MATH. SCAND. 4 (1956), 157–160

A REMARK ON THE CONSTRUCTION OF THE CENTRE OF A CIRCLE BY MEANS OF THE RULER

CHRISTIAN GRAM

It is well known that all problems of construction in the plane which can be solved by means of the ruler and the compasses can also be solved by means of the ruler alone if one circle and its centre are given. Here the centre cannot be dispensed with. D. Cauer has proved ([2], see also [1, § 5]) that even if two non-intersecting and non-concentric circles with unknown centres are given, it is impossible to construct their centres by means of the ruler. His proof runs as follows.

In a suitable rectangular coordinate system the equations of the given circles K_1 and K_2 may be written

$$(x-a)^2+y^2 = a^2-1, \qquad (x-b)^2+y^2 = b^2-1 ,$$

where $|a|>1$, $|b|>1$, $a \neq b$. The harmonic homology

$$\mathscr{H}: \qquad x = 1/x', \quad y = y'/x'$$

with centre $(-1, 0)$ and axis $x=1$ maps each of the circles and also the line joining their centres (the x-axis) onto itself, but the centres $(a, 0)$ and $(b, 0)$ go over into the new points $(1/a, 0)$ and $(1/b, 0)$. Suppose now that there exists a construction of one of the centres, say $(a, 0)$, by means of the ruler. Since $\mathscr{H}$ carries straight lines into straight lines and preserves incidence, the whole construction is transformed into a construction of exactly the same kind, which, however, would yield the point $(1/a, 0)$ instead of the centre.

As $\mathscr{H}$ maps the line joining the centres (briefly, the centre line) onto itself, the proof shows that even if two circles and their centre line are given, it is impossible to construct the centres by means of the ruler. Since $\mathscr{H}$ maps each circle of the pencil of circles determined by K_1 and K_2 onto itself, this holds even if finitely many circles of this pencil and their common centre line are given.

From the projective point of view the situation may be described as

Received May 3, 1956.

follows. Finding the centres of the circles is equivalent to finding their common polar with respect to the circles, that is, the line at infinity. The latter is determined by the two circular points at infinity, that is, by a pair of conjugate complex points of intersection of the circles. There are, however, two such pairs, the other one determining the radical axis of the circles. Hence, a construction of the centres by means of the ruler (if any) is necessarily ambiguous; besides the centres it must give the poles of the radical axis.

In the above argument it is assumed that no distinction is made between proper points and lines and the points and the line at infinity; for the homology $\mathscr{H}$ may carry parallel lines into intersecting lines and conversely. Further, $\mathscr{H}$ need not preserve the (Euclidean) order of points on a line. Consequently, if it is permitted, under the construction, to distinguish between pairs of parallel lines and pairs of intersecting lines or to decide whether or not a point, given or constructed, is between two other such points, Cauer's argument becomes invalid. It is the purpose of this note to demonstrate this by giving a construction of the centres of two given circles provided that a point of their centre line is known. Thus, we are dealing with constructions in the "oriented plane", in the terminology of Bierberbach [1, pp. 26, 151], or *ordered plane*, a term which seems more adequate. The question whether such a construction exists if only the two circles are given, is left open. So we are going to show:

In the ordered plane, let there be given two circles and a point of the line joining their centres. Then it is possible to construct the centres of the circles by means of the ruler.

The construction is carried out in several steps consisting in the following well-known constructions, which are briefly indicated for the convenience of the reader:

1. To construct the fixed points M_1 and M_2 of an hyperbolic involution on a line l of which two pairs of corresponding points A_1, B_1 and A_2, B_2 are known. From an arbitrary point S on one of the given circles A_1, B_1, A_2, B_2 are projected into the points a_1, b_1, a_2, b_2 on this circle. The line joining the points of intersection $L_1 = a_1b_2 \cdot b_1a_2$ and $L_2 = a_1a_2 \cdot b_1b_2$ intersects the circle in the points m_1 and m_2 whose projections on l from S are the fixed points M_1 and M_2 (see fig. 1).

2. Given two circles K_1 and K_2; to construct pointwise that circle C of the pencil of circles determined by K_1 and K_2 which passes through a given point P. Draw an arbitrary line through P which intersects K_1 in A_1, B_1 and K_2 in A_2, B_2. Construct the fixed points of the involution

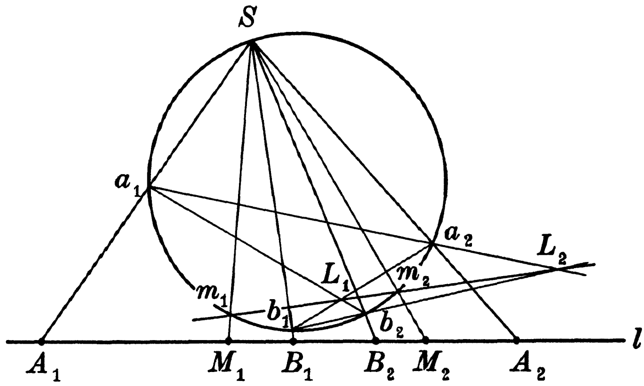

Fig. 1.

of which A_1, B_1 and A_2, B_2 are pairs. Then the harmonic conjugate to P with respect to these fixed points is a point of C.

3. To construct the polar p of a point P with respect to a circle, 5 points of which are known. Draw the lines joining P to two of the given points of the circle. By means of Pascal's theorem construct the second point of intersection of either line with the circle. Then the polar p of P is found by means of an inscribed complete quadrangle (see fig. 2).

4. Given 5 points of a circle and a line d; to construct the points of intersection of d with the circle. Construct the polars p and q of two points P and Q of d with respect to the circle (see fig. 2). Let P' and Q' be the points in which d intersects p and q, respectively. Then the fixed points of the involution determined by the pairs P, P' and Q, Q' are the points of intersection required.

Let now K_1 and K_2 be the given circles, and let D be a given point of their centre line.

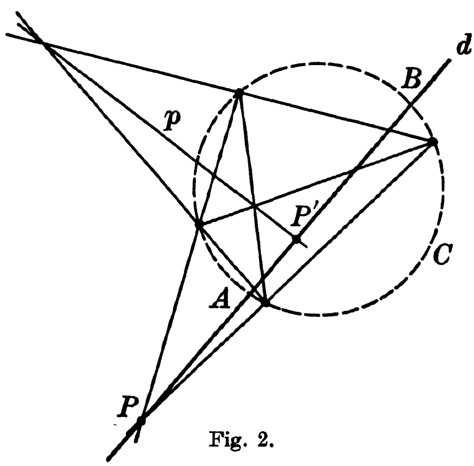

Fig. 2.

Firstly, we observe that the centre line d can be found as follows. The polars of D with respect to K_1 and K_2, which can be constructed (3), are parallel. Hence it is possible, by means of the ruler, to bisect any line segment parallel with these lines, in particular a chord of one of the circles. This gives a second point of the centre line d.

Secondly, we draw an arbitrary line l different from d intersecting both K_1 and K_2. The circles of the pencil determined by K_1 and K_2 intersect l in the pairs of a hyperbolic involution, two pairs of which are known, namely the intersections of l with K_1 and K_2. We construct the fixed points M_1 and M_2 of this involution (1).

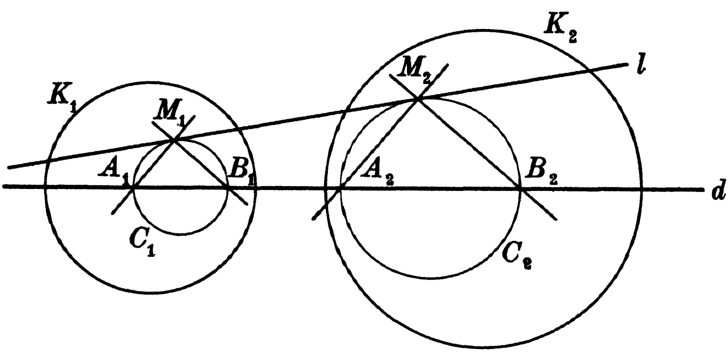

Fig. 3.

Thirdly, we construct pointwise the circles C_1 and C_2 of the pencil which pass through M_1 and M_2, respectively (2). The line l is a common tangent of C_1 and C_2.

Finally, we construct the pairs of points A_1, B_1 and A_2, B_2 in which d, the centre line, is intersected by C_1 and C_2, respectively (4) (see fig. 3). Using now the assumption that it is possible to recognize order relations, we choose the notations such that the oriented line segments A_1B_1 and A_2B_2 have the same or the opposite orientations according as l is an outer or an inner common tangent of C_1 and C_2. Then M_1A_1, M_2A_2 and M_1B_1, M_2B_2 are two pairs of parallel lines. Thus, any line segment, in particular the diameters on d of K_1 and K_2, can be bisected by means of the ruler.

REFERENCES

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2. D. Cauer, *Über die Konstruktion des Mittelpunktes eines Kreises mit dem Lineal allein*, Math. Ann. 73 (1913), 90–94. *Berichtigung*, Math. Ann. 74 (1913), 462–464.

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