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ON NEWTONIAN VECTOR FUNCTIONS

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1. Let the vector

$$\boldsymbol{V} \equiv \boldsymbol{V}(x) \equiv \boldsymbol{V}(x_1, x_2, x_3) \equiv X_1\boldsymbol{i}+X_2\boldsymbol{j}+X_3\boldsymbol{k}$$

be defined in a domain (non-null connected open set) D. If $\boldsymbol{V}$ has continuous partial derivatives of the first order in D, then $\boldsymbol{V}$ is said to be a Newtonian vector [4] provided its curl and divergence both vanish identically in D:

$$\text{(1)}\qquad \mathbf{curl}\,\boldsymbol{V} \equiv \nabla \times \boldsymbol{V} \equiv \left(\frac{\partial X_3}{\partial x_2}-\frac{\partial X_2}{\partial x_3}\right)\boldsymbol{i}+\left(\frac{\partial X_1}{\partial x_3}-\frac{\partial X_3}{\partial x_1}\right)\boldsymbol{j}+\left(\frac{\partial X_2}{\partial x_1}-\frac{\partial X_1}{\partial x_2}\right)\boldsymbol{k} = 0\,,$$

$$\text{(2)}\qquad \operatorname{div}\boldsymbol{V} \equiv \nabla\cdot\boldsymbol{V} \equiv \frac{\partial X_1}{\partial x_1}+\frac{\partial X_2}{\partial x_2}+\frac{\partial X_3}{\partial x_3} = 0\,.$$

It follows that every Newtonian vector is a harmonic vector.

We can consider (1) and (2) to be analogues of the Cauchy-Riemann equations. The analogy has been carried further by Fulton and Rainich, who obtained a "Cauchy integral formula" [4] and by Beckenbach, who obtained theorems of Morera type [1].

In this note we carry the analogy a little further by obtaining, for Newtonian vectors, analogues of results due to Fédoroff and the present author for analytic functions of one complex variable [3, 6]. We shall prove in detail only a special case of a more general result; the latter will be stated in full at the end of this note.

2. THEOREM 1. *Let $\boldsymbol{V}(x)$ be continuous in a domain D. Then a necessary and sufficient condition that $\boldsymbol{V}(x)$ be a Newtonian vector is that there exist a fixed null sequence $\{r_k\}$ of positive numbers with the property that*

$$\text{(3)}\qquad \lim_{k\to\infty}\frac{1}{r_k^5}\iiint\limits_{|x-x_k|\leqq r_k}(\boldsymbol{x}-\boldsymbol{x}_k)\cdot\boldsymbol{V}(x)\,dv(x) = 0\,,$$

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$$\lim_{k\to\infty} \frac{1}{r_k^5} \iiint_{|x-x_k|\leqq r_k} (\boldsymbol{x}-\boldsymbol{x}_k)\times \boldsymbol{V}(x)\, dv(x) = \boldsymbol{0} \tag{4}$$

both hold for every sequence $\{x_k\}$ in D that converges to a point of D.

NECESSITY. If $\boldsymbol{V}(x)$ is a Newtonian vector, then, as remarked above, $\boldsymbol{V}(x)$ has harmonic components. Hence if we expand each component of $\boldsymbol{V}(x)$ in a series of spherical harmonics, say about the point x_0, and if we make use of the orthogonality relations that exist among these harmonics, then we obtain the well-known relations

$$\iiint_{|x-x_0|\leqq r_k} (\boldsymbol{x}-\boldsymbol{x}_0)\cdot \boldsymbol{V}(x)\, dv(x) = \frac{4\pi}{15} r_k^5 \nabla\cdot \boldsymbol{V}(x_0)\,, \tag{5}$$

$$\iiint_{|x-x_0|\leqq r_k} (\boldsymbol{x}-\boldsymbol{x}_0)\times \boldsymbol{V}(x)\, dv(x) = \frac{4\pi}{15} r_k^5 \nabla\times \boldsymbol{V}(x_0)\,, \tag{6}$$

which hold for all sufficiently small r. Now (3) and (4) are trivial consequences of (1), (2), (5) and (6), for any null sequence $\{r_k\}$.

SUFFICIENCY. Now let $\{r_k\}$ be a fixed null sequence of positive numbers such that (3) and (4) hold for all sequences $\{x_k\}$ of points of D that converge to a point of D. The functions

$$f_k(y) \equiv \frac{1}{r_k^5} \iiint_{|x-y|\leqq r_k} (\boldsymbol{x}-\boldsymbol{y})\cdot \boldsymbol{V}(x)\, dv(x)\,, \tag{7}$$

$$\boldsymbol{W}_k(y) \equiv \frac{1}{r_k^5} \iiint_{|x-y|\leqq r_k} (\boldsymbol{x}-\boldsymbol{y})\times \boldsymbol{V}(x)\, dv(x) \tag{8}$$

are continuous on a certain subset D_k of D. Moreover, we can show that $\lim_{k\to\infty} f_k(y)=0$ and $\lim_{k\to\infty}\boldsymbol{W}_k(y)=\boldsymbol{0}$ uniformly on compact subsets of D. For, if the convergence were not uniform on say the compact set K in D, then there would exist a positive real number δ, and sequences $\{y_k'\}$, $\{y_k''\}$ of points of K such that $|f_k(y_k')| = \max_{y\in K}|f_k(y)|$ and $|\boldsymbol{W}_k(y_k'')| = \max_{y\in K}|\boldsymbol{W}_k(y)|$ and such that

$$\liminf_{k\to\infty}|f_k(y_k')| \geqq \delta > 0, \qquad \liminf_{k\to\infty}|\boldsymbol{W}_k(y_k'')| \geqq \delta > 0 \tag{9}$$

both hold. Since K is compact, we may assume that the sequences $\{y_k'\}$, $\{y_k''\}$ converge to points of K, and hence of D. But (9) contradicts (3) and (4). Hence the limits (3) and (4) are uniform limits for x_k on compact subsets of D. We can replace those equations by

$$\lim_{k\to\infty} f_k(y) = 0, \qquad \lim_{k\to\infty} \boldsymbol{W}_k(y) = \boldsymbol{0}, \tag{10}$$

which must hold uniformly on compact subsets of D.

We now assume that $\boldsymbol{V}(x)$ has continuous partial derivatives of the first order in D. If we use the finite Taylor expansion

$$\boldsymbol{V}(x) = \boldsymbol{V}(y) + (x_1 - y_1)\frac{\partial \boldsymbol{V}}{\partial x_1} + (x_2 - y_2)\frac{\partial \boldsymbol{V}}{\partial x_2} + (x_3 - y_3)\frac{\partial \boldsymbol{V}}{\partial x_3} + \boldsymbol{o}(|x-y|) \tag{11}$$

in (7) and (8), then we obtain

$$\begin{aligned} f_k(y) &= \frac{4\pi}{15}\nabla\cdot\boldsymbol{V}(y) + o(1), \\ \boldsymbol{W}_k(y) &= \frac{4\pi}{15}\nabla\times\boldsymbol{V}(y) + \boldsymbol{o}(1). \end{aligned} \tag{12}$$

From (10) and (12) we obtain (1) and (2).

Suppose now that $\boldsymbol{V}(x)$ is given to be only continuous in D. Then the mean-value functions [2]

$$\boldsymbol{V}^{(\varrho)}(x) \equiv \frac{3}{4\pi\varrho^2}\iiint\limits_{|y-x|\leqq\varrho} \boldsymbol{V}(y)\,dv(y)$$

have continuous partial derivatives of the first order in an open subset D_ϱ of D; moreover, it follows from (10) that $\boldsymbol{V}^{(\varrho)}(x)$ satisfies (3) and (4) uniformly on compact subsets of D_ϱ. Hence by the preceding argument, it follows that $\boldsymbol{V}^{\varrho}(x)$ is a (harmonic) Newtonian vector in D_ϱ. But $\boldsymbol{V}^{(\varrho)}(x) \rightrightarrows \boldsymbol{V}(x)$ on compact subsets of D, as $\varrho \to 0$, so it follows that $\boldsymbol{V}(x)$ must satisfy the equations (1) and (2) in D. This completes the proof.

3. If we examine the proof of Theorem 1, we see that a key to the proof is the pair of relations (5) and (6); essential use is made of the fact that the ellipsoid of inertia of the sphere is again a sphere. This last remark leads to a further generalization of Fédoroff's result, as follows.

Let $\{\Gamma_n\}$ denote a sequence of volumes homeomorphic to $|x| < 1$ and let δ_n and $|\Gamma_n|$ denote the diameter and volume of Γ_n. We say that $\{\Gamma_n\}$ is a null sequence of volumes if and only if for each $\varepsilon > 0$ there exists $m(\varepsilon)$ such that each Γ_n lies in $|x| < \varepsilon$ for all $n > m(\varepsilon)$. We also say that the sequence $\{\Gamma_n\}$ has the property Q if and only if there is a positive constant a such that the following relations hold for all Γ_n:

$$\iiint_{\Gamma_n} x_k \, dv(x) = \iiint_{\Gamma_n} x_k x_m \, dv(x) = 0, \quad k \neq m, \quad k, m = 1, 2, 3,$$

$$\frac{1}{a}\delta_n^2|\Gamma_n| \leqq \iiint_{\Gamma_n} x_1^2 \, dv(x) = \iiint_{\Gamma_n} x_2^2 \, dv(x) = \iiint_{\Gamma_n} x_3^2 \, dv(x) \leqq a\delta_n^2|\Gamma_n| .$$

If $\Gamma_n(x) \equiv [x+y \mid y \in \Gamma_n]$ denotes a translate of Γ_n, then we have the following result.

THEOREM 2. *Let $V(x)$ be continuous in a domain D. Then a necessary and sufficient condition that $V(x)$ be a Newtonian vector is that there exist a null sequence of volumes, with property Q, such that*

$$\lim_{n\to\infty} \frac{1}{\delta_n^2|\Gamma_n|} \iiint_{\Gamma_n(x_n)} (\boldsymbol{x}-\boldsymbol{x}_n)\cdot \boldsymbol{V}(x)\, dv(x) = 0 ,$$

$$\lim_{n\to\infty} \frac{1}{\delta_n^2|\Gamma_n|} \iiint_{\Gamma_n(x_n)} (\boldsymbol{x}-\boldsymbol{x}_n)\times \boldsymbol{V}(x)\, dv(x) = \mathbf{0}$$

hold for each sequence $\{x_n\}$ converging to a point in D.

PROOF. A proof can be given that parallels the proof given above for Theorem 1. We omit the proof.

4. It is clear that the preceding results have analogues in n dimensions, $n \geqq 2$. The author has given detailed proofs in [7] for the case $n=2$.

The basic idea, that of establishing the uniform limit (10) under the given hypothesis, is due to Müller [5].

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