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INTERSECTIONS OF TRANSLATES OF CONVEX BODIES

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1. In n-dimensional Euclidean space let K be a convex body, i.e. a compact convex set with interior points. Using a finite number of vectors $u_1, u_2, \ldots, u_m$ we translate K into the bodies $K+u_1, K+u_2, \ldots, K+u_m$. In this paper we shall be interested in the following set-up:

$$(K+u_k) \cap (K+u_l) \neq \emptyset \qquad (= \text{the void set})\,, \tag{1}$$

$$\bigcap_{k=1}^{m} (K+u_k) = \emptyset\,, \tag{2}$$

i.e. though the bodies $K+u_k$ pairwise meet, there is no point belonging to all of them.

Definition. For a convex body K let $I(K)$ denote the smallest integer m such that there exist m vectors $u_1, u_2, \ldots, u_m$ satisfying (1) and (2). If there exists no such integer m let $I(K)=\infty$.

From the definition we immediately get $I(K) \geqq 3$.

In this paper we are going to determine the value of $I(K)$ for different bodies K. I am indebted to B. v. Sz. Nagy for this problem. He has proved [3], that if K is a parallelepiped then $I(K)=\infty$, but that in all other cases $I(K) \leqq n+1$, where n is the dimension of the space. We are going to sharpen his result by proving that, if K is not a parallelepiped, then $I(K) \leqq 4$ and that $I(K)=4$ only for some convex polyhedra.

Main Theorem. *The number $I(K)$ has the following properties.*

A) *$I(K)$ is 3, 4, or ∞.*

B) *$I(K)>3$ if and only if*

1) *K is a convex polyhedron,*

2) *K is centrally symmetric, and*

3) *for any two disjoint faces L_1 and L_2 of K there are two distinct parallel supporting planes Π_1 and Π_2 of K such that $L_1 \subset \Pi_1$ and $L_2 \subset \Pi_2$.*

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C) $I(K) = \infty$ *if and only if* K *is a parallelepiped* (B. v. Sz. Nagy).

D) *For each dimension* n *there are only a finite number of affinely non-equivalent polyhedra* K *with* $I(K) > 3$.

After some preliminaries given in section 2 we prove the Main Theorem in sections 3–5. First we establish the necessity of the condition given in B) for a convex body K to have $I(K) > 3$. This is done in section 3, where we also prove D). In section 4 we prove A) and C), and in section 5 we give the proof of the sufficiency of the condition in B).

We may change our main problem by studying not only translations of K but general homotheties $\lambda_k K + u_k$, $\lambda_k > 0$, of K (cf. Nachbin [2]). This will be done in section 6. It turns out that $I(K)$ is also the minimum number of pairwise meeting homotheties of K with void intersection.

In section 7, finally, we give some examples of polyhedra with $I(K) > 3$. They are obtained out of lower-dimensional polyhedra by a general method. All known polyhedra with $I(K) > 3$ are obtained by this method, and for $n \leqq 5$ the method gives all possible polyhedra. Whether or not this is true for arbitrary n is still an unsolved problem.

The simplest convex polyhedron K with $I(K) = 4$ turns out to be the regular octahedron.

2. We shall use vector notations in Euclidean n-space and shall not strictly distinguish between points and vectors. The inner product $u_1x_1 + \ldots + u_nx_n$ of $u = (u_1, \ldots, u_n)$ and $x = (x_1, \ldots, x_n)$ is denoted by ux.

If K is a set and u a vector we denote by $K + u$ the set of all points $x + u$ with $x \in K$. Similarly, if K_1 and K_2 are two sets, $K_1 + K_2$ denotes the set of all points $x_1 + x_2$ with $x_1 \in K_1$ and $x_2 \in K_2$.

A linear manifold is defined as the set of points x satisfying a number of linear equations $u_k x = a_k$. If the linear manifold is $(n-1)$-dimensional it is called a plane.

A supporting plane of a convex body K is a plane which intersects K but does not intersect the interior of K. Two supporting planes of K are called opposite if they are distinct and parallel.

Let K_1 and K_2 be two convex bodies. There are three possibilities. First K_1 and K_2 may have interior points in common. Then there is a positive number δ such that, if u is any vector for which $|u| < \delta$, then $K_1 + u$ and K_2 have non-void intersection. Secondly K_1 and K_2 may have only boundary points in common. Then they have a common supporting plane containing $K_1 \cap K_2$ and separating the interior of K_1 from that of K_2. Finally $K_1 \cap K_2$ may be void. In this case there is a plane separating K_1 from K_2.

LEMMA 2.1. *Let Π_1 and Π_2 be two opposite supporting planes of a convex body K and let $y \in L_1 = K \cap \Pi_1$ and $z \in L_2 = K \cap \Pi_2$. Then*

$$K \cap (K+y-z) = L_1 \cap (L_2+y-z) \subset \Pi_1 . \tag{1}$$

PROOF. Let Π_1 be given by $ux=a$ and Π_2 by $ux=b$, where $a<b$. Then $uy=a$ and $uz=b$. For each point x of K we have $a \leqq ux \leqq b$. Hence for each point x of $K+y-z$ we get

$$a \leqq u(x-y+z) \leqq b ,$$

so that

$$2a-b \leqq ux \leqq a .$$

Hence all common points of K and $K+y-z$ satisfy $ux=a$, i.e. they lie in the plane Π_1. But $K \cap \Pi_1 = L_1$ and

$$(K+y-z) \cap \Pi_1 = (K+y-z) \cap (\Pi_2+y-z) = L_2+y-z .$$

This proves the lemma.

When a p-dimensional convex body is placed in Euclidean n-space, it determines a p-dimensional linear manifold containing it. We shall use the expressions interior point and boundary point of the body with respect to this manifold.

A convex body is called a convex polyhedron if it is the convex hull of a finite number of points. Equivalently it may be defined as a bounded set which is the intersection of a finite number of half-spaces $u_k x \leqq a_k$. A supporting plane of the polyhedron K meets K in a subpolyhedron of lower dimension called a face of K. A 1-dimensional face is called an edge. The $(n-1)$-faces $\big((n-1)$-dimensional faces$\big)$ together make up the whole boundary of K.

LEMMA 2.2. *Let K_1 and K_2 be two convex polyhedra (not necessarily n-dimensional). If there is no plane containing both K_1 and K_2 there is at most one pair of parallel planes Π_1 and Π_2 such that $K_1 \subset \Pi_1$ and $K_2 \subset \Pi_2$.*

PROOF. Suppose that there are two different such pairs of planes Π_1, Π_2 and Π_1', Π_2'. Then

$$K_1 \subset \Pi_1 \cap \Pi_1' \quad \text{and} \quad K_2 \subset \Pi_2 \cap \Pi_2' .$$

But $\Pi_1 \cap \Pi_1'$ and $\Pi_2 \cap \Pi_2'$ are two parallel $(n-2)$-dimensional linear manifolds. Therefore there is a plane containing them. Thus this plane contains K_1 and K_2, and we have a contradiction.

LEMMA 2.3. *Suppose that there is no plane containing two given disjoint convex polyhedra K_1 and K_2. Let* $\dim K_1 \leqq n-1$, *and assume that the linear* *manifold determined by* K_1 *does not meet* K_2. *Then there are two distinct parallel planes* Π_1 *and* Π_2 *and a polyhedron* K_2' *satisfying*:

a) $K_1 \subset \Pi_1$,

b) $K_2' = K_2 \cap \Pi_2 \neq \emptyset$,

c) *the open half-space bounded by* Π_2 *containing* Π_1 *and* K_1 *contains no point of* K_2,

d) *there is one and only one pair of parallel planes containing* K_1 *and* K_2', *namely* Π_1 *and* Π_2,

e) *no plane contains both* K_1 *and* K_2'.

Proof. Here d) is an immediate consequence of a), b), e), and Lemma 2.2. Since the linear manifold determined by K_1 does not meet K_2, there are planes Π_1 containing K_1 but not meeting K_2. For each such plane there is a unique parallel plane Π_2 satisfying b) and c). We still have to get e) satisfied. For that purpose restrict the plane Π_1 by requiring that the dimension of $K_2' = K_2 \cap \Pi_2$ reaches its maximum. Then suppose that K_1 and K_2' are contained in a plane Π. Since Π contains points of both Π_1 and Π_2, it is not parallel to these planes. Hence we have that $\Pi \cap \Pi_1$ and $\Pi \cap \Pi_2$ are $(n-2)$-dimensional.

By assumption there is no plane containing K_1 and K_2. Hence, since Π contains K_1, Π does not contain K_2. Therefore there is at least one open half-space Π^+ bounded by Π which contains points of K_2.

Now turn the two planes Π_1 and Π_2 around $\Pi \cap \Pi_1$ and $\Pi \cap \Pi_2$ keeping them parallel to each other. They are distinct and satisfy a), b), and c) until either they become parallel to Π (and therefore coincide) or the plane Π_2 meets a point of K_2 outside K_2'. Therefore if we turn the planes in such a way that $\Pi_2 \cap \Pi^+$ moves towards $K_2 \cap \Pi^+$, which is non-void, we may proceed with the motion until Π_2 contains some boundary point of K_2 outside of K_2'. However, this contradicts the fact that Π_1 was chosen such that $\dim(K_2 \cap \Pi_2)$ reached its maximum. This proves the lemma.

Before proceeding to the proof of our Main Theorem in the next three sections, let us here make some simple remarks about the number $I(K)$.

It follows immediately from the definition that $I(K)$ does not change if we replace K by an affine image.

When $n=1$ we have only one type of convex bodies, namely the closed intervals. For them we have $I(K)=\infty$. This follows from the following lemma.

Lemma 2.4. *Let there be given on the real line a finite number of pairwise meeting closed intervals* $[a_k, b_k]$, $1 \leq k \leq m$. *Then they have a point in common.*

PROOF. Put $a = \max a_k$ and $b = \min b_k$. Since $[a_{k_1}, b_{k_1}]$ meets $[a_{k_2}, b_{k_2}]$ we have $a_{k_1} \leqq b_{k_2}$. Hence, since k_1 and k_2 are arbitrary, we get $a \leqq b$, and for any k we have

$$a_k \leqq a \leqq b \leqq b_k .$$

Thus a and b belong to all the intervals.

We get a similar result when $n > 1$:

LEMMA 2.5. *Let $K_1, \ldots, K_m$ be pairwise meeting n-dimensional parallelepipeds with their $(n-1)$-faces parallel to the coordinate planes. Then they have a point in common.*

PROOF. Let K_k be defined by

$$a_{ik} \leqq x_i \leqq b_{ik}, \qquad 1 \leqq i \leqq n .$$

Keep i fixed. Since K_{k_1} meets K_{k_2} the interval $[a_{ik_1}, b_{ik_1}]$ must meet the interval $[a_{ik_2}, b_{ik_2}]$ Therefore, applying the preceding lemma, we can take a value a_i for which $a_{ik} \leqq a_i \leqq b_{ik}$ for all k. This may be done for each i. Then the point $a = (a_1, \ldots, a_i, \ldots, a_n)$ belongs to all K_k. This proves the lemma.

We close this section by giving an example of a polyhedron K with $I(K) = 3$, which will be of some interest in the sequel.

EXAMPLE 2.6. In Euclidean 6-space let K be the 5-dimensional polyhedron given by

$$\begin{aligned} &x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 0 , \\ &-3 \leqq x_i \leqq 3 , \qquad 1 \leqq i \leqq 6 . \end{aligned} \tag{2}$$

Evidently K has the origin as its centre of symmetry. Let us determine its vertices. In a vertex at least 5 of the coordinates x_i are 3 or -3. Because of (2) the only possibility is that three x_i are 3 and the remaining three x_i are -3.

That $I(K) = 3$ is seen by adding the three vectors

$$\begin{aligned} u_1 &= (\ \ 4, \ \ 4, -2, -2, -2, -2) , \\ u_2 &= (-2, -2, \ \ 4, \ \ 4, -2, -2) , \\ u_3 &= (-2, -2, -2, -2, \ \ 4, \ \ 4) , \end{aligned}$$

lying in the plane (2). We get $K_k = K + u_k$ given by (2) and the inequalities

$$\begin{aligned} K_1\colon\ &1 \leqq x_1, x_2 \leqq 7, & K_2\colon\ &-5 \leqq x_1, x_2 \leqq 1, & K_3\colon\ &-5 \leqq x_1, x_2 \leqq 1 , \\ &-5 \leqq x_3, x_4 \leqq 1, & &1 \leqq x_3, x_4 \leqq 7, & &-5 \leqq x_3, x_4 \leqq 1 , \\ &-5 \leqq x_5, x_6 \leqq 1, & &-5 \leqq x_5, x_6 \leqq 1, & &1 \leqq x_5, x_6 \leqq 7 . \end{aligned}$$

Here K_1 and K_2 contain the point $(1, 1, 1, 1, -2, -2)$ (and in fact a whole common edge), so that $K_1 \cap K_2 \neq \emptyset$. Similarly $K_1 \cap K_3 \neq \emptyset$ and $K_2 \cap K_3 \neq \emptyset$. The only point, however, satisfying all inequalities for K_1, K_2, and K_3 above is the point $(1, 1, 1, 1, 1, 1)$, which does not lie in (2). Hence $K_1 \cap K_2 \cap K_3 = \emptyset$.

3. A boundary point of a convex body K is called a point of strict convexity if K has a supporting plane meeting K only in this point. It is known (cf. [1, p. 88]) that every convex body has points of strict convexity and that a convex body with only a finite number of points of strict convexity is a polyhedron.

THEOREM 3.1. *If $I(K) > 3$, K is centrally symmetric.*

PROOF[1]. Let $y \in K$ be a point of strict convexity and Π_1 a corresponding supporting plane such that $K \cap \Pi_1 = y$. Take the opposite supporting plane Π_2 and a point $z \in K \cap \Pi_2$. Translate K into $K+y-z$. By Lemma 2.1 the intersection of K and $K+y-z$ lies in Π_1. The only point of K in Π_1, however, is y. Since $y = z + (y-z) \in K+y-z$ we get

$$K \cap (K+y-z) = y .$$

The two points y and z therefore have the property that $K-y$ and $K-z$ have a single point in common. We shall show that if $I(K) > 3$ this property implies that K has $\frac{1}{2}(y+z)$ as centre of symmetry.

To simplify the notations let the origin be the point $\frac{1}{2}(y+z)$. Then we have $z = -y$. We know that $K-y$ and $K+y$ have a single point in common, and since y and $-y$ both lie in K the common point must be the origin, i.e.

$$(K-y) \cap (K+y) = 0 .$$

Now let x be an arbitrary point of K. We have to prove that $-x \in K$. That $x \in K$ can be expressed by $0 \in K-x$. Hence the three translates

$$K_1 = K-y, \qquad K_2 = K+y, \qquad K_3 = K-x$$

have the origin in common. Now introduce $K_4 = K+x$. Then

$$K_1 = K_3+x-y \quad \text{and} \quad K_4 = K_2+x-y ,$$

and since K_3 and K_2 meet, so do K_1 and K_4. Similarly

$$K_2 = K_3+x+y \quad \text{and} \quad K_4 = K_1+x+y ,$$

and since K_3 and K_1 meet so do K_2 and K_4. Since K_1 and K_2 meet we

[1] An idea of Th. Bang has made it possible for me to simplify the proof of this theorem.

therefore know that K_1, K_2 and K_4 pairwise meet. Hence, since $I(K) > 3$, they have a common point. But 0 is the only common point of K_1 and K_2. Thus we must have $0 \in K_4$ or, equivalently, $-x \in K$. This proves the theorem.

Whenever $I(K) > 3$, and therefore K is centrally symmetric, two points of K (or two faces of K, when K is proved to be a polyhedron) will be called opposite if they lie symmetrically with respect to the centre of symmetry of K.

THEOREM 3.2. *If $I(K) > 3$, K is a polyhedron.*

PROOF. We are going to show that if K is a convex body with an infinite number of points of strict convexity then $I(K) = 3$. From this it will follow that, if K is a convex body with $I(K) > 3$, then K must have only a finite number of points of strict convexity and therefore must be a polyhedron. Thus the theorem will be proved.

Let therefore y_k, $k = 1, 2, \ldots$, be an infinite sequence of distinct points of strict convexity for K, and let z_k for each k be the point opposite to y_k. As in the proof of Theorem 3.1 we get

$$K \cap (K + y_k - z_k) = y_k .$$

Since K is bounded we may assume that the sequences y_k and z_k converge. Let $k \neq l$ and put

$$K_1 = K, \qquad K_2 = K + y_k - z_k, \qquad K_3 = K + y_l - z_l .$$

Then

$$K_1 \cap K_2 = y_k, \qquad K_1 \cap K_3 = y_l , \tag{1}$$

and, since $y_k \neq y_l$,

$$K_1 \cap K_2 \cap K_3 = \emptyset . \tag{2}$$

Furthermore $K_3 = K_2 + u$ where

$$u = (y_l - y_k) + (z_k - z_l) ,$$

and if u is small enough, K_2 and K_3 have common interior points. Hence, since $u \to 0$ when $k, l \to \infty$, we have for sufficiently large k and l

$$K_2 \cap K_3 \neq \emptyset . \tag{3}$$

But (1), (2), and (3) show that $I(K) = 3$. This completes the proof of the theorem.

THEOREM 3.3. *Let K be a polyhedron with $I(K) > 3$. Then for any two disjoint faces L_1 and L_2 of K there are two opposite supporting planes Π_1 and Π_2 of K such that $L_1 \subset \Pi_1$ and $L_2 \subset \Pi_2$.*

PROOF. Let K be a polyhedron with $I(K) > 3$. Assume that there exist L_1 and L_2 satisfying 1) and 2) below.

1) *L_1 and L_2 are two disjoint faces of K.*

2) *There is no pair of opposite supporting planes Π_1 and Π_2 of K such that $L_1 \subset \Pi_1$ and $L_2 \subset \Pi_2$.*

We shall show that this assumption will lead to $I(K) = 3$, a contradiction which will prove our theorem.

Choose L_1 and L_2 such that

3) $\dim L_1 - \dim L_2$ *is maximal under conditions* 1) *and* 2).

Before proceeding with the proof let us point out two examples which the reader may find illustrative in the rest of the proof. The first example is the 2-dimensional body, the boundary of which is a regular hexagon. Here L_1 will be a side and L_2 a vertex. The other example is the 5-dimensional polyhedron given in Example 2.6. Here we may take as L_1 the 3-dimensional face defined by $x_1 = 3$, $x_2 = 3$ and as L_2 the 1-dimensional edge joining $(3, -3, -3, -3, 3, 3)$ and $(-3, 3, -3, -3, 3, 3)$.

Now let us continue the proof by concluding from 1), 2), and 3) several facts about L_1 and L_2.

4) *Every face L_1' of K containing L_1 as a proper subset meets L_2.*

For if $L_1' \cap L_2$ is void, L_1' and L_2 satisfy 1). They also satisfy 2) since a plane containing L_1' also contains L_1. But

$$\dim L_1' - \dim L_2 > \dim L_1 - \dim L_2$$

contradicting 3).

5) *Every plane Π which contains L_1 and is disjoint to L_2, is a supporting plane of K.*

For let Π_1 be a supporting plane intersecting K in L_1. Then, by 1), Π_1 is disjoint to L_2. Therefore it is possible to turn a plane Π_2 from Π_1 to Π around the intersection $\Pi_1 \cap \Pi$ (which contains L_1) so that Π_2 never intersects L_2. To prove 5) it will be sufficient to prove that in all positions the plane Π_2 satisfies $\Pi_2 \cap K = L_1$ and therefore is a supporting plane of K. When $\Pi_2 = \Pi_1$ we have in fact $\Pi_2 \cap K = L_1$ and in general at least $\Pi_2 \cap K \supset L_1$. If $\Pi_2 \cap K \neq L_1$ for some position of Π_2, take the first position for which this happens. Π_2 is then still a supporting plane but intersects K in a face having L_1 as a proper subset. Since this is impossible because of 4), we must have $\Pi_2 \cap K = L_1$.

6) *There is no plane Π containing both L_1 and L_2.*

For, again, let Π_1 be a supporting plane meeting K in L_1. Then Π_1

is disjoint to L_2. Suppose that there is a plane Π containing both L_1 and L_2. Let y be any point outside of Π. Take the plane Π_2 containing y and $\Pi \cap \Pi_1$. Then $\Pi \cap \Pi_2 = \Pi \cap \Pi_1$. Since $L_2 \subset \Pi$, we conclude

$$\Pi_2 \cap L_2 = \Pi_2 \cap \Pi \cap L_2 \subset \Pi_1 \cap L_2 = \emptyset .$$

But Π_2 contains L_1. Hence, by 5), Π_2 is a supporting plane of K. Thus y is not an interior point of K. The point y, however, was arbitrarily chosen outside of Π, so that all interior points of K must lie in Π. This is a contradiction since K is n-dimensional.

7) *There is one and only one pair of parallel planes, Π_1' and Π_2', such that $L_1 \subset \Pi_1'$ and $L_2 \subset \Pi_2'$. Π_1' is a supporting plane of K, but Π_2' is not a supporting plane of K.*

For, since L_1 and L_2 are disjoint faces of the same polyhedron, the linear manifold determined by L_1 does not meet L_2. Hence, because of 6), we may apply Lemma 2.3. We get two distinct parallel planes Π_1' and Π_2' and a polyhedron L_2' satisfying:

a) $L_1 \subset \Pi_1'$,
b) $L_2' = L_2 \cap \Pi_2' \neq \emptyset$,
c) the open half-space bounded by Π_2' containing Π_1' and L_1 contains no point of L_2,
d) there is one and only one pair of parallel planes containing L_1 and L_2', namely Π_1' and Π_2'.

Let us show that $L_2' = L_2$. In fact, if $L_2' \neq L_2$, so that L_2 is not contained in Π_2', we see from c) that Π_2' is not a supporting plane of K. Thus, because of d), L_1 and L_2' satisfy 2). Since they obviously also satisfy 1), the fact that

$$\dim L_1 - \dim L_2' > \dim L_1 - \dim L_2$$

contradicts 3). Hence $L_2' = L_2$ and $L_2 \subset \Pi_2'$.

It remains to be proved that Π_1' is a supporting plane of K and that Π_2' is not a supporting plane of K. The first fact, however, follows readily from 5), and thereafter the second fact is a consequence of 2). This proves 7).

By the aid of 1)–7) we shall now show that $I(K) = 3$, contrary to the assumption $I(K) > 3$.

By the central symmetry we have a unique face L_3 opposite to L_2. In each of the faces L_k, $k = 1, 2, 3$, take an interior point x_k (with respect to the linear manifold determined by L_k). Then a supporting plane of K through x_k contains L_k. Consider

$$K_1 = K - x_1 + u, \qquad K_2 = K - x_2, \qquad K_3 = K - x_3,$$

where u is a small vector, still to be determined.

The origin lies in both $K - x_1$ and $K_2 = K - x_2$. If these two polyhedra have no common interior point, they will have a common supporting plane Π through the origin. Then $\Pi + x_1$ is a supporting plane of K containing x_1 and therefore also L_1, and similarly $\Pi + x_2$ is a supporting plane of K containing x_2 and therefore also L_2. Since $\Pi + x_1$ and $\Pi + x_2$ are parallel, this contradicts 2) or 6). This contradiction shows that $K - x_1$ and K_2 have in fact common interior points. Hence for sufficiently small u also K_1 and K_2 have common interior points. In particular $K_1 \cap K_2 \neq \emptyset$.

Each supporting plane of K containing L_3 corresponds to a parallel supporting plane containing L_2. Therefore we may repeat the arguments above and show, that if u is small enough, we also have $K_1 \cap K_3 \neq \emptyset$.

Let us find $K_2 \cap K_3$. Since the origin belongs to this set, we have $K_2 \cap K_3 \neq \emptyset$. Let us apply Lemma 2.1. Since L_2 and L_3 are two opposite faces of K, there are two opposite supporting planes Π_2 and Π_3 such that $L_2 = K \cap \Pi_2$ and $L_3 = K \cap \Pi_3$. Hence, by Lemma 2.1,

$$K \cap (K + x_2 - x_3) \subset L_2$$

and, by the translation $-x_2$,

$$K_2 \cap K_3 \subset L_2 - x_2 .$$

We translate the two planes Π_1' and Π_2' of 7) into $\Pi_1' - x_1$ and $\Pi_2' - x_2$. We get two parallel planes both of which contain the origin. Hence they coincide. Since $L_2 - x_2 \subset \Pi_2' - x_2$ we therefore obtain

$$K_2 \cap K_3 \subset \Pi_1' - x_1 .$$

Now we get from 7) that $\Pi_1' - x_1$ is a supporting plane of $K - x_1$. Hence by giving u a suitable direction we can assure that

$$K_1 \cap (\Pi_1' - x_1) = \emptyset .$$

Hence

$$K_1 \cap K_2 \cap K_3 = \emptyset .$$

With u so chosen the three translates K_1, K_2, and K_3 show that $I(K) = 3$, contrary to our assumption $I(K) > 3$.

Hence for a convex polyhedron K with $I(K) > 3$ there is, for any pair of disjoint faces L_1 and L_2, always a pair of opposite supporting planes Π_1 and Π_2 such that $L_1 \subset \Pi_1$ and $L_2 \subset \Pi_2$. This proves Theorem 3.3.

Theorems 3.1, 3.2, and 3.3 together prove that the condition given in B) of the Main Theorem is necessary for $I(K) > 3$.

COROLLARY 3.4. *If $I(K) > 3$, any two opposite $(n-1)$-faces together contain all vertices of K.*

PROOF. Let L_1 be an $(n-1)$-face and L_2 a vertex not contained in L_1. By Theorem 3.3 there are two opposite supporting planes Π_1 and Π_2, such that $L_1 \subset \Pi_1$ and $L_2 \subset \Pi_2$. Since L_1 is $(n-1)$-dimensional, we have $L_1 = K \cap \Pi_1$. Therefore the face opposite to L_1 is $K \cap \Pi_2$, which contains L_2.

PROOF OF D) OF THE MAIN THEOREM. By Theorems 3.1 and 3.2 we know that, if $I(K) > 3$, K is a centrally symmetric polyhedron. Take n linearly independent supporting planes of K, each meeting K in an $(n-1)$-face. Make an affine transformation so that the centre of symmetry is the origin and these n planes are the planes $x_i = 1$, $1 \leqq i \leqq n$, in an orthogonal coordinate system. The opposite planes then have the equations $x_i = -1$, $1 \leqq i \leqq n$. Denote the cube $\{x \mid -1 \leqq x_i \leqq 1\}$ by C.

Take an arbitrary vertex of K. By Corollary 3.4 this vertex must, for each i, lie in the plane $x_i = 1$ or in the plane $x_i = -1$. Hence it has all its coordinates equal to 1 or -1. Hence it is a vertex of C.

Thus we have shown that any n-dimensional convex polyhedron with $I(K) > 3$ is affinely equivalent to the convex hull of some of the vertices of a fixed cube. Therefore there can exist only a finite number of affinely non-equivalent polyhedra with $I(K) > 3$.

REMARK 3.5. We know that for $n = 1$ we always have $I(K) = \infty$. Let us consider $n = 2$. Take some vertices of a square. If we take one or two vertices, the convex hull is not 2-dimensional, and if we take three vertices, the convex hull is not centrally symmetric. Hence, except the square and its affine images, which have $I(K) = \infty$, all 2-dimensional convex bodies have $I(K) = 3$ (in accordance with B. v. Sz. Nagy's result $I(K) \leqq n+1$). The same arguments show for $n = 3$ that if there is some K with $I(K) > 3$ which is not affinely equivalent to a cube, it has to be affinely equivalent to a regular octahedron.

REMARK 3.6. Let us consider centrally symmetric convex polyhedra having the property described in Corollary 3.4: Any pair of opposite $(n-1)$-faces together contain all vertices. An inspection of the proof of D) of the Main Theorem shows that there are only a finite number of affinely non-equivalent polyhedra of this type. Among them we have all K with $I(K) > 3$. However, some of them have $I(K) = 3$. An example of such a polyhedron is the 5-dimensional polyhedron given in Example 2.6. There is no such polyhedron for $n \leqq 4$.

4. Now we know enough about the consequences of the assumption

$I(K) > 3$ to be able to prove A) and C) of the Main Theorem. Since we see from Lemma 2.5 that a parallelepiped has $I(K) = \infty$, it will be sufficient in order to prove both A) and C) to prove the following theorem.

THEOREM 4.1. *If $I(K) > 3$ and K is not a parallelepiped, then $I(K) = 4$.*

The proof depends on a lemma.

LEMMA 4.2. *If an n-dimensional convex polyhedron L is not a simplex, we can find some vertex y of L and some point z belonging to an $(n-2)$-face of L such that the segment from y to z contains interior points of L.*

PROOF. Since L is not a simplex it has more than $n+1$ vertices. Take $n+1$ vertices of L spanning an n-dimensional simplex and let y be one of the remaining vertices of L. Then the convex hull of the set of all vertices of L except y is n-dimensional. Therefore, if we let M be an $(n-1)$-face of L not containing y, there is some vertex $y' \neq y$ of L not contained in M. I claim that for some point z in the boundary of M the segment from y to z contains interior points of L.

In fact let H be the convex hull of y and M. Assume that all segments from y to the points $z \in \operatorname{bdry} M$ belong to $\operatorname{bdry} L$. Then, since $\operatorname{bdry} H$ consists of all these segments together with M we have

$$\operatorname{bdry} H \subset \operatorname{bdry} L .$$

Hence, since H and L are both n-dimensional convex bodies, we conclude $H = L$. The point y', however, is a vertex of L, and since H is the convex hull of some other vertices of L, y' is not contained in H. Hence we cannot have $H = L$. This contradiction shows that for some point z of $\operatorname{bdry} M$ the segment from y to z contains interior points of L. Since a boundary point of M lies in some $(n-2)$-face of L, this proves the lemma.

Let us point out that the regular octahedron provides an illustrative example for the proof that follows.

PROOF OF THEOREM 4.1. Let K be an n-dimensional convex body with $I(K) > 3$. Assume that K is not a parallelepiped. Then by Remark 3.5, $n \geqq 3$. Since $I(K) > 3$, we get from Theorems 3.1 and 3.2 that K is a centrally symmetric polyhedron. Hence, since K is not a parallelepiped, its number of $(n-1)$-faces is greater than $2n$. Let x_0 be a vertex of K. Because of Corollary 3.4, one of each pair of opposite $(n-1)$-faces contains x_0. Therefore the number of $(n-1)$-faces containing x_0 is greater than n.

Cut K with a plane Π separating x from the remaining vertices of K.

Put $L = K \cap \Pi$. L is $(n-1)$-dimensional, and since the number of $(n-1)$-faces of K which contain x_0 is greater than n, the number of $(n-2)$-faces of L is greater than n. Hence it is not a simplex. Apply Lemma 4.2 to L. We get a vertex y of L and a point z belonging to an $(n-3)$-face of L, where the segment from y to z contains interior points of L (with respect to Π). All interior points of L are interior points of K. Thus the segment from y to z contains interior points of K.

From the fact that Π separates x_0 from the remaining vertices of K, we conclude that the two segments from x_0 to y and z can be continued in K beyond y and z respectively. Hence a supporting plane of K containing y or z must contain x_0. From this it follows that, if two supporting planes of K contain y and z respectively, they cannot be parallel. For both of them must contain x_0, so that if they are parallel they must coincide. But then they both contain y and z and therefore the whole segment from y to z. This segment, however, contains interior points of K. Therefore it cannot lie in a supporting plane.

Furthermore, since y is a vertex of L, y belongs to an edge L_1 of K containing x_0 and since z belongs to an $(n-3)$-face of L, z belongs to an $(n-2)$-face L_2 of K containing x_0.

Now consider the four translates of K:

$$K_1 = K-y, \qquad K_2 = K-y', \qquad K_3 = K-z+u, \qquad K_4 = K-z'+u ,$$

where y' and z' are the points of K opposite to y and z respectively, and u is a small vector. Let us show that u can be determined such that K_1, K_2, K_3, and K_4 pairwise meet but have void intersection.

First consider the intersection of K_1 and K_3. Since K_1 and $K-z$ have the origin in common they have non-void intersection. As proved above there is no pair of parallel supporting planes of K containing y and z respectively, neither distinct nor coinciding. Therefore no plane through the origin can be a supporting plane of both $K-y=K_1$ and $K-z$. Hence K_1 and $K-z$ have common interior points, and for sufficiently small u we have $K_1 \cap K_3 \neq \emptyset$.

Since a supporting plane of K through y' or z' corresponds to a parallel supporting plane through the opposite point y or z, the arguments above also show that for sufficiently small u we have $K_1 \cap K_4 \neq \emptyset$, $K_2 \cap K_3 \neq \emptyset$, and $K_2 \cap K_4 \neq \emptyset$.

The origin is contained in K_1 and K_2. Hence $K_1 \cap K_2 \neq \emptyset$. Similarly, u is contained in K_3 and K_4, so that $K_3 \cap K_4 \neq \emptyset$.

Since y and y' are opposite boundary points of the centrally symmetric polyhedron K and y is contained in L_1, we get from Lemma 2.1

$$K \cap (K+y-y') \subset L_1 .$$

Hence, by adding the vector $-y$,

$$K_1 \cap K_2 \subset L_1 - y .$$

Similarly, since z and z' are opposite and z is contained in L_2,

$$K \cap (K + z - z') \subset L_2 ,$$

and, by adding the vector $-z+u$,

$$K_3 \cap K_4 \subset L_2 - z + u .$$

Determine u such that $L_1 - y$ and $L_2 - z + u$ are disjoint. This is possible with arbitrarily small u. In fact $L_1 - y$ and $L_2 - z$ both contain the origin, and the sum of their dimensions is $n-1$. Hence they are contained in a plane. If u is not parallel to this plane, $L_1 - y$ and $L_2 - z + u$ are disjoint. With u so chosen

$$K_1 \cap K_2 \cap K_3 \cap K_4 \subset (L_1 - y) \cap (L_2 - z + u) = \emptyset .$$

Hence we have shown that K_1, K_2, K_3, and K_4 pairwise meet but have void intersection. This proves $I(K) \leqq 4$. Since $I(K) > 3$ we get $I(K) = 4$. This proves Theorem 4.1.

5. In this section we shall prove that the condition given in B) of the Main Theorem is sufficient for a convex body K to have $I(K) > 3$. This is done by proving the following theorem.

THEOREM 5.1. *If K is a centrally symmetric n-dimensional polyhedron with $I(K) = 3$, then there are two disjoint faces L_1 and L_2 of K such that there is no pair of opposite supporting planes Π_1 and Π_2 with $L_1 \subset \Pi_1$ and $L_2 \subset \Pi_2$.*

We need a lemma.

LEMMA 5.2. *Let M_1, M_2, and M_3 be three compact convex sets in Euclidean n-space and suppose that* $\dim M_1 = \dim M_2 = n$. *If M_1, M_2, and M_3 pairwise meet but have void intersection, there is a translate $M_1 + u$ of M_1 such that also $M_1 + u$, M_2, and M_3 pairwise meet and have void intersection and that, furthermore, $M_1 + u$ and M_2 have no common interior point.*

PROOF. Take a point y in $M_1 \cap M_3$. Since $M_1 \cap M_2 \cap M_3$ is void, we can separate M_2 from y by a plane Π_1. Choose the coordinate system such that Π_1 has the equation $vx = 0$ and such that $vy > 0$. Then if Π_1^- denotes the open half-space $vx < 0$, we have $M_2 \subset \Pi_1^-$. Similarly, let Π_1^+ denote the open half-space $vx > 0$, which contains y.

Now we move M_1 by adding a vector $u=u(t)$, $t \geqq 0$, which is continuous in the variable t. During this motion we want to have

$$(1) \qquad y \in M_1+u\ ,$$

so that $M_3 \cap (M_1+u) \neq \emptyset$, and we want to have

$$(2) \qquad (M_1+u) \cap M_2 \cap M_3 = \emptyset\ .$$

We are going to choose $u(t)$ such that $u(0)=0$, and such that for some t we have $(M_1+u(t)) \cap M_2 = \emptyset$. Then, since $u(t)$ is continuous, there will be a t such that $M_1+u(t)$ and M_2 meet but have no common interior point, and the lemma will be proved. Starting with $u(t)=0$ we define $u(t)$ successively for the intervals $k \leqq t \leqq k+1$, $k=0, 1, \ldots$, until we reach an integer k such that $M_1+u(k)$ and M_2 are disjoint. For each k let $u(t)$ vary linearly from $u(k)$ to a suitably chosen $u(k+1)$. We require

$$(3) \qquad v\big(u(k+1)-u(k)\big) \geqq vy\ .$$

If (3) is satisfied for all k we have, since $u(0)=0$,

$$vu(k) \geqq k(vy)\ .$$

Hence, since M_1 and M_2 are bounded and $vy>0$, $M_1+u(k)$ and M must be disjoint for sufficiently large k.

Therefore, let $u(k)$ be given such that $u=u(k)$ satisfies (1) and (2) and

$$(4) \qquad \big(M_1+u(k)\big) \cap M_2 \neq \emptyset\ .$$

Then, in order to complete the proof of the lemma, we have to find $u(k+1)$ such that (3) is satisfied and that (1) and (2) are satisfied for all u on the segment from $u(k)$ to $u(k+1)$.

Because of (2) for $u=u(k)$, we may separate $M_2 \cap M_3$ from $M_1+u(k)$ by a plane Π_2, which we choose not parallel to Π_1. Denote by Π_2^+ the open half-space bounded by Π_2 containing $M_1+u(k)$ and by Π_2^- the open half-space bounded by Π_2 containing $M_2 \cap M_3$.

Let us now turn a plane Π around $\Pi_1 \cap \Pi_2$. Starting with the position Π_2, the plane Π shall move so that one of the two half-planes in which $\Pi_1 \cap \Pi_2$ divides Π is contained in $\Pi_1^+ \cap \Pi_2^-$, where it meets neither $M_1+u(k)$ nor $M_2 \cap M_3$. Then the other half-plane is contained in $\Pi_1^- \cap \Pi_2^+$ and does not meet $M_2 \cap M_3$. Now $M_2 \subset \Pi_1^-$, and $M_1+u(k) \subset \Pi_2^+$. Hence, because of (4), there are points of $M_1+u(k)$ in $\Pi_1^- \cap \Pi_2^+$. We turn Π until the half-plane of Π in $\Pi_1^- \cap \Pi_2^+$ meets $M_1+u(k)$. Then we get a supporting plane Π_3 of $M_1+u(k)$ separating the interior of $M_1+u(k)$ from $M_2 \cap M_3$. We can take a point z in $\Pi_3 \cap \big(M_1+u(k)\big) \cap \Pi_1^-$.

Now put $u(k+1)=u(k)+y-z$. Then, since $z \in \Pi_1^-$, we have

$$v(u(k+1)-u(k)) = v(y-z) > vy ,$$

which proves (3). Since both y and z lie in $M_1+u(k)$, we have for each λ, $0 \leqq \lambda \leqq 1$,

$$(1-\lambda)y+\lambda z \in M_1+u(k) .$$

Hence

$$y \in M_1+u(k)+\lambda(y-z) ,$$

proving (1) for all points u on the segment from $u(k)$ to $u(k+1)$. Finally, since Π_3 is a supporting plane of $M_1+u(k)$ and since $z \in \Pi_3$ and $y \in M_1+u(k)$, the two bodies $M_1+u(k)$ and $M_1+u(k)+\lambda(y-z)$ for $\lambda>0$ are contained in the same closed half-space bounded by Π_3. But $M_2 \cap M_3$ is contained in the complementary open half-space. This proves (2) for all u on the segment from $u(k)$ to $u(k+1)$.

This completes the proof of the lemma.

Proof of Theorem 5.1. Since $I(K)=3$ we can take three translates K_1, K_2, and K_3 of K pairwise meeting but with void intersection.

Assume that K_1 and K_2 have no common interior point. Let y be the midpoint of the segment connecting the centers of K_1 and K_2. Since K_1 and K_2 are translates of each other and centrally symmetric, K_1 is symmetric to K_2 with respect to y. Hence the intersection $K_1 \cap K_2$ is symmetric with respect to y, and since $K_1 \cap K_2$ is convex and non-void, $y \in K_1 \cap K_2$. Hence, since K_1 and K_2 have no interior point in common, y is a boundary point of both K_1 and K_2.

Let M_1 and M_2 be the faces of K_1 and K_2 respectively containing y as an interior point. By symmetry a supporting plane Π of K_1 through y intersecting K_1 in M_1 is a supporting plane of K_2 intersecting K_2 in M_2. The plane Π separates the centers of K_1 and K_2 hence also the interiors of K_1 and K_2. Therefore

$$K_1 \cap K_2 = M_1 \cap M_2 \subset \Pi .$$

M_1 and M_2 are symmetric to each other with respect to y, and $y \in M_1 \cap M_2$. Hence M_1 and M_2 have the same dimension, say m, and they both determine the same m-dimensional linear manifold Λ.

Now among all possible K_1, K_2, and K_3 pairwise meeting and having void intersection let us take K_1, K_2, and K_3 such that $\dim(K_1 \cap K_2)$ is as low as possible. Then, because of Lemma 5.2, K_1 and K_2 will have no common interior point, and we have with the notations above

$$m = \dim(M_1 \cap M_2) = \dim(K_1 \cap K_2) .$$

Let us prove that if m is minimal, then $K_3 \cap \Lambda = \emptyset$.

In fact, put $M_3 = K_3 \cap \Lambda$ and suppose that M_3 is non-void. There will be two cases to consider:

a) M_3 meets both M_1 and M_2,

b) M_3 is disjoint to at least one of M_1 and M_2, say M_2.

In both cases we shall get a contradiction by showing that we can move K_1 by adding a vector u parallel to Λ such that K_1+u meets K_2 and K_3 but not $K_2 \cap K_3$ and such that $\dim\big((K_1+u)\cap K_2\big) < m$.

When u is parallel to Λ, it is also parallel to the plane Π. Hence Π separates the interiors of K_1+u and K_2, so that $(K_1+u)\cap K_2$ is contained in Π. Since $K_2 \cap \Pi = M_2 \subset \Lambda$, we get

$$(K_1+u) \cap K_2 \subset \Lambda \,. \tag{5}$$

a) M_3 meets M_1 and M_2. Since M_1, M_2, and M_3 are subsets of K_1, K_2, and K_3 respectively, their intersection is void. Hence, since M_1 and M_2 have in Λ the point y as common interior point, we can apply Lemma 5.2 to M_1, M_2, and M_3 in Λ. We get a vector u parallel to Λ such that M_1+u meets M_2 and M_3 but not $M_2 \cap M_3$, and that

$$\dim\big((M_1+u) \cap M_2\big) < m \,.$$

Then K_1+u meets K_2 and K_3. Since u is parallel to Λ, we get from (5) that

$$(K_1+u) \cap K_2 \;=\; \big((K_1+u) \cap \Lambda\big) \cap (K_2 \cap \Lambda) \;=\; (M_1+u) \cap M_2 \,,$$

so that

$$\dim\big((K_1+u) \cap K_2\big) < m \,. \tag{6}$$

Finally, because of (5),

$$\begin{aligned}(K_1+u) \cap K_2 \cap K_3 &= \big((K_1+u) \cap \Lambda\big) \cap (K_2 \cap \Lambda) \cap (K_3 \cap \Lambda) \\ &= (M_1+u) \cap M_2 \cap M_3 \;=\; \emptyset \,.\end{aligned} \tag{7}$$

b) M_3 is disjoint to M_2. In Λ separate M_2 and M_3 by an $(m-1)$-dimensional linear manifold Λ'. It divides Λ into two parts; the part of Λ (taken closed) containing M_3 is called Λ^+. Let y' be an arbitrary point in M_3. Since M_2 is disjoint to Λ^+ but meets M_1, there are points of M_1 outside of Λ^+. Let z be such a point with a maximal distance to Λ'. Then $M_1 + (y'-z)$ is contained in Λ^+ and it is therefore disjoint to M_2. Put $u = \lambda(y'-z)$, where $0<\lambda<1$. Then we can take such a λ that M_1+u and M_2 meet but have no common interior point. Then K_1+u meets K_2 and, as in case a), we get (6) and (7). It remains to be proved that K_1+u meets K_3. Let z' be any point in $K_1 \cap K_3$. Then K_1 contains z and z' and therefore also $\lambda z + (1-\lambda)z'$, and K_3 contains y' and z' and therefore also $\lambda y' + (1-\lambda)z'$. Hence K_1+u and K_3 both contain the point

$$\lambda z + (1-\lambda)z' + u = \lambda y' + (1-\lambda)z' .$$

This proves the case b).

From now on let m be minimal so that we have $K_3 \cap \Lambda = \emptyset$. Hence, since M_1 is a face of K_1 and therefore $\dim M_1 \leqq n-1$, we are now able to apply Lemma 2.3 to M_1 and K_3. We get two distinct parallel planes Π_1' and Π_2' such that

a) $M_1 \subset \Pi_1'$,

b) $K_3' = K_3 \cap \Pi_2' \neq \emptyset$,

c) the open half-space bounded by Π_2' containing Π_1' and M_1 contains no point of K_3,

d) there is one and only one pair of parallel planes containing M_1 and K_3', namely Π_1' and Π_2'.

By b) and c) we get that Π_2' is a supporting plane of K_3. Hence K_3' is a face of K_3. Let us prove that Π_1' is not a supporting plane of K_1. In fact Π_1' contains M_1 by a) and therefore the point y. Hence, if Π_1' is a supporting plane of K_1, it is also a supporting plane of K_2 and separates the interiors of K_1 and K_2. But this is impossible. For by c) K_3 lies in one of the open half-spaces bounded by Π_1', and since K_3 meets both K_1 and K_2, the bodies K_1 and K_2 both have points in the same open half-space bounded by Π_1'. Hence their interiors cannot be separated by Π_1'. Thus Π_1' is not a supporting plane of K_1.

Finally suppose that we have $K_1 = K + u_1$ and $K_3 = K + u_3$. We put

$$L_1 = M_1 - u_1 \quad \text{and} \quad L_2 = K_3' - u_3 .$$

I claim that L_1 and L_2 are disjoint faces of K such that there is no pair of opposite supporting planes Π_1 and Π_2 with $L_1 \subset \Pi_1$ and $L_2 \subset \Pi_2$.

Since M_1 is a face of K_1 and since K_3' is a face of K_3, we see that L_1 and L_2 are faces of K. Since L_1 is a translate of M_1 and L_2 a translate of K_3', we get from d) that there is only one pair of parallel planes containing L_1 and L_2. These planes are $\Pi_1' - u_1$ and $\Pi_2' - u_3$. Since Π_1' is not a supporting plane of $K + u_1$, the plane $\Pi_1' - u_1$ is not a supporting plane of K, and since Π_2' is a supporting plane of K_3, the plane $\Pi_2' - u_3$ is a supporting plane of K. Thus we see that $\Pi_1' - u_1$ and $\Pi_2' - u_3$ are distinct planes. Since they are parallel and contain L_1 and L_2 respectively, L_1 and L_2 are disjoint. Since $\Pi_1' - u_1$ and $\Pi_2' - u_3$ are the only pair of parallel planes containing L_1 and L_2 respectively, and $\Pi_1' - u_1$ is not a supporting plane of K, Theorem 5.1 is proved.

We have now completely proved the Main Theorem.

6. In this section we shall change our main problem. Instead of translates of a convex body K we shall consider bodies $\lambda K+u$ homothetic with K, where $\lambda>0$. Consider numbers $\lambda_1, \ldots, \lambda_m>0$ and vectors $u_1, \ldots, u_m$. Let us have

$$(1) \qquad (\lambda_k K+u_k) \cap (\lambda_l K+u_l) \neq \emptyset ,$$

$$(2) \qquad \bigcap_{k=1}^{m} (\lambda_k K+u_k) = \emptyset .$$

DEFINITION. For a convex body K let $I'(K)$ denote the smallest integer m such that there exist m numbers $\lambda_1, \ldots, \lambda_m>0$ and m vectors $u_1, \ldots, u_m$ satisfying (1) and (2). If there exists no such integer m let $I'(K)=\infty$.

THEOREM 6.1. *For any convex body K we have $I(K)=I'(K)$.*

PROOF. Since we can choose all $\lambda_k=1$, we have $I'(K)\leqq I(K)$. Therefore we have to show that if $I'(K)=m$ with m finite, then $I(K)\leqq m$.

Let λ_k and u_k be chosen satisfying (1) and (2), where $1\leqq k\leqq m$ and $m=I'(K)$. Let λ_0 be a number for which $\lambda_k\leqq\lambda_0$ for all k. I claim that for fixed k we can change λ_k and u_k in such a way that (1) and (2) are still satisfied and the new λ_k is equal to λ_0. When this is proved, we complete the proof of the theorem in the following way. We change successively for each $k=1, \ldots, m$ the number λ_k and the vector u_k, so that we get $\lambda_k=\lambda_0$, and keep each time all the remaining λ_l and u_l fixed. After m steps we have $\lambda_k=\lambda_0$ for all k. But then the sets $\lambda_0 K+u_k$ show that $I(\lambda_0 K)\leqq m$. Since obviously $I(\lambda_0 K)=I(K)$, we have $I(K)\leqq m$.

Therefore we only have to show that we can change λ_k and u_k as described above. Put

$$K_k = \lambda_k K+u_k \quad \text{and} \quad L = \bigcap_{l\neq k} (\lambda_l K+u_l) .$$

Since $I'(K)=m$, we must have $L\neq\emptyset$. By (2), K_k and L are disjoint, and we can separate them by a plane Π. Let z be a point in K_k with minimal distance to Π. We consider the new body

$$(3) \qquad K_k' = \lambda_0 K+\frac{\lambda_0}{\lambda_k}u_k+\left(1-\frac{\lambda_0}{\lambda_k}\right)z ,$$

which is of the form $\lambda_0 K+u$. We have

$$K_k'-z = \frac{\lambda_0}{\lambda_k}(K_k-z) ,$$

so that K_k' is obtained from K_k by an enlarging homothety keeping z

in K_k fixed. Hence K_k' contains K_k, and because of the choice of z, the body K_k' does not meet Π, so that Π separates K_k' from L. Thus K_k' meets all the sets $\lambda_l K + u_l$, $l \neq k$, but not their intersection.

Hence, if we replace $\lambda_k K + u_k$ by the set K_k', we still have (1) and (2) satisfied, and we can complete the proof of the theorem in the way described above.

7. To an n-dimensional convex body K with the origin as interior point there corresponds the dual convex body K^*. It is defined to be the set of all points $u = (u_1, \ldots, u_n)$ such that $ux \leqq 1$ for all $x \in K$. If K is symmetric with respect to the origin, so is K^*. If K is a polyhedron, K^* is also a polyhedron, and if K is defined by a finite number of inequalitites $u_k x \leqq 1$, K^* is the convex hull of the finite number of points u_k given in these inequalities.

In this section we are going to give examples of convex bodies K with $I(K) > 3$. By the Main Theorem, these K have to be centrally symmetric polyhedra. Let us make the convention always to place K with the origin as its centre of symmetry.

Theorem 7.1. *$I(K) > 3$ implies $I(K^*) > 3$.*

Proof. Since K is a centrally symmetric polyhedron with the origin as its centre of symmetry, the same facts hold for K^*. Thus of the necessary and sufficient condition for $I(K) > 3$ given in B) of the Main Theorem the two parts 1) and 2) are true for K^* when they are true for K. To prove our theorem it will be sufficient to show that also 3) is true for K^* when it is true for K. The dual of 3), however, is as follows: If L_1^* and L_2^* are any two faces of K^* not together contained in a supporting plane, there are two opposite boundary points u_1 and u_2 of K^* such that $u_1 \in L_1^*$ and $u_2 \in L_2^*$. Let L_1 and L_2 be two disjoint faces of K^*. Take the face L_2' opposite to L_2. Since L_1 and L_2 are disjoint, there do not exist two opposite boundary points u_1 and u_2 such that $u_1 \in L_1$ and $u_2 \in L_2'$. Hence, by the dual of 3), L_1 and L_2' must together lie in a supporting plane, so that L_1 and L_2 are contained respectively in two opposite supporting planes. This shows that 3) holds for K^* and proves Theorem 7.1.

Let $n = n_1 + n_2$ and let the n-dimensional Euclidean space E be equal to $E_1 + E_2$, where the two E_i, $i = 1, 2$, are orthogonal linear manifolds through the origin of E, and $\dim E_i = n_i$. Let K_i be an n_i-dimensional polyhedron in E_i.

Theorem 7.2. *$I(K_1 + K_2) = \min\big(I(K_1), I(K_2)\big)$.*

PROOF. Put $K=K_1+K_2$ and let $f_i(x)$ be the projection of a point $x\in E$ into E_i. Then, since $f_i(K)=K_i$, we have

$$f_i(K+u) = K_i+f_i(u)\,,$$
$$\big(K_1+f_1(u)\big)+\big(K_2+f_2(u)\big) = K+u\,.$$

Thus $K+u$ is determined by its two projections. Let us prove:

a) *A system of translates*

$$K+u_1,\ \ldots,\ K+u_m \tag{1}$$

with $m\geqq 2$ *has non-void intersection if and only if each of the two systems* (2) *and* (3) *of projected sets*

$$K_1+f_1(u_1),\ \ldots,\ K_1+f_1(u_m)\,, \tag{2}$$
$$K_2+f_2(u_1),\ \ldots,\ K_2+f_2(u_m) \tag{3}$$

has non-void intersection.

In fact

$$f_i\Big(\bigcap_k (K+u_k)\Big) \subset \bigcap_k f_i(K+u_k) = \bigcap_k \big(K_i+f_i(u_k)\big)\,.$$

Hence if (1) has non-void intersection so has each of (2) and (3). Conversely, if x_1 is contained in all sets (2) and x_2 in all sets (3), x_1+x_2 is contained in all sets (1).

Now, if $I(K)=m$ is finite and the sets (1) pairwise meet but have void intersection, we get from a) that both systems of sets (2) and (3) pairwise meet and at least one of them has void intersection. Hence

$$I(K) \geqq \min\big(I(K_1), I(K_2)\big)\,,$$

which is true even if $I(K)=\infty$.

On the other hand, if for instance $I(K_1)\leqq I(K_2)$ with $m=I(K_1)$ finite, take $u_k=f_1(u_k)$ such that the sets (2) pairwise meet but have void intersection. Then $f_2(u_k)=0$. Hence by a) the sets (1) pairwise meet but have void intersection. Thus

$$I(K) \leqq \min\big(I(K_1), I(K_2)\big)\,,$$

which is true even if $I(K_1)=I(K_2)=\infty$. This proves Theorem 7.2.

Let now each K_i be a centrally symmetric polyhedron in E_i with the origin as its centre of symmetry.

THEOREM 7.3. *If* $K=K_1+K_2$ *then* K^* *is the convex hull* $H(K_1^*\cup K_2^*)$ *of* K_1^* *and* K_2^*, *where* K_i^* *is the dual polyhedron of* K_i *in* E_i.

Proof. Let K_1 be defined by some inequalities

$$-1 \leqq u_1x_1 + \ldots + u_{n_1}x_{n_1} \leqq 1 , \tag{4}$$

and let K_2 be defined by some inequalities

$$-1 \leqq v_{n_1+1}x_{n_1+1} + \ldots + v_nx_n \leqq 1 . \tag{5}$$

K consists of all points $(x_1, \ldots, x_n)$ satisfying all inequalities (4) and (5). Therefore K^* is the convex hull of the points

$$\pm (u_1, \ldots, u_{n_1}, 0, \ldots, 0) \tag{6}$$

and the points

$$\pm (0, \ldots, 0, v_{n_1+1}, \ldots, v_n) . \tag{7}$$

But the convex hull of the points (6) is $K_1{}^*$ and the convex hull of the points (7) is $K_2{}^*$. Hence the theorem is proved.

With the same notations as in Theorem 7.2 we get

Theorem 7.4. *If $I(K_1) > 3$ and $I(K_2) > 3$, then $I(K_1 + K_2) > 3$ and $I\big(H(K_1 \cup K_2)\big) > 3$, where $H(K_1 \cup K_2)$ denotes the convex hull of K_1 and K_2.*

Proof. That $I(K_1 + K_2) > 3$, follows from Theorem 7.2. Because of Theorem 7.1, $I(K_1{}^*) > 3$ and $I(K_2{}^*) > 3$. Hence $I(K_1{}^* + K_2{}^*) > 3$. Since by Theorem 7.3

$$(K_1{}^* + K_2{}^*)^* = H(K_1{}^{**} \cup K_2{}^{**}) = H(K_1 \cup K_2) ,$$

we get by Theorem 7.1 that $I\big(H(K_1 \cup K_2)\big) > 3$. Thus the theorem is proved.

Theorem 7.4 gives a general method of getting polyhedra K with $I(K) > 3$. We start with the 1-dimensional convex polyhedron, the interval, which has $I(K) = \infty$, and get successively for $n = 2, 3, \ldots$ new polyhedra with $I(K) > 3$ by taking as K_1 and K_2 in Theorem 7.4 the polyhedra already obtained.

For $n = 2$ we get only one polyhedron, the square, which has $I(K) = \infty$. The square can be given by either $\max(|x_1|, |x_2|) \leqq 1$ or $|x_1| + |x_2| \leqq 1$.

For $n = 3$ we get two polyhedra, the cube and the octahedron. The cube has $I(K) = \infty$ and can be given by $\max(|x_1|, |x_2|, |x_3|) \leqq 1$. The octahedron, which by Theorem 7.4 has $I(K) > 3$ but which is no parallelepiped, has $I(K) = 4$ by the Main Theorem. It can be given by $|x_1| + |x_2| + |x_3| \leqq 1$.

For $n = 4$ we get 4 polyhedra. They can be given by:

$$\max(|x_1|, |x_2|, |x_3|, |x_4|) \leqq 1 ,$$
$$\max(|x_1|, |x_2|, |x_3|) + |x_4| \leqq 1 ,$$
$$\max(|x_1| + |x_2| + |x_3|, |x_4|) \leqq 1 ,$$
$$|x_1| + |x_2| + |x_3| + |x_4| \leqq 1 .$$

Here the first one is the 4-dimensional cube, which has $I(K) = \infty$. The remaining three polyhedra have $I(K) = 4$.

For $n = 5$ we get in this way 8 polyhedra, and for $n = 6$ we get 18 polyhedra. We can continue in this way and get polyhedra with arbitrary dimension. All these polyhedra can be given as above by combining the two procedures of taking the sum and the maximum of the numbers $|x_i|$.

All polyhedra which we get in this way and those that are affinely equivalent to them have $I(K) > 3$. Whether there are other polyhedra with $I(K) > 3$ is an open question; however for $n \leqq 5$ there is no other such polyhedron. This last result has been obtained by simple computation (cf. Remark 3.5).

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