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PROBLEMS ON CONVEX BODIES

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The theory of Minkowski spaces leads to various problems concerning the plane sections through the center of a symmetric convex body. Many of these problems share with questions in number theory the great appeal of being understandable to a layman (at least in three dimensions) without being easy. Contributions to these problems would not only *advance the theory of Minkowski spaces, but lead the way to a new direction of research on convex bodies.*

This note lists some of the problems, stating what is known about them (if anything), and interpreting them in terms of Minkowskian geometry. These interpretations presuppose familiarity with the concepts created by one of the authors in [4] and may be omitted by readers interested only in convex bodies.

The following notations will be used throughout. We consider an n-dimensional euclidean space E^n, $n \geqq 3$, with rectangular coordinates $x_1, \ldots, x_n$ whose origin is denoted by z. The symbol u will always stand for a unit vector with origin z and components $u_1, \ldots, u_n$. The end point of u traverses the unit sphere Ω. If $f(u)$ is positive and defined on Ω the "surface $uf(u)$" is given by the equations

$$x_i = u_i f(u_1, \ldots, u_n), \qquad u \in \Omega .$$

The hyperplane $\Sigma x_i u_i = \Sigma x_i(-u_i) = 0$ is denoted by $H(u)$ or $H(-u)$. K is a convex body with z as interior point and *center*; V is the volume of K. The intersection of K and $H(u)$ is $K(u)$ and $A(u)$ is the $(n-1)$-dimensional measure or area of $K(u)$.

The first problem refers to two such convex bodies K and K' with center z. Let V' and $A'(u)$ be defined for K' as V and $A(u)$ for K.

PROBLEM 1: *Does $A'(u) \leqq A(u)$ for all $u \in \Omega$ imply $V' \leqq V$?*

The answer is affirmative if K' is an ellipsoid because, see [5],

$$(1) \qquad V^{n-1} \geqq \varkappa_n{}^{n-2}\, \varkappa_{n-1}{}^{-n}\, n^{-1} \int_\Omega A(u)^n \, d\omega_u$$

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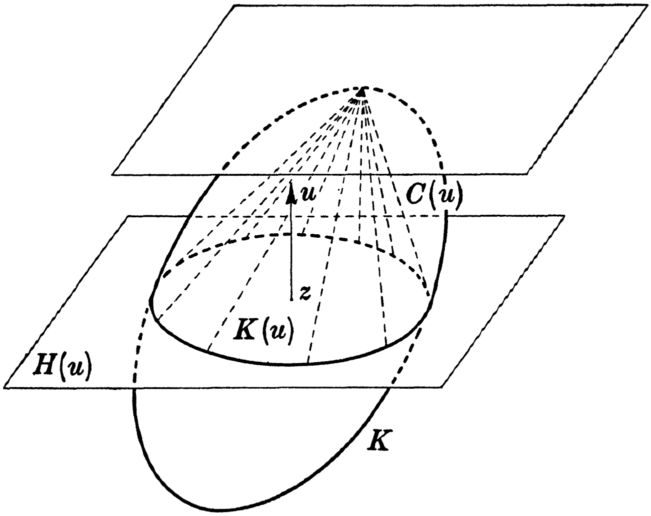

with equality only when K is an ellipsoid. Here $\varkappa_\nu = \pi^{\nu/2}/\Gamma(\nu/2+1)$ (equal to the volume of the solid unit sphere in E^ν) and $d\omega_u$ is the area element of Ω at u. The only other known special case is when $A'(u) = cA(u)$ for all $u \in \Omega$. It then follows from Funk's spherical integration theorem, see [2, pp. 136–138], that K' and K are homothetic.

This problem is equivalent to the following question on area in Minkowski spaces: Let $F(x-y)$, $F'(x-y)$ define two Minkowski metrics in E^n (i.e. $F(x)$ satisfies the conditions:

$$F(x) > 0 \text{ for } x \neq (0, \ldots, 0), \quad F(\mu x) = |\mu| F(x), \quad F(x) \text{ is convex;}$$

similarly for $F'(x)$, compare [6, p. 100]). If for some r, $2 \leqq r \leqq n-1$, the area of any (sufficiently smooth) r-dimensional surface with respect to the metric F does not exceed its area with respect to F', does then the same hold for s-dimensional surfaces with $s > r$? The answer is, in general, negative for $s < r$, even when both metrics are euclidean, i.e. K and K' are both ellipsoids, and is trivially affirmative if $r = 1$, because then $K' \subset K$.

The inequality (1) states that the ellipsoids maximize $V^{1-n} \int_\Omega A(u)^n \, d\omega_u$. This suggests

Problem 2: *To find a convex body K* (with center z) *which minimizes* $V^{1-n} \int_\Omega A(u)^n \, d\omega_u$.

The greatest lower bound of this expression is positive (see [5, p. 11]) and the existence of K can be deduced by an affine form of Blaschke's Selection theorem.

It is doubtful whether the solution of this problem would have interesting applications in Minkowskian geometry (unless the answer is an interesting body). However the following modification is important:

PROBLEM 3: *To find an estimate from above for V in terms of $A(u)$ such that the equality sign characterizes the ellipsoids.*

Experimenting with this problem led the authors to conjecture that such an estimate exists but involves the partial derivatives of

$$A(u) = A(u_1, \ldots, u_n) .$$

Any characterization of the ellipsoids is significant for Minkowskian geometry because it means a characterization of the euclidean geometry. A solution of Problem 3 would, most likely, in addition give important information about the solutions of the isoperimetric problem, which are the surfaces homothetic to the polar reciprocal of the surface $uA(u)$ with respect to Ω. These surfaces are, in general, not homothetic to the spheres (i.e. to the boundary of K), see the comments after Problem 5. However, they play in many Minkowskian theorems the role of spheres in the analogous euclidean theorems, see [4].

The inequality (1) implies further that

$$V^{n-1} \geqq \varkappa_n{}^{n-2} \varkappa_{n-1}{}^{-n} n^{-1} \min_u A(u)^n$$

with equality only for spheres, which therefore yield

$$\max_K \min_u A(u)^n V^{1-n} .$$

This leads to

PROBLEM 4: *Which K yields* $\min_K \max_u A(u)^n V^{1-n}$?

(The corresponding minmin and maxmax problems obviously have no solution.)

An affirmative answer to Problem 1 in the case where K is a sphere would solve Problem 4. For putting

$$r^{n-1} \varkappa_{n-1} = \max_u A'(u) ,$$

we would have

$$\varkappa_{n-1}{}^n \varkappa_n{}^{1-n} \leqq \max A'(u)^n (V')^{1-n}$$

for all K' so that Problem 4 would be solved by spheres. Its significance for Minkowskian geometry concerns again the solutions of the isoperimetric problem. The answer to Problem 4 would in particular decide the question whether the so-called isoperimetrix, i.e. the solution of the isoperimetric problem with center z and Minkowski area equal to n times its Minkowskian volume, can be properly contained in the unit sphere K. Problem 6, which is related to Problem 4, would also settle this question.

For a given u we construct the cone of maximal volume $C(u)$ with base $K(u)$ and apex in K. The apex of such a cone is any point of K on a supporting plane of K parallel to $H(u)$.

PROBLEM 5: *Are the ellipsoids characterized by the property that $C(u)$ is constant* (i.e. independent of u for fixed K)?

This problem is of particular interest owing to its connection with certain well known results of Radon [8] and Blaschke [1]. Radon discovered in [8] that symmetry of perpendicularity for lines in a Minkowski plane does not imply that the metric is euclidean. The plane geometries with symmetric perpendicularity are exactly those with unit circles for which the triangles, corresponding to $C(u)$, have constant area, see [4] and [6, p. 104]. Blaschke [1], compare also [6, p. 103], showed that symmetry of perpendicularity for lines in a Minkowski space of dimension $n \geqq 3$ implies that the metric is euclidean. This seemed to indicate that there is no analogue to Radon's curves in higher dimensions. However, in [4] it is shown that there is a natural way of defining perpendicularity of a line to a hyperplane and of a hyperplane to a line. The K with constant $C(u)$ are exactly those, which as unit spheres of Minkowskian geometries lead to the case where the two pairings between lines and hyperplanes are identical. Moreover, the surface $uC(u)^{-1}$ is a solution of the isoperimetric problem, so that Problem 5 also inquires whether in higher dimensional Minkowskian geometries, other than the euclidean, the solutions of the isoperimetric problem can be homothetic to the spheres.

PROBLEM 6: *To find a convex body for which $V^{-1} \max_u C(u)$ is minimal.*

The corresponding minmin and maxmax problems are simple and are solved by cylinders and double cones. The maxmin problem is also unsolved for higher dimensions, but in the plane it is solved by regular affine hexagons, see [7].

It is shown in [7] that the ellipse solves Problem 6 in two dimensions, so that the ellipsoid may be conjectured as answer to this problem. In terms of Minkowskian geometry the problem means: for each Minkowskian geometry we consider the maximum of the Minkowski sine between line and hyperplane, find the Minkowskian geometries for which this maximum is minimal.

PROBLEM 7: *Find a convex body K for which the integral $\int_{\partial K} A(u)^{-1}\, ds_u$*

is minimal, where ds_u is the area element of the boundary ∂K of K at a point of contact of a tangent plane parallel to $H(u)$.

For 2 dimensions, the anwer is a regular affine hexagon, see [7]. *The corresponding maximum problem is solved by the parallelepiped.* For a sketch of the proof, let $E(u)$ be the supporting function of K. Then, since $A(u)$ is maximal among the areas of parallel sections of K normal to u, we have

$$\int_{\partial K} A(u)^{-1}\, ds_u = \int_{\partial K} E(u)\,[E(u)A(u)]^{-1}\, ds_u \leqq 2V^{-1} \int_{\partial K} E(u)\, ds_u = 2n\,.$$

To find the condition on K for equality, we observe that there can be at most $2n$ distinct tangent planes to K for which $E(u)A(u) = V/2$. But since this condition must hold for almost all points of ∂K, K is a polyhedron with at most $2n$ faces. However, all polyhedra with center have at least $2n$ faces and the only polyhedron with center and $2n$ faces is a parallelepiped.

The Minkowskian interpretation of Problem 7 is to find a Minkowski metric for which the area of the unit sphere is minimal.

The difficulty in all these extremal problems derives from the fact that none of the symmetrization processes can be applied, because they do not transform plane sections into plane sections.

PROBLEM 8: *Are the ellipsoids characterized by the fact that the Gauss curvature at a point of contact with a tangent plane parallel to $H(u)$ is proportional to $A(u)^{-(n+1)}$?*

The answer is affirmative for two dimensions. The Minkowskian interpretation involves the Minkowski curvature of surfaces and asks: is the geometry euclidean if the Minkowski spheres have constant curvature?

For a given point p on the boundary ∂K of K, determine u such that the cone with apex p and base $K(u)$ has maximal volume (in general p is not an apex of $C(u)$). Call $P(u)$ the $(n-1)$-dimensional area of the projection of K on $H(u)$ parallel to the line pz. Clearly $P(u) \geqq A(u)$.

PROBLEM 9: *Can the ratio $P(u)/A(u)$ be constant for K that are not ellipsoids?*

If the value of this constant is one, the answer to Problem 9 is negative, because it is then merely a reformulation of Blaschke's theorem, see [1] and [6, p. 93], that the ellipsoids are the only convex bodies with plane shadow boundaries.

In the language of Minkowskian geometry with K as unit sphere, the plane $H(u)$ is normal to the line pz or pz is transversal to $H(u)$. The ratio $\varkappa_{n-1}P(u)/A(u)$ is the area of the tranversal projection of K on $H(u)$, hence the analogue to the "äussere Quermass" of K in the euclidean case, see [2, pp. 30–31]. Problem 9 inquires whether in a Minkowskian geometry, other than the euclidean, a sphere can have constant "äusseres Quermass". In Minkowski spaces, whose spheres solve Problem 9, there is an analogue to Cauchy's formula for area, see [2, p. 48], and hence to its implications, in particular in integral geometry.

The last problem is of somewhat different nature. It is proved in [3] that the surface $uA(u)$ is convex. Calling an r-dimensional linear subspace of E^n briefly an r-flat, our problem is this:

PROBLEM 10: *If $n \geqq 4$ and $2 \leqq r < n-1$, does a result analogous to the convexity of the surface $uA(u)$ hold for the sections of K by r-flats through z?*

Here it is not even quite clear what the term "analogous" may mean, because the very formulation of convexity presupposes an underlying linear range, whereas the r-flats through z fail to form a linear variety, if $2 \leqq r < n-1$. However, the following suggests itself: The convexity of the surface $uA(u)$ may be reformulated by introducing for an arbitrary vector V with origin z the function

$$F(v) = |v| A(v/|v|)^{-1} \quad \text{if} \quad v \neq (0, \ldots, 0), \quad F(0) = 0 .$$

Then $F(v)$ is a convex function of v. Since $F(v)$ is positive homogeneous of degree 1, the inequality

$$F(v) \leqq \sum_{i=1}^{k} \lambda_i F(v^i) \quad \text{for} \quad v = \sum_{i=1}^{k} \lambda_i v^i, \quad \lambda_i \geqq 0 ,$$

is equivalent to the convexity of $F(v)$.

Introduce Plücker coordinates $p = (p_1, \ldots, p_N)$ for the r-flats in E^n (since we only consider r-flats through z, we may also use Plücker coordinates for the $(r-1)$-flats in $(n-1)$-dimensional projective space). If

$$|p| = \left(\sum_{j=1}^{N} p_j^2\right)^{\frac{1}{2}} = 1 ,$$

denote by $B(p)$ the r-dimensional area of the intersection of K with the r-flat p through z. If q represents any r-flat through z put

$$F(q) = |q| \, B(q/|q|)^{-1} \quad \text{and} \quad F(0) = 0 .$$

The following appears then as a natural analogue to the convexity of the surface $uA(u)$:

If q, and $q^1, \ldots, q^k$ represent r-flats through z and

$$q = \sum_{i=1}^{k} \lambda_i q^i, \qquad \lambda_i \geqq 0 ,$$

then

$$F(q) \leqq \sum_{i=1}^{k} \lambda_i F(q^i) .$$

The question on Minkowski area which leads to Problem 10 is this: A straight segment is a shortest connection of its endpoints in Minkowskian geometry. Let a sufficiently smooth surface S, homeomorphic to an $(r-1)$-sphere, in an r-flat L_r be given and bound in L_r the set W homeomorphic to the solid r-dimensional sphere. The convexity of $F(v)$ implies, see [4], that the r-dimensional area of W is not greater than that of any surface W' bounded by S and homeomorphic to W, provided W' lies in an $(r+1)$-flat. (The last condition is automatically satisfied for $r = n-1$.) Problem 10 inquires whether this statement remains true without the last restriction.

REFERENCES

1. W. Blaschke, *Räumliche Variationsprobleme mit symmetrischer Transversalitätsbedingung*, Ber. Sächs. Akad. Wiss. Leipzig 68 (1916), 50–55.
2. T. Bonnesen und W. Fenchel, *Theorie der konvexen Körper*, Berlin 1934, New York 1948.
3. H. Busemann, *A theorem on convex bodies of the Brunn-Minkowski type*, Proc. Nat. Acad. Sci. U.S.A. 35 (1949), 27–31.
4. H. Busemann, *The foundations of Minkowskian geometry*, Comment. Math. Helv. 24 (1950), 156–186.
5. H. Busemann, *Volume in terms of concurrent cross-sections*, Pacific J. Math. 3 (1953), 1–12.
6. H. Busemann, *The geometry of geodesics*, New York 1955.
7. C. M. Petty, *On the geometry of the Minkowski plane*, to appear in Rivista Mat. Univ. Parma.
8. J. Radon, *Über eine besondere Art ebener konvexer Kurven*, Ber. Sächs. Akad. Wiss. Leipzig 68 (1916), 131–134.

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