, p_2\}$ shows that $p \in A \cup \eta$. Thus $S \cap L_\eta \subset A \cup \eta$, and since the reverse inclusion is evident the two sets are the same.

We wish, finally, to show that if $y \in \eta$ and $p \in L_\eta \sim \{\Phi\}$, then

$$[y, y + tp] \subset \eta$$

for some $t > 0$. If $p \in S$, then (since S is a convex cone) $y + 2p \in S$ and $y \prec y + p$. Then if also $p \in L_\eta$ we have $y + p \in A \cup \eta$ and $y + p \prec y$, whence $[y, y+p] \subset \eta$. On the other hand, if $p \in L_\eta \sim (S \cup \{\Phi\})$ then

$$\{-p, -2p\} \subset S \cap L_\eta = A \cup \eta ,$$

whence $-2p \prec y$ and $ry + (1-r)(-2p) \in S$ for some $r > 1$. It follows that $z = y + 2r^{-1}(r-1)p \in S$, whence $[y, z[\subset \eta$ and the proof of (2.4) is complete.

We can now prove the first main result.

(2.5) THEOREM. *If S is a semispace in L, there are a set $\mathfrak{F}$ of linear functionals on L and a linear order r on $\mathfrak{F}$ such that $S = S(\mathfrak{F}, r)$.*

PROOF. Let $\mathfrak{C}$ and $\prec'$ be as in (2.4). It follows from (2.4) that for each $\eta \in \mathfrak{C}$, L_η admits a linear functional g_η such that $g_\eta = 0$ on $\sigma\{\xi : \xi \prec' \eta\}$ and $g_\eta > 0$ on η. Let f_η be a linear functional on L with $g_\eta \subset f_\eta$, and let r be the order for the f_η's induced in the natural way by the order $'\succ$ (the inverse of $\prec'$). It can be verified that $S(\{f_\eta : \eta \in \mathfrak{C}\}, r)$ is a semispace in L which contains S, and thus must be identical with S. The proof is complete.

The following remark is easily verified.

(2.6) *Suppose* $S(\mathfrak{F}, r)$ *is a semispace in* L. *Then for* $x, y \in S(\mathfrak{F}, r)$, $x \approx y$ *if and only if* $f_x = f_y$ (*where for* $p \in L, f_p$ *denotes the first* $f \in \mathfrak{F}$ *for which* $fp \neq 0$).

(2.7) THEOREM. *The linear space* L *is of dimension* $\leqq \aleph_0$ *if and only if every semispace in* L *is basic.*

PROOF. For the "only if" part of the theorem it suffices (in view of (2.3)) to show that if $\dim L = \aleph_0$ and S is a semispace in L, then S is basic. Let L_α be an increasing sequence of finite-dimensional subspaces of L whose union is L. For each n, let $S_n = S \cap L_n$; S_n is a semispace in L_n. Let B_1 be a representative set from S_1; and having chosen B_n as a representative set from S_n, observe that B_n must be contained in a representative set B_{n+1} from S_{n+1}. Let $\prec$ be the order in S, and for each n let $\prec_n$ be the order in S_n. Then by (2.3), B_n is a basis for L_n and $S_n = S[B_n, \succ_n]$. It then follows that $S = S[B, \succ]$, where $B = \bigcup_1^\infty B_n$ and is a basis for L. The proof is complete in one direction. (Note that the last assertion of (2.3) does not carry over to spaces of dimension $\aleph_0$).

Now suppose $\dim L > \aleph_0$; we wish to exhibit in L a nonbasic semispace. Observe first that there are complementary subspaces L_1 and L_2 of L such that $\aleph_0 < \dim L_1 \leqq 2^{\aleph_0}$. Since the linear space of all real sequences is of dimension $2^{\aleph_0}$, L_1 must be isomorphic with a subspace and hence L_1 admits a countable total set $\mathfrak{F}_1$ of linear functionals. With u a well-ordering of $\mathfrak{F}_1$, the set $S_1 = S(\mathfrak{F}_1, u)$ is a semispace in L_1. Let S_2 be a semispace in L_2 and $S = (S_1 + L_2) \cup S_2$. Then S is a semispace in L and we shall show it is not basic.

Suppose there are a basis B for L and an order r on B such that $S = S[B, r]$. From (2.6) it follows that B is a representative set from S. But since $\mathfrak{F}_1$ is countable, only countably many equivalence classes of S intersect $S_1 + L_2$, and hence all but countably many members of B are in $S_2 \subset L_2$. Thus some subspace of L_2 admits in L a complementary subspace of countable dimension, contradicting the fact that $\dim L_1 > \aleph_0$. The proof of (2.7) is complete.

Two semispaces S_1 and S_2 in L will be called *isomorphic* provided there is an isomorphism of L onto L taking S_1 onto S_2. We shall determine, in terms of the dimension of L, the number of isomorphism-types represented by the semispaces in L. For this we need the following result, whose proof was supplied by Professor Tarski and does not use the continuum hypothesis.

(2.8) *If* n *is a cardinal* $\geqq \aleph_0$, *there are* 2^n *different order-types represented by the linearly ordered sets of cardinality* n.

PROOF. If X is a set of cardinality n, then $X \times X$ is of cardinality n and each linear order on X may be regarded as a subset of $X \times X$. Thus the number of order-types under discussion is $\leqq 2^n$. Now let β be the first ordinal whose set P of predecessors has cardinality n and $\mathfrak{K}$ the class of all subsets of P which have cardinality n. For each $K \in \mathfrak{K}$, let L_K be the linearly ordered set $\Sigma_{\alpha \in K}(\alpha + R)$, where R is the set of rational numbers in its usual order. It can be verified that L_K is isomorphic with L_M only if $M = K$. Since $\mathfrak{K}$ is of cardinality 2^n and each L_K is of cardinality n, this completes the proof.

(2.9) THEOREM. *Let k be the number of isomorphism-types represented by the semispaces in L. Then $k = 1$ when* $\dim L < \aleph_0$ *and* $k = 2^{\dim L}$ *when* $\dim L = \aleph_0$ *or* $\dim L \geqq 2^{\aleph_0}$.

PROOF. Note from (2.6) that two basic semispaces $S[B, r]$ and $S[C, t]$ are isomorphic if and only if the linearly ordered sets (B, r) and (C, t) are isomorphic. Now when $\dim L \leqq \aleph_0$, all semispaces in L are basic (2.7). Thus when $\dim L < \aleph_0$, the fact that $k = 1$ follows from the fact that two finite linearly ordered sets are isomorphic if they have the same cardinality. And when $\dim L = \aleph_0$, the fact that $k = 2^{\aleph_0}$ follows from (2.8). When $\dim L \geqq 2^{\aleph_0}$, L is of cardinality $\dim L$, so certainly $k \leqq 2^{\dim L}$; and it follows from (2.8) that $k \geqq 2^{\dim L}$. The proof is complete, though it would still be of interest to treat the case $\aleph_0 < \dim L < 2^{\aleph_0}$ without using the continuum hypothesis.

3. Intersections of countable families of semispaces. We start with

(3.1) LEMMA. *In a finite-dimensional variety V, if a family $\mathfrak{K}$ of convex sets has empty intersection, then so has some countable subfamily.*

PROOF (by induction on the dimension n of V). For $n = 0$, the assertion is obvious. Suppose it is true for $n = k$ and consider a $(k+1)$-dimensional V. By the Lindelöf property of $V \sim \pi\mathfrak{K}'$ (with $\mathfrak{K}' = \{\mathrm{cl}K : K \in \mathfrak{K}\}$), there is a countable subfamily $\mathfrak{L}_1$ of $\mathfrak{K}$ such that $\pi L_1' = \pi\mathfrak{K}'$. It is easy to check that if $\pi\mathfrak{K}'$ has an interior point p, then $p \in \pi\mathfrak{K}$. Hence, since $\pi\mathfrak{K}$ is empty, $\pi\mathfrak{K}'$ has no interior point and must be contained in a k-dimensional subvariety W of V. By the inductive hypothesis, there is a countable subfamily $\mathfrak{L}_2$ of $\mathfrak{K}$ such that $\pi\{K \cap W : K \in \mathfrak{L}_2\} = \Lambda$. Then with $\mathfrak{L} = \mathfrak{L}_1 \cup \mathfrak{L}_2$, we have $\pi\mathfrak{L} = \Lambda$, and the proof is complete.

This result should be compared with Helly's theorem [3] that if a *finite* family of convex sets in E^n has empty intersection, then so has some at-most-$(n+1)$-membered subfamily, and with the fact that if an arbitrary family of closed sets in E^n has empty intersection, then so has some countable subfamily. It may be remarked that if an infinite family of convex sets in E^n satisfies certain topological conditions and has empty intersection, then so has some finite subfamily [3, 4].

Now for each variety V, $\mathfrak{S}V$ will denote the class of all sets $C \subset V$ such that either $C = V$ or C is a semispace at some point of V, and $\mathfrak{J}V$ will denote the class of all intersections of countable subfamilies of $\mathfrak{S}V$. It is easily verified that $\mathfrak{J}V$ is closed under countable intersections and includes all open halfspaces in V; hence, when V is finite-dimensional, $\mathfrak{J}V$ includes all closed convex subsets of V. But for V at least two-dimensional, $\mathfrak{J}V$ omits all strictly convex open subsets of V.

The basic property of $\mathfrak{J}V$ is as follows:

(3.2) THEOREM. *For a variety V of countable dimension and a convex subset C of V, the following assertions are equivalent*:

(i) $C \in \mathfrak{J}V$;

(ii) *whenever $\mathfrak{K}$ is a family of convex sets in V for which $\pi\mathfrak{K} \subset C$, then $\pi\mathfrak{L} \subset C$ for some countable $\mathfrak{L} \subset \mathfrak{K}$*;

(iii) *whenever $\mathfrak{K}$ is a family of convex sets in V for which $\pi\mathfrak{K} = C$, then $\pi\mathfrak{L} = C$ for some countable $\mathfrak{L} \subset \mathfrak{K}$.*

PROOF. That (ii) implies (iii) is obvious; and since each convex set is an intersection of semispaces, it is clear that (iii) implies (i). That (i) implies (ii) will be proved by induction on the dimension n of V, being obvious when $n = 0$. Suppose it has been proved for $n = k$ (where $k < \aleph_0$) and consider a $(k+1)$-dimensional V. Let $\mathfrak{M}V$ be the class of all convex sets C in V for which (ii) is true. We wish to show that $\mathfrak{J}V \subset \mathfrak{M}V$, and for this it clearly suffices to show that $\mathfrak{S}V \subset \mathfrak{M}V$.

Note first that if Q is an open halfspace in V, then $Q \in \mathfrak{M}V$. For consider a family $\mathfrak{K}$ of convex sets in V having $\pi\mathfrak{K} \subset Q$. If $\pi\mathfrak{K} = \Lambda$, we obtain a countable $\mathfrak{L} \subset \mathfrak{K}$ with $\pi\mathfrak{L} \subset Q$ by applying the Lemma (3.1) to $\mathfrak{K}$; if $\pi\mathfrak{K} \neq \Lambda$, such an $\mathfrak{L}$ is obtained by applying the Lemma to obtain a countable $\mathfrak{L} \subset \mathfrak{K}$ with $\pi\{K \cap H : K \in \mathfrak{L}\} = \Lambda$, where H is the bounding hyperplane of Q. Now since $\mathfrak{M}V$ includes all open halfspaces in V and is closed under countable intersection, $\mathfrak{M}V$ includes all closed halfspaces in V. Consider a semispace S in V with bounding hyperplane H, and a family $\mathfrak{K}$ of convex sets in V with $\pi\mathfrak{K} \subset S$. Since $S \cup H \in \mathfrak{M}V$, there is a countable $\mathfrak{L}_1 \subset \mathfrak{K}$ with $\pi L_1 \subset S \cup H$; and since $S \cap H \in \mathfrak{S}H$, by the inductive hypothesis there is a countable $\mathfrak{L}_2 \subset \mathfrak{K}$ with $\pi\{K \cap H : K \in \mathfrak{L}_2\} \subset S \cap H$. But then $\pi(\mathfrak{L}_1 \cup \mathfrak{L}_2) \subset S$, and the proof is complete for the finite-dimensional case.

It remains to consider an $\aleph_0$-dimensional V, and to show that $\mathfrak{S}V \subset \mathfrak{M}V$. To this end, let S be a semispace at the point p of V, and let $\{V_1\}_1^\infty$ be a sequence of finite-dimensional varieties through p such that $\bigcup_1^\infty V_i = V$. Consider a family $\mathfrak{K}$ of convex sets with $\pi\mathfrak{K} \subset S$. For each i, $S \cap V_i \in \mathfrak{S}V_i$ and hence by the finite-dimensional result there is a countable $\mathfrak{L}_i \subset \mathfrak{K}$ with $\pi\mathfrak{L}_i \cap V_i \subset S \cap V_i$. But then $\pi \bigcup_1^\infty \mathfrak{L}_i \subset S$ and the proof of Theorem 1 is complete.

We have use for

(3.3) COROLLARY. *If W is a subvariety of the countable-dimensional variety V, then $\mathfrak{I}W = \{A \cap W : A \in \mathfrak{I}V\}$.*

PROOF. It is not hard to show that $W \in \mathfrak{I}V$. That $\mathfrak{I}W \subset \{A \cap W : A \in \mathfrak{I}V\}$ follows from this and the additional observation that

$$\mathfrak{S}W \subset \{S \cap W : S \in \mathfrak{S}V\}.$$

The reverse inclusion follows from the fact that $W \in \mathfrak{I}V$ and the characterization provided by (3.2).

For the finite-dimensional case, the structure of the members of $\mathfrak{I}V$ may be described recursively by means of the following

(3.4) THEOREM. *Suppose V is a finite-dimensional variety and C is a convex subset of V. Then $C \in \mathfrak{I}V$ if and only if there is a countable family $\mathfrak{H}$ of supporting hyperplanes of C in V such that $\operatorname{cl} C \sim C \subset \sigma\mathfrak{H}$ and $C \cap \mathfrak{H} \in \mathfrak{I}H$ for each $H \in \mathfrak{H}$.*

PROOF. Suppose first that $C \in \mathfrak{I}V$; we wish to produce the family $\mathfrak{H}$ as described. Let $\mathfrak{K}$ be a countable family of semispaces in V having $\pi\mathfrak{K} = C$, let $\mathfrak{L}$ be the set of all members of $\mathfrak{K}$ whose bounding hyperplanes actually support C, and let $\mathfrak{H}$ be the set of such hyperplanes. For each $H \in \mathfrak{H}$, $H \cap C \in \mathfrak{I}H$ by the Corollary. And clearly $\operatorname{cl} C \sim \sigma\mathfrak{H} \subset \pi\mathfrak{K} = C$, whence $\operatorname{cl} C \sim C \subset \sigma\mathfrak{H}$ and the proof is complete in one direction.

Suppose conversely that $\mathfrak{H}$ exists as described; we must show that $C \in \mathfrak{I}V$. This is obvious if C is actually contained in some $H \in \mathfrak{H}$, so suppose not and for each H let Q_H be the open half-space bounded by H and containing $C \sim H$. For each H, $C \cap H \in \mathfrak{I}H$ and hence $C \cap H = \bigcap_1^\infty T_H{}^i$, where each $T_H{}^i$ is a semispace in H. With

$$\mathfrak{F} = \{T_H{}^i \cup Q_H : 1 \leqq i < \infty, H \in \mathfrak{H}\},$$

$\mathfrak{F}$ is a countable family of semispaces in V for which $C \subset \pi\mathfrak{F}$ and $\pi\mathfrak{F} \cap \operatorname{cl} C \subset C$. But clearly there is a countable family $\mathfrak{G}$ of semispaces in V such that $\pi G = \operatorname{cl} C$, and we then have $\pi(\mathfrak{F} \cup \mathfrak{G}) = C$, whence $C \in \mathfrak{I}V$ and (3.4) has been proved.

In answer to some natural questions concerning possible extensions of Theorem (3.2) to higher-dimensional linear spaces, we record the following two observations: (1) In an n-dimensional linear space (where n is an arbitrary cardinal number), there is a family of n semispaces whose intersection is empty, even though for $m<n$ each m-membered subfamily has nonempty intersection. (2) If E is a linear space of dimension $n \geqq 2^{\aleph_0}$ and $\mathfrak{K}$ is an arbitrary family of subsets of E, there is an at-most-n-membered subfamily $\mathfrak{L}$ of $\mathfrak{K}$ for which $\pi\mathfrak{L}=\pi\mathfrak{K}$. However, this is a trivial assertion, for E is of cardinality n.

4. An analogous result for (F)-spaces. An *(F)-space* in the sense of Banach [1] is merely a topological linear space whose topology is generated by a complete invariant metric. The present section extends Theorem (3.2) to separable spaces of this sort. Since such a space, if infinite-dimensional, is of uncountable dimension, it is clear that the convex sets involved must be subjected to some topological restrictions. We proceed with the relevant definitions.

An *inside point* of a convex set C in a topological linear space is a point p such that if K is the union of all line segments in C having p as an inner point, then $\bigcup_1^\infty n(K-p)$ is dense in $C-p$. The set C is *admissible* provided for each closed linear variety L, $L\cap C$ includes all inside points of its closure. It is not hard to verify that if a convex set is closed, open, or finite-dimensional, then it is admissible. A closely related family of sets is utilized by Michael [6].

(4.1) THEOREM. *If E is a separable space of type (F) and $\mathfrak{A}$ is a family of admissible convex sets in E whose intersection is the closed convex set C, then some countable subfamily of $\mathfrak{A}$ has intersection C.*

PROOF. (For a family $\mathfrak{X}$ of sets, $\mathfrak{X}'$ will here denote $\{\mathrm{cl}\,X : X\in\mathfrak{X}\}$.) By transfinite induction or Zorn's lemma, it is easy to produce a pair of functions $(\mathfrak{B}, F)$ such that 1) the domain of $\mathfrak{B}$ and of F is the set $I=[0, \Omega[$ of all countable ordinal numbers; 2) $F_0=E$, and for each $\alpha\in I$,

$$\mathfrak{B}_\alpha = \{A\cap F_\alpha : A\in\mathfrak{A}\};$$

3) for each $\alpha\in I$, $F_{\alpha+1}$ is the smallest closed linear variety containing $\pi\mathfrak{B}_\alpha'$; 4) if α is a limit ordinal in I, $F_\alpha=\pi\{F_\beta' : \beta<\alpha\}$.

It is clear that always $F_\alpha\supset F_{\alpha+1}$, and we now show that if $\pi\mathfrak{B}_\alpha'\neq\pi\mathfrak{B}_\alpha$, then $F_\alpha\neq F_{\alpha+1}$. Note that $\pi\mathfrak{B}_\alpha$ is the closed convex set $C\cap F_\alpha$, and hence by a result in [5], $\pi\mathfrak{B}_\alpha$ must have an inside point p. Now suppose there is a point $q\in\pi\mathfrak{B}_\alpha'\sim\pi\mathfrak{B}_\alpha$, and let $y\in]p, q[\sim\pi\mathfrak{B}_\alpha$. Then y is an inside point of $\pi\mathfrak{B}_\alpha'$, and thus if K is the union of all line segments in $\pi\mathfrak{B}_\alpha'$ having y as an inner point, the set $L=\bigcup_1^\infty n(K-y)$ is dense in $\pi\mathfrak{B}_\alpha'=y$. It is easily verified that L is a linear subspace of E, and hence $F_{\alpha+1}$ is the closure of $L+y$. Thus if $F_{\alpha+1}=F_{\alpha}$, it follows that y is an inside point of each member of $\mathfrak{B}_{\alpha}'$, and (since each member of $\mathfrak{B}_{\alpha}$ is an admissible set) that $y\in\pi\mathfrak{B}_{\alpha}$. This contradiction shows that if $\pi\mathfrak{B}_{\alpha}'\neq\pi\mathfrak{B}_{\alpha}$, then $F_{\alpha}\neq F_{\alpha+1}$.

Now if for each $\alpha\in I$ it is true that $\pi\mathfrak{B}_{\alpha}'\neq\pi\mathfrak{B}_{\alpha}$, there is a function p on I such that always $p_{\alpha}\in F_{\alpha}\sim F_{\alpha+1}$. But the range of p is then an uncountable separable metric space, necessarily having a condensation point p_{α}, which then is an accumulation point of $F_{\alpha+1}$. Since this is impossible, there must be a countable ordinal $\gamma\in I$ for which $\pi\mathfrak{B}_{\gamma}'=\pi\mathfrak{B}_{\gamma}$.

For each $\alpha\in I$, there is by the Lindelöf property a countable $\mathfrak{A}_{\alpha}\subset\mathfrak{A}$ for which $\pi\{A\cap F_{\alpha}: A\in A_{\alpha}\}'=\pi\mathfrak{B}_{\alpha}'$. With $\mathfrak{D}=\sigma\{\mathfrak{A}_{\alpha}: 0\leqq\alpha\leqq\gamma\}$, $\mathfrak{D}$ is a countable subfamily of $\mathfrak{A}$. It can be verified that for each $\beta\in I$,

$$\pi\sigma\{\mathfrak{A}_{\alpha}: 0 \leqq \alpha \leqq \beta\} \sim C \subset \pi\mathfrak{B}_{\beta}' \sim C \subset F_{\beta+1} ,$$

and with $\beta=\gamma$ this gives

$$\pi D \sim C \subset \pi\mathfrak{B}_{\gamma}' \sim C = \pi\mathfrak{B}_{\gamma} \sim C = \Lambda ,$$

so the proof of (4.1) is complete.

As in the previous section, easy examples show that the result just proved probably has no significant extension to nonseparable spaces. However, *for the separable case we do not know whether the condition of completeness is necessary.* The following additional remarks (dealing with certain natural-seeming extensions of the Theorem) may be of interest:

(1) If a convex set G has an interior point and intersects each of its supporting hyperplanes in an admissible set, then G is admissible; this is true, in particular, if G is the union of a strictly convex open set with an arbitrary subset of its boundary. On the other hand, Hilbert space contains a convex precompact G_{δ} set which is not admissible.

(2) Hilbert space contains a family $\mathfrak{K}$ of convex sets, each having an interior point and being simultaneously an $F_{\sigma\delta}$ set and a $G_{\delta\sigma}$ set, such that $\mathfrak{K}$ has empty intersection but every countable subfamily of $\mathfrak{K}$ has nonempty intersection.

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