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ON AN INEQUALITY FOR THE HYPERBOLIC MEASURE AND ITS APPLICATIONS IN THE THEORY OF FUNCTIONS

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Introduction. Let D be a simply or multiply connected domain with more than two boundary points in the extended z-plane (i.e. the plane including the point $z=\infty$), and let D^* be the universal covering surface of D. A classical theorem states that there is a one-one conformal mapping of D^* onto the circle $|w|<1$. This paper contributes to the general deformation theory for such mappings. In order to state the main result as simply as possible, we take as a standard domain not the unit circle but the strip S: $|v|<\frac{1}{4}\pi$ ($w=u+iv$, u and v real). Let z_1 and z_2 be finite and belong to D, and let the analytic function $z=f(w)$ map S onto D^* and take the values z_1 and z_2 for two real w-values u_1 and u_2. If D is multiply connected there is an infinity of such mappings with distinct values of the difference u_2-u_1, but as is well known, the ratio $|f'(u_1)/f'(u_2)|$ will be the same for all of them. We give bounds for this ratio in terms of the geometrical configuration in the z-plane. If F and f are the greatest and smallest values of $|(z-z_1)/(z-z_2)|$ for z on the boundary of D, our result is

$$f^2 \leqq |f'(u_1)/f'(u_2)| \leqq F^2 ,$$

the signs of equality holding only for $F=f$.

If we let the two points z_1 and z_2 coincide, we get a relation that can be used to estimate the change in curvature under the conformal mapping of a curve. It is a simple general property of conformal mapping that if a function $w=g(z)$ is regular for $z=z_0$ and $|g'(z_0)|=1$, then the curvatures of all (regular) curves tangent to a given line in $z=z_0$ increase by the same amount under the transformation $w=g(z)$. We give bounds for this amount in terms of the geometrical configuration in D.

In these introductory remarks we have referred to a standard domain in the w-plane, but in general we use instead the usual non-Euclidean geometry in D. One may say that the object of our study is the interplay

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of Euclidean and non-Euclidean measure in an arbitrary domain. In this study the partial differential equation

$$(1) \qquad \Delta u = 4e^{2u}, \quad \Delta \equiv \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}, \qquad z = x+iy$$

plays an important part; our main results are theorems on the solutions of this equation.

If the domain D is simply connected, some of the results are known. In particular we mention a theorem by J. L. Ullman [5, Theorem 2] and a related one by J. L. Walsh [6, the concluding remark]. In § 3 we give new, simple proofs of these theorems.

§ 1. The partial differential equation (1).

1.1. Transformation of the solutions. The property of (1) important for the theory of functions may be stated in this way: If the domain D_z is mapped conformally onto a domain D_w by an analytic function $z=f(w)$ and if $u(z)$ is a solution of (1) in D_z, then the function

$$u\big(f(w)\big)+\log|f'(w)|$$

is a solution of the corresponding equation in D_w. For the proof we refer to [3, p. 51]. We may use this to find the behaviour of a solution for $z\to\infty$ when $z=\infty$ is an inner point of D_z. Putting $w=z^{-1}$, we see that

$$u(w^{-1})+\log|w^{-2}|$$

defines a solution in a domain that includes the point $w=0$. This shows that we have

$$(2) \qquad u(z) = u_1(z)-2\log r\,,$$

where $u_1(z)$ tends to a limit for $|z|=r\to\infty$.

1.2. A lemma. *Let u and u_1 be two solutions of* (1), *bounded above and satisfying the inequality*

$$u > u_1$$

in the circle $r<R$. Then there is a positive constant k such that the inequality

$$u-u_1 > k(R-r)$$

holds in the circle.

Proof. Putting $u-u_1=v$, we have

$$(3) \qquad \Delta v = 4(e^{2u}-e^{2u_1}) = 4e^{2u_1}(e^{2v}-1) < Kv$$

for a suitable positive constant K.

We now introduce the elementary function

$$v_1 = c\big((R/r)^n - 1\big).$$

We are going to show that, by choosing suitable values of n and c, we can make v_1 a minorant of v in the annulus $\frac{1}{2}R \leqq r < R$. Differentiating, we find

$$\frac{dv_1}{dr} = -cnR^n r^{-n-1}$$

and

$$\Delta v_1 \equiv \frac{1}{r}\frac{d}{dr}\left(r\frac{dv_1}{dr}\right) = cn^2R^n r^{-n-2} = v_1\frac{n^2}{r^2\big(1-(r/R)^n\big)} > v_1\frac{n^2}{R^2}.$$

We now take $n > RK^{\frac{1}{2}}$ and get for $0 < r < R$

$$\Delta v_1 > Kv_1. \tag{4}$$

The positive constant c is taken so small that

$$v - v_1 > 0$$

on the circle $r = \frac{1}{2}R$. For $r \to R$ we have $\liminf(v - v_1) \geqq 0$ since v is positive and v_1 tends to zero. It follows from this that if the continuous function $v - v_1$ were to take negative values in $\frac{1}{2}R < r < R$, it would have a negative minimum. But $v - v_1$ is positive at any minimum point, for (3) and (4) give

$$K(v - v_1) > \Delta(v - v_1),$$

and at a minimum we have $\Delta(v - v_1) \geqq 0$.

Consequently the inequality $v \geqq v_1$ holds for $\frac{1}{2}R < r < R$. The statement now follows since we have $dv_1/dr < 0$ for $r = R$ and $v > 0$ for $0 \leqq r \leqq \frac{1}{2}R$.

From the Lemma we draw this conclusion: *If two non identical solutions u and u_1 of* (1) *in a domain D satisfy the inequality*

$$u \geqq u_1,$$

then the sign of equality will hold in no finite point of D.

PROOF. Let z_0 be a point in which we have $u(z_0) > u_1(z_0)$. We draw the greatest circle about z_0 in the interior of which we have $u > u_1$. This circle must pass through a boundary point of D, for otherwise there would be a zero z_1 of the difference $u - u_1$ on the periphery and, according to the Lemma, $\operatorname{grad}\big(u(z_1) - u_1(z_1)\big)$ would not be zero; but then $u - u_1$ would take negative values in the vicinity of z_1, and a contradiction is obtained. Now it follows in the usual way that any finite point in D can be connected to z_0 by a chain of circles in which we have $u > u_1$.

1.3. The hyperbolic measure. The word hyperbolic is abbreviated h. In a domain D with more than two boundary points (h. domain) there exist infinitely many solutions of (1); this was proved by Picard [4]. One of these, u_0, has the extremal property that at any finite point of D it takes a greater value than any other solution in D. It also takes a greater value at the point than any other solution on D^*. For by a conformal mapping all the solutions on D^* can be transformed into the solutions in the unit circle $|w|<1$, and inequalities between the values at a particular point of D^* are not disturbed by this transformation. In $|w|<1$ the extremal solution is $-\log(1-|w|^2)$ (see [1, p. 360]), and it is well known that this solution transformed to D^* is single-valued in D (see later, section 2.1). The function

$$\lambda(z) = e^{u_0(z)}$$

is called the h. measure in D. We shall make use of the following property of λ: *When z tends to a finite boundary point of D, λ tends to infinity.* If D has only three boundary points, this follows from the properties of the elliptic modular function (for an elementary proof see [1]), and from the extremal property it follows that the addition of more boundary points increases the values of λ.

1.4. The fundamental inequality. THEOREM 1. *If the h. domain D contains the half-plane $y<0$ and if z belongs to D and to the half-plane $y>0$, then we have*

$$\lambda(z) \geqq \lambda(\bar{z}) , \tag{5}$$

the sign of equality holding only when the boundary of D belongs entirely to the real axis.

PROOF. We may suppose $z=\infty$ to be an interior point of D; otherwise we can achieve this by applying the transformation $w=-(z-a)^{-1}$ for a real a in D, under which the two terms in (5) are multiplied by the same positive factor. First we consider the case when D has no boundary points on the real axis. The part of D that belongs to the half-plane $y>0$ is called D_1. In D_1 we consider the function

$$\varphi(z) = u_0(z)-u_0(\bar{z}) .$$

It is zero on the real axis, and it tends to zero for $z \to \infty$ according to (2) and to infinity when z tends to a boundary point of D. It follows that if the continuous function $\varphi(z)$ were to take negative values in D_1, it would have a negative minimum. At a minimum point z_0 we have, however, $\Delta\varphi(z_0) \geqq 0$ and from

$$\Delta\varphi(z) = 4\left(e^{2u_0(z)} - e^{2u_0(\bar{z})}\right)$$

we get $\varphi(z_0) \geqq 0$. Hence $\varphi(z) \geqq 0$ in D_1. This is also true if D has boundary points on the real axis; for, on applying a translation $i\varepsilon$ ($\varepsilon > 0$), we obtain from what has already been proved

$$u_0(z) \geqq u_0(\bar{z} - 2i\varepsilon)\ ,$$

and $\varepsilon \to 0$ gives the statement.

If the boundary of D is entirely on the real axis, $u_0(z) = u_0(\bar{z})$ follows from the extremal property of u_0. If D has at least one boundary point in $y > 0$, the two functions $u_0(z)$ and $u_0(\bar{z})$ are not identical in D_1 and, according to 1.2, the sign of equality can hold at no point of D_1. This concludes the proof.

Letting z tend to a point of D on the real axis, we find $\partial\varphi/\partial y \geqq 0$ for $y = 0$, and, according to the Lemma in 1.2, equality holds only if the boundary of D is entirely on the real axis. We state this in a slightly more general form as a theorem:

THEOREM 2. *Let E be a h. domain and l a straight line such that one of the open half-planes bounded by l belongs to E and the other contains at least one boundary point of E. Then for any point z_0 on l and belonging to E the vector* grad $\lambda(z_0)$ *is different from zero and points to that side of l on which boundary points are situated.*

§ 2. Applications.

2.1. Hyperbolic geometry. The usual non-Euclidean geometry in the unit circle $|w| < 1$ is defined by the metric

$$d\sigma = \frac{|dw|}{1 - |w|^2} \equiv \lambda_w |dw|\ ; \tag{6}$$

$d\sigma$ remains invariant when the circle is mapped conformally onto itself. By means of a conformal mapping of $|w| < 1$ onto D^* the metric (6) is transferred to D^* in such a way that $d\sigma$ is the same for corresponding elements dz and dw. Thus $d\sigma$ is invariant under conformal mappings of D^* onto itself. Putting $d\sigma = \mu|dz|$ we find

$$\mu = |dw/dz|\, \lambda_w\ ,$$

and this is the h. measure on D^* according to 1.1 and 1.3. If D^* is mapped conformally onto itself in such a way that a point P goes over into a point P_1 with the same coordinate z, then neither $d\sigma$ nor $|dz|$ will change and hence μ has the same value at P and P_1. Consequently μ is equal to λ, the h. measure in D, and we have the formula

$$d\sigma = \lambda |dz| \tag{7}$$

for any h. domain.

In the model of the h. plane in $|w| < 1$ the geodesics are the arcs of circles orthogonal to the fundamental circle $|w| = 1$. Any two points in $|w| < 1$ can be connected by one such arc, and this is the curve of shortest h. length connecting the points. On D^* the geodesics have the same property; but if D is multiply connected, there is an infinity of geodesics connecting any two points in D. Among these there is always one with minimal h. length, and this is a h. shortest curve connecting the points. This is seen at once when D^* is mapped onto the circle as the points in $|w| < 1$ that correspond to a point in D do not accumulate in the interior of the circle.

Let D contain the half-plane $y < 0$; *then this half-plane is a h. convex domain.* At first it follows from Theorem 1 that if, for a negative constant k, a curve c in D lies in the half-plane $y > k$, then the h. length of c is greater than the h. length of the curve symmetric to c with respect to the line $y = k$. Now let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, where $0 > y_1 \geqq y_2$, be joined by a curve a of minimal h. length. Then $y \leqq y_1$ at all points on a; otherwise a h. shorter curve joining z_1 to z_2 would be obtained by reflecting an arc of a contained in the half-plane $y > y_1$ in the line $y = y_1$.

It now follows that the curve of minimal h. length joining two points in $y < 0$ is uniquely determined, for as two such curves would belong to $y < 0$, they would correspond to two geodesics connecting the same two points on D^*.

We now prove Ullman's theorem generalised to multiply connected domains.

THEOREM 3. *Let the h. domain D contain the half-plane $y < 0$ and have at least one boundary point in the half-plane $y > 0$. Then a curve c_1 joining the points x_1 (on the real axis) and z_2 ($y_2 > 0$) in D is h. longer than the h. shortest curve c_2 from x_1 to $\bar{z}_2$.*

PROOF. The part of c_1 situated in $y > 0$ is reflected in the line $y = 0$ and the rest of the curve is retained. In this way we obtain a curve connecting x_1 and $\bar{z}_2$, and this curve is h. shorter than c_1, but not than c_2.

If in Theorem 3 we take z_2 close to x_1 and transfer to a sheet of D^*, it is seen that the geodesic which is the locus of points having the same h. distance from the points $P(z_2)$ and $P(\bar{z}_2)$ on D^* has, in the sheet considered, no points that are projected into the half-plane $y \leqq 0$. Letting

$P(z_2)$ tend to $P(x_1)$, it is seen that the geodesic through $P(x_1)$ that touches the line l corresponding to the real axis, is a h. supporting line of l. This is in accordance with the convexity property just proved.

2.2. Hyperbolic curvature. For a curve in D having a Euclidean curvature we can also define a h. curvature with the following characteristic properties (see [2, p. 22]): If the domain is the unit circle and the point its centre, the two curvatures are equal; and the h. curvature is invariant under conformal mappings. It is seen that the lines of h. curvature zero are the h. geodesics. As usual we fix the sign of the curvature in accordance with the direction chosen on the tangent; the angle from this direction to the direction of the normal is $+\frac{1}{2}\pi$.

For the Euclidean curvature $\varkappa_e$ and the h. curvature $\varkappa_h$ of a curve c at a point of an arbitrary h. domain D we are going to prove the formula

$$\varkappa_e = \lambda\varkappa_h + \frac{\partial \log \lambda}{\partial n}. \tag{8}$$

If we apply a similarity to D, $\varkappa_h$ remains invariant, while the other three quantities are divided by the ratio of the similarity. It follows that we may suppose $\lambda = 1$. The point may be $z=0$ and the tangent the x-axis.

We map D^* conformally onto $|w|<1$ by a function $z=f(w)$ having at $w=0$ the development

$$z = f(w) = w + a_2 w^2 + \ldots .$$

For c we have a parametric representation of the form

$$x = t + o(t)$$
$$y = \tfrac{1}{2}\varkappa_e t^2 + o(t^2) .$$

Putting $a_2 = \alpha_2 + i\beta_2$ we find

$$u = x - \alpha_2 x^2 + 2\beta_2 xy + \alpha_2 y^2 + \ldots$$
$$v = y - \beta_2 x^2 - 2\alpha_2 xy + \beta_2 y^2 + \ldots$$

which gives for the image c_1 of the curve c

$$u = t + o(t)$$
$$v = \tfrac{1}{2}(\varkappa_e - 2\beta_2)t^2 + o(t^2) .$$

This shows that we have

$$\varkappa_e = \varkappa_h + 2\beta_2 . \tag{9}$$

For $\lambda = \lambda(z)$ we have

$$\lambda(z) = \left|\frac{dw}{dz}\right| (1-|w|^2)^{-1} = \left|\frac{dw}{dz}\right| (1+|w|^2+\ldots) .$$

From

$$\left|\frac{dw}{dz}\right|^2 = 1 - 2a_2 z - 2\bar{a}_2 \bar{z} + \ldots$$

we get

$$\lambda(z) = 1 - 2\alpha_2 x + 2\beta_2 y + \ldots$$

which gives for $z = 0$

$$\frac{\partial \log \lambda}{\partial y} = \frac{\partial \lambda}{\partial y} = 2\beta_2 . \tag{10}$$

From (9) and (10) the formula (8) follows.

We see that, at the point z_0 on the line l considered in Theorem 2, the h. curvature of l is different from zero; the curvature vector is directed to that side of l which contains no boundary points. This is in accordance with the fact that this half-plane is a h. convex domain.

2.3. The general form of the theorems. Hitherto we have considered a half-plane free from boundary points; we may instead take an arbitrary circle. Theorem 1 then takes the form:

Theorem 4. *Let G be a h. domain, and let z_1 and z_2 be finite and in G. Then we have for z on the boundary of G*

$$\min \left|\frac{z-z_1}{z-z_2}\right|^2 \leqq \frac{\lambda(z_2)}{\lambda(z_1)} \leqq \max \left|\frac{z-z_1}{z-z_2}\right|^2 , \tag{11}$$

the signs of equality holding only when $|(z-z_1)/(z-z_2)|$ has the same value for all boundary points z.

Proof. We put $\max|(z-z_1)/(z-z_2)| = F$ and consider the circle c: $|(z-z_1)/(z-z_2)| = F$. Let z_0 be a point on c and in G. We use the transformation

$$w = (z-z_0)^{-1} \tag{12}$$

which tranforms G into a domain G_w and c into a straight line l. The points w_1 and w_2 corresponding to z_1 and z_2 are symmetrical with respect to l, and there are no boundary points on the side of l containing w_2. Theorem 1 then gives

$$\lambda_w(w_2) \leqq \lambda_w(w_1) , \tag{13}$$

the sign of equality holding only when the boundary of G is on c. From (7) and (12) we get

$$\lambda_w = \lambda_z \left| \frac{dz}{dw} \right| = \lambda_z |z - z_0|^2$$

and (13) gives

$$\lambda_z(z_2)|z_2 - z_0|^2 \leqq \lambda_z(z_1)|z_1 - z_0|^2 ,$$

which is the right side of (11) because z_0 is on c.

The left side of (11) follows from the right side by interchanging z_1 and z_2.

From Theorem 2 we get

THEOREM 5. *Let z_0 be finite and in a h. domain G, and let the vector t define a direction through z_0. The greatest and smallest curvatures of circles touching t in z_0 and having boundary points of G on the periphery are called K and k. Then we have*

$$k \leqq \frac{\partial \log \lambda(z_0)}{\partial n} \leqq K , \tag{14}$$

the signs of equality holding only if $k = K$.

PROOF. We prove the inequality to the left; the one to the right is then obtained by reversing the direction of t. We may suppose $z_0 = 0$ and t on the positive real axis. If we apply the transformation

$$w = (z^{-1} + \tfrac{1}{2}ki)^{-1} ,$$

the circle with curvature k will go over into the line $v = 0$, and the domain G_w will contain the half-plane $v < 0$. At the point $w = 0$ that corresponds to $z = 0$ we have by Theorem 2

$$\frac{\partial \log \lambda_w}{\partial v} \geqq 0 ,$$

the sign of equality holding only when the two circles coincide. At $w = 0$ the expression

$$\frac{\partial \log \lambda_w}{\partial v} \equiv \frac{\partial}{\partial v} \left(\log \lambda_z + \log \left| \frac{dz}{dw} \right| \right)$$

reduces to

$$\frac{\partial \log \lambda_z}{\partial y} - k$$

since we have

$$\left(\frac{dz}{dw} \right)_{w=0} = 1$$

and

$$\left|\frac{dz}{dw}\right| = |1-\tfrac{1}{2}kwi|^{-2} = (1+kv+\tfrac{1}{4}k^2v^2+\tfrac{1}{4}k^2u^2)^{-1} .$$

This concludes the proof.

The geometrical meaning of the theorem is clear from formula (8). If in particular we consider a h. geodesic, we find

$$k \leqq \varkappa_e \leqq K .$$

This means that *the Euclidean circle of curvature at any point of a h. geodesic divides the boundary or contains the whole boundary on the periphery.* For simply connected domains this theorem is due to Walsh.

§ 3. Simply connected domain.

3.1. Theorems of Ullman and Walsh. Applying the reflection principle of 1.4 to a harmonic function, we get simple proofs of the theorems of Ullman and Walsh. Hyperbolic geometry is not used.

THEOREM (Ullman). *Let the simply connected h. domain D, containing the point $z=\infty$, have boundary points in the half-plane $y>0$, but not in $y<0$, and let the schlicht function $w=f(z)$ map D conformally onto the circle $|w|<1$ in such a way that $z=\infty$ corresponds to $w=0$. Then, if z in D is finite and belongs to the half-plane $y>0$, we have*

$$|f(z)| > |f(\bar{z})| .$$

PROOF. For large values of $|z|\equiv r$ we may suppose $f(z)$ to have the development

$$f(z) = a_1z^{-1}+a_2z^{-2}+\dots ,$$

a_1 being real and positive. The part of D belonging to the half-plane $y>0$ is an open point set D_1 consisting of one or more domains. In D_1 we consider the harmonic function

$$\mu = \log\left|\frac{f(z)}{f(\bar{z})}\right| .$$

Putting $x=r\cos\theta$, $y=r\sin\theta$ we find

$$\mu = 2r^{-1}a_1^{-1}\operatorname{Im}a_2\sin\theta+\sum_{n=2}^{\infty}b_nr^{-n}\sin n\theta , \tag{15}$$

and it is seen that $\mu\to 0$ for $r\to\infty$. For z tending to a point on the real axis we also have $\lim\mu=0$, and for z tending to any other boundary point the limit is positive. It follows that $\mu > 0$ holds in D_1, and this completes the proof. As an immediate consequence (15) gives

$$\operatorname{Im} a_2 > 0 . \tag{16}$$

THEOREM (Walsh). *Let γ be a closed analytic Jordan curve in the circle $|w| < 1$ passing through the point $w=0$. If there exists a schlicht analytic function in $|w| < 1$ that maps γ onto a circle, then the curvature vector of γ at $w=0$ is different from zero and points into the interior of the curve.*

PROOF. We may suppose that γ touches the u-axis at $w=0$ and that the direction of the v-axis points into the interior of the curve at that point. We shall prove, then, that the centre of curvature corresponding to the point $w=0$ is on the positive v-axis.

According to the hypothesis there is a function

$$z = g(w) = w^{-1} + (\alpha + i\beta) + a_1 w + \dots , \tag{17}$$

regular and schlicht in $0 < |w| < 1$ and mapping the interior of γ onto the half-plane $y < 0$. The inverse function has the development

$$w = z^{-1} + (\alpha + i\beta) z^{-2} + \dots$$

for large values of $|z|$, and from (16) we get the inequality

$$\beta > 0 . \tag{18}$$

Instead of (17) we introduce the function

$$z = w^{-1} + (\alpha + i\beta) \tag{19}$$

that maps the circle

$$\left| w - \frac{i}{2\beta} \right| = \frac{1}{2\beta} \tag{20}$$

onto the line $y=0$. Comparing (17) and (19) it is seen that (19) must map γ onto a curve having $y=0$ as an asymptote. But this shows that (20) is the circle of curvature of γ at $w=0$. As the centre of (20) is on the positive v-axis, the theorem is proved.

It should be remarked that γ may have points on the circle $|w| = 1$, but must not consist of a diameter and a semicircle; only in this case the curvature vanishes at $w=0$.

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