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A NOTE ON "THE LAW OF SUPPLY AND DEMAND"

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1. Introduction. In a recent paper [1], Gale has shown the existence of an equilibrium in a simple but extremely flexible model of a competitive market. The features of this model that are relevant can be described very concisely. It makes provision for n goods $G_1, \ldots, G_n$ at relative prices $\pi_1, \ldots, \pi_n$ (which are non-negative and sum to one). The activity of the various economic units in the market is summarized by an aggregate supply function S which assigns to each price vector $p=(\pi_1, \ldots, \pi_n)$ a set $S(p)$ of commodity bundles $x=(\xi_1, \ldots, \xi_n)$. In each bundle $x \in S(p)$, the component ξ_j measures an aggregated supply of G_j (i.e., the total amount supplied by producing units diminished by the total amount demanded by consuming units) that might occur at the prices p. Each bundle x associated with p is assumed to satisfy the budget inequality, $x \cdot p = \xi_1 \pi_1 + \ldots + \xi_n \pi_n \geqq 0$; this holds necessarily if each economic unit receives at least as much income from the goods that it produces as it pays for the goods it consumes. Equilibrium for this model consists of a price vector p and a bundle $x \in S(p)$ such that $x \geqq 0$. (Throughout this note, vector inequalities such as $x \geqq a$ are to hold in all components).

With the preceding identifications as its economic interpretation, the central mathematical result of [1] is the following lemma.

Principal Lemma. *Let S be a bounded continuous set-valued function from the unit $(n-1)$-simplex P into R_n such that*

(a) *$S(p)$ is non-empty and convex for all $p \in P$,*

(b) *if $x \in S(p)$ then $x \cdot p \geqq 0$.*

Then there exists $p \in P$ and $x \in S(p)$ such that $x \geqq 0$.

The proof of the principal lemma given by Gale is in two parts. The first deals with the special case in which the function S is single-valued and hence continuous in the ordinary sense; the fundamental tool used is the lemma of Knaster, Kuratowski, and Mazurkiewicz [2] preliminary

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to the Brouwer fixed point theorem. The second part handles the general case by approximating a set-valued S with single-valued functions in a manner similar to that used by Kakutani [3] in extending the Brouwer fixed point theorem. Our object in this note will be to prove the principal lemma directly by means of the fixed point theorem of Eilenberg and Montgomery [4]. Although it uses a stronger topological result, the starting point of this proof seems closer to the usual verbal explanation of economic equilibrium in a competitive market.

A final section demonstrates that the main theorem of matrix games is a corollary of the principal lemma, thus extending the range of application of Gale's model.

2. Proof of the principal lemma. The explanation commonly given by literary economists for the tendency toward equilibrium in a competitive market is that "the prices of goods in short supply increase relative to the prices of those goods for which the supply is adequate." To translate this into analytic terms, associate with each vector $x=(\xi_1, \ldots, \xi_n)$ a vector $d=(\delta_1, \ldots, \delta_n)$ defined by $\delta_j=\xi_j$ if $\xi_j<0$ and $\delta_j=0$ if $\xi_j \geqq 0$, for $j=1, \ldots, n$. Evidently, δ_j measures the (negative) *deficit* of G_j in x; if the dependence upon x is in question, we shall write $d=d(x)$. We then define

$$p' = f_x(p) = \frac{p-d}{1-\Sigma\delta_j} \quad \text{for } p \in P \quad \text{and} \quad x \in R_n \, .$$

It is clear that, for each x, f_x is a continuous function from P into P that parallels the verbal prescription given above.

LEMMA 1. *If $x \cdot p \geqq 0$ then $p=f_x(p)$ if and only if $x \geqq 0$.*

PROOF. The sufficiency is immediate since $x \geqq 0$ implies $d=0$.

To prove the necessity, note that there exists an index k with $\xi_k \geqq 0$ and $\pi_k>0$ (since otherwise $x \cdot p<0$). For this index, $\delta_k=0$ and

$$\pi_k' = \pi_k = \pi_k(1-\Sigma\delta_j) \, .$$

Cancelling π_k, we have $\Sigma\delta_j=0$ but $d \leqq 0$, hence $d=0$. Therefore, $x \geqq 0$ and the proof is complete.

LEMMA 2. *If x and x' are such that $x \cdot p \geqq 0$, $x' \cdot p \geqq 0$, and $d(x) \neq d(x')$ then $f_x(p) \neq f_{x'}(p)$.*

PROOF. Let $d=(\delta_j)=d(x)$ and $d'=(\delta_j')=d(x')$ with $d' \neq d$. If

$$p' = f_x(p) = f_{x'}(p)$$

then

$$(1-\Sigma\delta_j)p' = p-d \quad \text{and} \quad (1-\Sigma\delta_j')p' = p-d' \, .$$

Subtracting $(\Sigma\delta_j' - \Sigma\delta_j)p' = d' - d \neq 0$ and hence $d' - d = kp'$ with $k \neq 0$. We may assume $k > 0$, possibly interchanging d and d', and hence $0 \geqq d' \geqq d$ with $\Sigma\delta_j' > \Sigma\delta_j$.

Consider any index k with $\xi_k \geqq 0$. Then $\delta_k = 0$ and hence $\delta_k' = 0$. Therefore $(1 - \Sigma\delta_j)\pi_k' = \pi_k = (1 - \Sigma\delta_j')\pi_k'$ and hence $\pi_k = \pi_k' = 0$ since $\Sigma\delta_j' > \Sigma\delta_j$. This contradicts $x \cdot p \geqq 0$ and hence the assumption $f_x(p) = f_{x'}(p)$ is false.

Lemma 3. *Let S be a closed convex set in R_n and let*

$$D = \{d \mid d = d(x) \quad \text{for} \quad x \in S\}.$$

Then D is contractible in itself to a point.

Proof. The lemma will be proved by induction on n; for $n = 1$, both S and D are closed intervals and the lemma is obvious. For $n > 1$, define the following sets:

$$S^* = \{x^* \mid x^* = (\xi_1, \ldots, \xi_{n-1}, 0) \quad \text{for} \quad x = (\xi_1, \ldots, \xi_{n-1}, \xi_n) \in S\}$$
$$D^* = \{d^* \mid d^* = (\delta_1, \ldots, \delta_{n-1}, 0) \quad \text{for} \quad d = (\delta_1, \ldots, \delta_{n-1}, \delta_n) \in D\}.$$

It is clear that S^* and D^* satisfy the conditions of the lemma in R_{n-1} and hence D^* is contractible in itself to a point by the induction hypothesis.

For each $d = (\delta_1, \ldots, \delta_{n-1}, \delta_n) \in D$, the non-positive numbers δ for which $(\delta_1, \ldots, \delta_{n-1}, \delta) \in D$ form a closed interval. Let $\bar{\delta} = \max \delta$; then the set D is seen to be retractible to the set

$$\bar{D} = \{\bar{d} \mid \bar{d} = (\delta_1, \ldots, \delta_{n-1}, \bar{\delta}) \quad \text{for} \quad d = (\delta_1, \ldots, \delta_{n-1}, \delta_n) \in D\}.$$

by the mapping

$$h(d, t) = t\bar{d} + (1-t)d \qquad \text{for} \quad 0 \leqq t \leqq 1 \quad \text{and} \quad d \in D.$$

Since $\bar{D}$ is obviously homeomorphic to D^*, the proof is complete.

Proof of the Principal Lemma. Define a set-valued mapping F of P into P by

$$F(p) = \{p' \mid p' = f_x(p) \quad \text{for} \quad x \in S(p)\}.$$

To verify the continuity of F, let $\{p_k\}$ and $\{p_k'\}$ be two sequences such that $\lim p_k = p_0$, $\lim p_k' = p_0'$, and $p_k' \in F(p_k)$ for $k = 1, 2, \ldots$. Then there is a sequence $\{x_k\}$ such that $p_k' = f_{x_k}(p_k)$ and $x_k \in S(p_k)$ for $k = 1, 2, \ldots$. Since S is bounded, we may assume $\lim x_k = x_0$, choosing subsequences throughout, if necessary. Then, by the continuity of S, $x_0 \in S(p_0)$. Furthermore, by the continuity of f as a function of x and p, $p_0' = f_{x_0}(p_0)$. Therefore, $p_0' \in F(p_0)$ and F is continuous.

If $D(p)$ is associated with $S(p)$ as in Lemma 3, then Lemma 2 proves that $F(p)$ is homeomorphic to $D(p)$, since the latter is compact. Hence $F(p)$ is closed and contractible by Lemma 3. Therefore, the following fixed point theorem of Eilenberg and Montgomery [4], applies:

Let P be an acyclic absolute neighborhood retract and $F: P \to P$ a continuous multivalued function such that for every $p \in P$ the set $F(p)$ is acyclic. Then F has a fixed point $p_0 \in F(p_0)$.

By Lemma 1, $p_0 = f_x(p_0)$ and $x \cdot p_0 \geqq 0$ imply $x \geqq 0$ and thus the equilibrium is established.

3. The main theorem of matrix games.

The starting point for the theory of zero-sum two-person games [5] is the following theorem:

THEOREM. *Given any* $m \times n$ matrix $A = (a_{ij})$ *there exist two probability vectors,* $r = (\varrho_1, \ldots, \varrho_m)$ *with* $\varrho_i \geqq 0$ *and* $\sum \varrho_i = 1$, *and* $p = (\pi_1, \ldots, \pi_n)$ *with* $\pi_j \geqq 0$ *and* $\sum \pi_j = 1$, *and a real number* v *such that*

$$rA = \left(\sum \varrho_i a_{ij}\right) \geqq v$$

and

$$Ap = \left(\sum a_{ij} \pi_j\right) \leqq v\,.$$

PROOF. For each probability vector p, let $v(p) = \max_i \sum a_{ij} \pi_j$ and define $R(p)$ to be the set of all probability vectors r such that

$$rAp = \sum \varrho_i a_{ij} \pi_j = v(p)$$

(thus, $\varrho_i > 0$ only if $\sum a_{ij} \pi_j = v(p)$). Finally, define

$$S(p) = \left\{x \mid \xi_j = \sum \varrho_i a_{ij} - v(p) \quad \text{for} \quad r \in R(p) \quad \text{and} \quad j = 1, \ldots, n\right\}.$$

Note that, if $x \in S(p)$ then $x \cdot p = rAp - v(p) = 0$ since $r \in R(p)$. Since all of the conditions of the Principal Lemma are met, there exists p with $x \in S(p)$ and $x \geqq 0$. Hence, $Ap \leqq v(p)$ by definition, while $x \geqq 0$ means $rA \geqq v(p)$ for some $r \in R(p)$.

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