MATH. SCAND. 4 (1956), 193–208

UNIVERSAL RELATIONAL SYSTEMS

BJARNI JÓNSSON

Introduction. As a typical example of the kind of problems treated in this paper we mention the following question: Given an ordinal α, does there exist a group $\mathfrak{G}$ with the cardinal $\aleph_\alpha$, having the property that every group with the same cardinal is isomorphic to a subgroup of $\mathfrak{G}$? Such a group $\mathfrak{G}$, if it exists, will be referred to as an $\aleph_\alpha$ universal group. Similar questions can be asked for various other systems, such as groupoids, lattices and partially ordered systems. In § 2 we introduce the notion of an ($\aleph_\alpha$, $\boldsymbol{K}$) universal relational system, where $\boldsymbol{K}$ is some class of relational systems. Assuming the Generalized Continuum Hypothesis, we then prove the existence of ($\aleph_\alpha$, $\boldsymbol{K}$) universal systems for all classes $\boldsymbol{K}$ satisfying certain conditions (one of which depends on α). In § 3 we show that the conditions imposed upon $\boldsymbol{K}$ are satisfied if $\alpha > 0$, and if $\boldsymbol{K}$ is the class of all groups, the class of all groupoids, the class of all lattices, or the class of all partially ordered systems. The answers to the questions raised above therefore turn out to be affirmative for the case when $\alpha > 0$, provided the Generalized Continuum Hypothesis holds.

Certain special cases of our result are known from the literature. Thus it is easy to show that there exists, for every ordinal α, an $\aleph_\alpha$ universal Abelian group. In Hausdorff [3, Chapter 6] it is shown that if the Generalized Continuum Hypothesis holds, and if α is not a limit ordinal, then there exists an $\aleph_\alpha$ universal simply ordered system. In Mostowski [6] an $\aleph_0$ universal partially ordered system is constructed, and in Johnston [4] it is shown that if the Generalized Continuum Hypothesis holds, and if α is cofinal with ω, then there exists an $\aleph_\alpha$ universal partially ordered system. While these authors describe explicitly their universal systems, we shall in the general case have to be content with a non-constructive existence proof.

1. Preliminaries. Given an ordinal μ, the μ-termed sequence, or μ-tuple, whose successive terms are $x_0, x_1, \ldots, x_\xi, \ldots$ will be denoted by $\langle x_0, x_1, \ldots, x_\xi, \ldots \rangle$; if μ is finite, then the sequence will also be written

Received June 14, 1956.

$\langle x_0, x_1, \ldots, x_{\mu-1}\rangle$. If A is any set, then the set of all μ-termed sequences all of whose terms belong to A will be denoted by A^μ. If μ is a finite ordinal, then by a *μ-ary relation*, or a *relation of rank μ*, is understood any set R of μ-termed sequences; R is called a *relation* if it is a μ-ary relation for some finite ordinal μ. By a *relational system* or, more briefly, a *system*, is understood a sequence

$$\mathfrak{A} = \langle A, R_0, R_1, \ldots, R_{\varkappa-1}\rangle ,$$

such that A is a non-empty set, $\varkappa$ is a finite ordinal, $R_0, R_1, \ldots, R_{\varkappa-1}$ are relations, and each relation R_τ is included in A^{μ_τ} where μ_τ is the rank of R_τ. The sequence $\langle \mu_0, \mu_1, \ldots, \mu_{\varkappa-1}\rangle$ is called the *similarity type* of $\mathfrak{A}$, and two relational systems having a common similarity type are said to be *similar*. The notion of isomorphism between similar relational systems is defined in an obvious manner.

Suppose $\mathfrak{A}=\langle A, R_0, R_1, \ldots, R_{\varkappa-1}\rangle$ is a system with the similarity type $\langle \mu_0, \mu_1, \ldots, \mu_{\varkappa-1}\rangle$. If B is a non-empty subset of A, then the new system

$$\langle B, R_0 \cap B^{\mu_0}, R_1 \cap B^{\mu_1}, \ldots, R_{\varkappa-1} \cap B^{\mu_{\varkappa-1}}\rangle$$

is called the *restriction* of $\mathfrak{A}$ to B, and is denoted by $\mathfrak{A} \mid B$. If $\mathfrak{B}=\mathfrak{A}\mid B$ for some non-empty subset B of A, then $\mathfrak{B}$ is said to be a *subsystem* of $\mathfrak{A}$, and $\mathfrak{A}$ is said to be an *extension* of $\mathfrak{B}$, in symbols, $\mathfrak{B} \prec \mathfrak{A}$. By an element of the system $\mathfrak{A}$ is meant an element of the set A, and the cardinal of the set A will also be referred to as the cardinal of the system $\mathfrak{A}$, and will be denoted alternatively by ${}^\star A$ or by ${}^\star\mathfrak{A}$.

When the operations of union and intersection are applied to systems having a common similarity type, it is understood that these operations are to be performed on corresponding terms of the systems involved. Thus if α is a positive ordinal and if similar systems

$$\mathfrak{A}_\xi = \langle A_\xi, R_{\xi,0}, R_{\xi,1}, \ldots, R_{\xi,\varkappa-1}\rangle$$

are associated with all the ordinals $\xi < \alpha$, then

$$\bigcup_{\xi<\alpha}\mathfrak{A}_\xi = \langle \bigcup_{\xi<\alpha} A_\xi, \bigcup_{\xi<\alpha} R_{\xi,0}, \bigcup_{\xi<\alpha} R_{\xi,1}, \ldots, \bigcup_{\xi<\alpha} R_{\xi,\varkappa-1}\rangle .$$

It is easy to see that if $\mathfrak{A}_\xi \prec \mathfrak{A}_\eta$ whenever $\xi < \eta < \alpha$, then $\mathfrak{A}_\eta \prec \bigcup_{\xi<\alpha}\mathfrak{A}_\xi$ for every $\eta < \alpha$. If the systems

$$\mathfrak{A} = \langle A, R_0, R_1, \ldots, R_{\varkappa-1}\rangle \quad \text{and} \quad \mathfrak{B} = \langle B, S_0, S_1, \ldots, S_{\varkappa-1}\rangle$$

are similar, then their intersection

$$\mathfrak{A}\cap\mathfrak{B} = \langle A\cap B, R_0\cap S_0, R_1\cap S_1, \ldots, R_{\varkappa-1}\cap S_{\varkappa-1}\rangle$$

is again a relational system, provided the sets A and B are not disjoint.

Assuming that $A \cap B$ is not empty, it can be shown without difficulty that in order for $\mathfrak{A} \cap \mathfrak{B}$ to be a subsystem of $\mathfrak{A}$ and of $\mathfrak{B}$ it is necessary and sufficient that the systems $\mathfrak{A} \mid (A \cap B)$ and $\mathfrak{B} \mid (A \cap B)$ be equal to each other, and hence equal to $\mathfrak{A} \cap \mathfrak{B}$.

2. ($\aleph_\alpha$, $\boldsymbol{K}$) universal systems. Generalizing the notion of an $\aleph_\alpha$ universal group, mentioned above, we introduce the following:

DEFINITION 2.1. *Given a class $\boldsymbol{K}$ of systems and an ordinal α, a system $\mathfrak{A} \in \boldsymbol{K}$ is said to be ($\aleph_\alpha$, $\boldsymbol{K}$) universal if $^\star\mathfrak{A} = \aleph_\alpha$ and if every system $\mathfrak{B} \in \boldsymbol{K}$ with $^\star\mathfrak{B} \leqq \aleph_\alpha$ is isomorphic to a subsystem of $\mathfrak{A}$.*

For convenience we also introduce two auxiliary concepts:

DEFINITION 2.2. *Two extensions $\mathfrak{B}$ and $\mathfrak{C}$ of a system $\mathfrak{A}$ are said to be equivalent modulo $\mathfrak{A}$ if there exists a function φ which maps $\mathfrak{B}$ isomorphically onto $\mathfrak{C}$ in such a way that $\varphi(x) = x$ for every element x of $\mathfrak{A}$.*

DEFINITION 2.3. *Given a class $\boldsymbol{K}$ of systems, an ordinal α, and a system $\mathfrak{A} \in \boldsymbol{K}$ with $^\star\mathfrak{A} < \aleph_\alpha$, by an ($\aleph_\alpha$, $\boldsymbol{K}$) universal extension of $\mathfrak{A}$ is meant an extension $\mathfrak{B} \in \boldsymbol{K}$ of $\mathfrak{A}$ such that every extension $\mathfrak{C} \in \boldsymbol{K}$ of $\mathfrak{A}$ with $^\star\mathfrak{C} < \aleph_\alpha$ is equivalent modulo $\mathfrak{A}$ to a subsystem of $\mathfrak{B}$.*

We shall be concerned with a class $\boldsymbol{K}$ of relational systems, subject to certain conditions. To avoid repetition, we list these conditions here:

I. *There exist $\mathfrak{A}$, $\mathfrak{B} \in \boldsymbol{K}$ such that $\mathfrak{A}$ and $\mathfrak{B}$ are not isomorphic.*

II. *If $\mathfrak{A} \in \boldsymbol{K}$ and $\mathfrak{A} \cong \mathfrak{B}$, then $\mathfrak{B} \in \boldsymbol{K}$.*

III. *For every $\mathfrak{A}$, $\mathfrak{B} \in \boldsymbol{K}$ there exists $\mathfrak{C} \in \boldsymbol{K}$ such that $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic to subsystems of $\mathfrak{C}$.*

IV. *For every $\mathfrak{A}$, $\mathfrak{B} \in \boldsymbol{K}$, if $\mathfrak{A} \cap \mathfrak{B} \in \boldsymbol{K}$, $\mathfrak{A} \cap \mathfrak{B} \prec \mathfrak{A}$ and $\mathfrak{A} \cap \mathfrak{B} \prec \mathfrak{B}$, then there exists $\mathfrak{C} \in \boldsymbol{K}$ such that $\mathfrak{A} \prec \mathfrak{C}$ and $\mathfrak{B} \prec \mathfrak{C}$.*

V. *If λ is a positive ordinal, if $\mathfrak{A}_\xi \in \boldsymbol{K}$ for every $\xi < \lambda$, and if $\mathfrak{A}_\xi \prec \mathfrak{A}_\eta$ whenever $\xi < \eta < \lambda$, then $\bigcup_{\xi < \lambda} \mathfrak{A}_\xi \in \boldsymbol{K}$.*

VI$_\alpha$. *If $\mathfrak{A} \in \boldsymbol{K}$, $\mathfrak{B} \prec \mathfrak{A}$ and $^\star\mathfrak{B} < \aleph_\alpha$, then there exists $\mathfrak{C} \in \boldsymbol{K}$ such that $\mathfrak{B} \prec \mathfrak{C} \prec \mathfrak{A}$ and $^\star\mathfrak{C} < \aleph_\alpha$.*

To illustrate these conditions, consider the case in which $\boldsymbol{K}$ is the class of all groups. We may regard a group as a system $\langle A, \cdot, ^{-1} \rangle$ consisting of a non-empty set A, a binary operation $\cdot$, and a unary operation $^{-1}$, satisfying certain well-known conditions. Since a μ-ary operation may be considered as a special kind of $(\mu+1)$-ary relation, we thus conceive of a group as a relational system having the similarity type $\langle 3, 2 \rangle$.

It is obvious that the class $\boldsymbol{K}$ of all groups satisfies the conditions I, II, III and V. If α is a positive ordinal, $\mathfrak{A} = \langle A, \cdot, ^{-1} \rangle$ is a group, and

B is any subset of A with $^\star B < \aleph_\alpha$, then B generates a subgroup $\mathfrak{C}$ of $\mathfrak{A}$ with $^\star\mathfrak{C} < \aleph_\alpha$. Hence VI_α holds in case $\alpha > 0$. On the other hand, VI_0 is not satisfied, since a finite subset of A may generate an infinite subgroup of $\mathfrak{A}$. Finally consider the condition IV. Given two groups

$$\mathfrak{A} = \langle A, \cdot, ^{-1}\rangle \quad \text{and} \quad \mathfrak{B} = \langle B, \cdot', ^{-1'}\rangle ,$$

the assumption that $\mathfrak{A} \cap \mathfrak{B} \in \boldsymbol{K}$ means that the set $A \cap B$ is non-empty, and that

$$x \cdot y = x \cdot' y \in A \cap B \quad \text{and} \quad x^{-1} = x^{-1'} \in A \cap B \quad \text{for every} \quad x, y \in A \cap B .$$

It is clear that if this condition is satisfied, then $\mathfrak{A} \cap \mathfrak{B}$ is a subgroup of $\mathfrak{A}$ and of $\mathfrak{B}$, so that the last two formulae in the hypothesis of IV are actually superfluous in this case. Now if $\mathfrak{A} \cap \mathfrak{B} \in \boldsymbol{K}$, then we may form (cf. Schreier [8]) the so-called free product $\mathfrak{C}$ of $\mathfrak{A}$ and $\mathfrak{B}$ with the amalgamated subgroup $\mathfrak{A} \cap \mathfrak{B}$, and it is well known that $\mathfrak{C}$ is a group which contains $\mathfrak{A}$ and $\mathfrak{B}$ as subgroups. Hence the condition IV is also satisfied in this case.

LEMMA 2.4. *If α is any ordinal, and if $\boldsymbol{K}$ is a class of systems which satisfies the conditions* I–V *and* $\mathrm{VI}_{\alpha+1}$, *then there exists* $\mathfrak{A} \in \boldsymbol{K}$ *with* $^\star\mathfrak{A} = \aleph_\alpha$.

PROOF. According to I there exist two non-isomorphic systems $\mathfrak{B}', \mathfrak{B}'' \in \boldsymbol{K}$, and by III there exists $\mathfrak{C} \in \boldsymbol{K}$ such that $\mathfrak{B}'$ and $\mathfrak{B}''$ are isomorphic to subsystems $\mathfrak{C}'$ and $\mathfrak{C}''$ of $\mathfrak{C}$. Since $\mathfrak{C}'$ and $\mathfrak{C}''$ are not isomorphic, one of the equations $\mathfrak{C}' = \mathfrak{C}$ and $\mathfrak{C}'' = \mathfrak{C}$ must fail; we may assume that $\mathfrak{C}' \neq \mathfrak{C}$. It follows from II that $\mathfrak{C}' \in \boldsymbol{K}$, and using II again we can associate with all the ordinals $\xi < \omega_\alpha$ extensions $\mathfrak{C}_\xi \in \boldsymbol{K}$ of $\mathfrak{C}'$ which are equivalent to $\mathfrak{C}$ modulo $\mathfrak{C}'$, in such a way that $\mathfrak{C}_\xi \cap \mathfrak{C}_\eta = \mathfrak{C}'$ whenever $\xi < \eta < \omega_\alpha$. Using II, IV and V we easily obtain systems $\mathfrak{A}_\xi \in \boldsymbol{K}$, associated with all the ordinals $\xi < \omega_\alpha$, such that

$$\mathfrak{C}_\xi \prec \mathfrak{A}_\xi \prec \mathfrak{A}_\eta \quad \text{and} \quad \mathfrak{A}_\xi \cap \mathfrak{C}_\eta = \mathfrak{C}' \quad \text{whenever} \quad \xi < \eta < \omega_\alpha .$$

Letting

$$\mathfrak{A}' = \bigcup\nolimits_{\xi < \omega_\alpha} \mathfrak{A}_\xi ,$$

we see by V that $\mathfrak{A}' \in \boldsymbol{K}$. Furthermore, the sequence $\langle \mathfrak{A}_0, \mathfrak{A}_1, \ldots, \mathfrak{A}_\xi, \ldots \rangle$ is strictly increasing, whence $^\star\mathfrak{A}' \geqq \aleph_\alpha$. Choosing a subsystem $\mathfrak{A}''$ of $\mathfrak{A}'$ with $^\star\mathfrak{A}'' = \aleph_\alpha$, we infer from $\mathrm{VI}_{\alpha+1}$ that there exists $\mathfrak{A} \in \boldsymbol{K}$ such that $\mathfrak{A}'' \prec \mathfrak{A} \prec \mathfrak{A}'$ and $^\star\mathfrak{A} = \aleph_\alpha$.

LEMMA 2.5. *If α is any ordinal,* $\boldsymbol{K}$ *is a class of systems which satisfies the condition* VI_α, $\mathfrak{A} \in \boldsymbol{K}$ *and* $^\star\mathfrak{A} = \aleph_\alpha$, *then there exist subsystems* $\mathfrak{A}_\xi \in \boldsymbol{K}$ *of*

$\mathfrak{A}$, *associated with all the ordinals* $\xi < \omega_\alpha$, *such that* ${}^*\mathfrak{A}_\xi < \aleph_\alpha$ *for every* $\xi < \omega_\alpha$, $\mathfrak{A}_\xi \prec \mathfrak{A}_\eta$ *whenever* $\xi < \eta < \omega_\alpha$, *and* $\mathfrak{A} = \bigcup_{\xi < \omega_\alpha} \mathfrak{A}_\xi$.

PROOF. Letting β be the smallest ordinal such that ω_β is cofinal with ω_α, we can associate subsystems $\mathfrak{B}_\mu$ of $\mathfrak{A}$ with all the ordinals $\mu < \omega_\beta$ in such a way that ${}^*\mathfrak{B}_\mu < \aleph_\alpha$ for every $\mu < \omega_\beta$, $\mathfrak{B}_\mu \prec \mathfrak{B}_\nu$ whenever $\mu < \nu < \omega_\beta$, and $\mathfrak{A} = \bigcup_{\mu < \omega_\beta} \mathfrak{B}_\mu$. Using VI_α and the definition of β we obtain subsystems $\mathfrak{C}_\mu \in \boldsymbol{K}$ of $\mathfrak{A}$, associated with all the ordinals $\mu < \omega_\beta$, such that $\mathfrak{B}_\mu \prec \mathfrak{C}_\mu$ and ${}^*\mathfrak{C}_\mu < \aleph_\alpha$ for every $\mu < \omega_\beta$, and $\mathfrak{C}_\mu \prec \mathfrak{C}_\nu$ whenever $\mu < \nu < \omega_\beta$. It follows that $\mathfrak{A} = \bigcup_{\mu < \omega_\beta} \mathfrak{C}_\mu$. From the definition of β we infer that there exists a non-decreasing function φ on the set of all ordinals $\xi < \omega_\alpha$ onto the set of all ordinals $\mu < \omega_\beta$, and letting $\mathfrak{A}_\xi = \mathfrak{C}_{\varphi(\xi)}$ for every $\xi < \omega_\alpha$ we readily see that the systems $\mathfrak{A}_\xi$ satisfy the required conditions.

LEMMA 2.6. *If* α *is any ordinal and* $\boldsymbol{K}$ *is a class of systems which satisfies the conditions* V *and* VI_α, *then* $\boldsymbol{K}$ *satisfies the condition* VI_β *for every ordinal* $\beta > \alpha$.

PROOF. First assume that $\beta = \gamma + 1$, and that $\boldsymbol{K}$ satisfies the condition VI_γ. Suppose $\mathfrak{A} \in \boldsymbol{K}$, $\mathfrak{B} \prec \mathfrak{A}$ and ${}^*\mathfrak{B} < \aleph_\beta$. Then ${}^*\mathfrak{B} \leqq \aleph_\gamma$, and letting δ be the smallest ordinal such that ω_δ is cofinal with ω_γ we can associate subsystems $\mathfrak{B}_\xi$ of $\mathfrak{B}$ with all the ordinals $\xi < \omega_\delta$ in such a way that ${}^*\mathfrak{B}_\xi < \aleph_\gamma$ for every $\xi < \omega_\delta$, $\mathfrak{B}_\xi \prec \mathfrak{B}_\eta$ whenever $\xi < \eta < \omega_\delta$, and $\mathfrak{B} = \bigcup_{\xi < \omega_\delta} \mathfrak{B}_\xi$. Using VI_γ we can then associate subsystems $\mathfrak{C}_\xi \in \boldsymbol{K}$ of $\mathfrak{A}$ with all the ordinals $\xi < \omega_\delta$ in such a way that $\mathfrak{B}_\xi \prec \mathfrak{C}_\xi$ and ${}^*\mathfrak{C}_\xi < \aleph_\gamma$ for every $\xi < \omega_\delta$, and $\mathfrak{C}_\xi \prec \mathfrak{C}_\eta$ whenever $\xi < \eta < \omega_\delta$. Letting $\mathfrak{C} = \bigcup_{\xi < \omega_\delta} \mathfrak{C}_\xi$, we conclude that $\mathfrak{B} \prec \mathfrak{C}$ and ${}^*\mathfrak{C} \leqq \aleph_\gamma < \aleph_\beta$, while $\mathfrak{C} \in \boldsymbol{K}$ by V. Thus $\boldsymbol{K}$ satisfies the condition VI_β.

It is obvious that if $\beta > \alpha$, β is a limit ordinal, and $\boldsymbol{K}$ satisfies the condition VI_γ for every ordinal γ with $\alpha \leqq \gamma < \beta$, then $\boldsymbol{K}$ also satisfies the condition VI_γ. The proof is therefore easily completed by transfinite induction.

LEMMA 2.7. *Suppose* α *is an ordinal and* $\boldsymbol{K}$ *is a class of systems which satisfies the conditions* II, IV *and* VI_α. *If* $\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2 \in \boldsymbol{K}$, $\mathfrak{A}_0 \prec \mathfrak{A}_1 \prec \mathfrak{A}_2$, ${}^*\mathfrak{A}_1 < \aleph_\alpha$, *and* $\mathfrak{A}_2$ *is an* $(\aleph_\alpha, \boldsymbol{K})$ *universal extension of* $\mathfrak{A}_1$, *then* $\mathfrak{A}_2$ *is an* $(\aleph_\alpha, \boldsymbol{K})$ *universal extension of* $\mathfrak{A}_0$.

PROOF. Consider any system $\mathfrak{B} \in \boldsymbol{K}$ with $\mathfrak{A}_0 \prec \mathfrak{B}$ and ${}^*\mathfrak{B} < \aleph_\alpha$. By II we may assume that $\mathfrak{A}_1 \cap \mathfrak{B} = \mathfrak{A}_0$, and it follows by IV and VI_α that there exists $\mathfrak{C} \in \boldsymbol{K}$ such that $\mathfrak{A}_1 \prec \mathfrak{C}$, $\mathfrak{B} \prec \mathfrak{C}$ and ${}^*\mathfrak{C} < \aleph_\alpha$. Hence there exists a function φ which maps $\mathfrak{C}$ isomorphically onto a subsystem of $\mathfrak{A}_2$ in such a way that $\varphi(x) = x$ for every element x of $\mathfrak{A}_1$. Consequently φ maps

$\mathfrak{B}$ isomorphically onto a subsystem of $\mathfrak{A}_2$ in such a way that $\varphi(x)=x$ for every element x of $\mathfrak{A}_0$. Thus $\mathfrak{A}_2$ is an $(\aleph_\alpha, \boldsymbol{K})$ universal extension of $\mathfrak{A}_0$.

LEMMA 2.8. *If α is an ordinal with the property that $\mathfrak{m}<\aleph_\alpha$ always implies that $2^{\mathfrak{m}}\leqq\aleph_\alpha$, if $\boldsymbol{K}$ is a class of systems which satisfies the conditions* I–V *and* $\mathrm{VI}_{\alpha+1}$, *and if $\mathfrak{A}_0\in\boldsymbol{K}$ and ${}^\star\mathfrak{A}_0<\aleph_\alpha$, then there exists an $(\aleph_\alpha, \boldsymbol{K})$ universal extension $\mathfrak{A}$ of $\mathfrak{A}_0$ with ${}^\star\mathfrak{A}\leqq\aleph_\alpha$.*

PROOF. It follows from III that all the systems in $\boldsymbol{K}$ have a common similarity type $\langle\mu_0, \mu_1, \ldots, \mu_{\varkappa-1}\rangle$. Hence $\mathfrak{A}_0$ is of the form

$$\mathfrak{A}_0 = \langle A_0, R_0, R_1, \ldots, R_{\varkappa-1}\rangle ,$$

where ${}^\star A_0<\aleph_\alpha$ and $R_\tau\subseteq A_0{}^{\mu_\tau}$ for $\tau=0, 1, \ldots, \varkappa-1$. With each cardinal $\mathfrak{n}<\aleph_\alpha$ associate a set $B_{\mathfrak{n}}$ with ${}^\star B_{\mathfrak{n}}=\mathfrak{n}$ which has no element in common with A_0, and let $\boldsymbol{L}_{\mathfrak{n}}$ be the class of all extensions $\mathfrak{B}\in\boldsymbol{K}$ of $\mathfrak{A}_0$ which are of the form

$$\mathfrak{B} = \langle A_0\cup B_{\mathfrak{n}}, S_0, S_1, \ldots, S_{\varkappa-1}\rangle .$$

Since for each system $\mathfrak{B}\in\boldsymbol{L}_{\mathfrak{n}}$ and for $\tau=0, 1, \ldots, \varkappa-1$ we have $S_\tau\subseteq(A_0\cup B_{\mathfrak{n}})^{\mu_\tau}$, we see that

$${}^\star\boldsymbol{L}_{\mathfrak{n}} \leqq \Pi_{\tau<\varkappa}2^{\mathfrak{n}_\tau} \quad \text{where} \quad \mathfrak{n}_\tau = ({}^\star A_0+\mathfrak{n})^{\mu_\tau} \quad \text{for} \quad \tau = 0, 1, \ldots, \varkappa-1 .$$

Since $\mathfrak{n}_\tau<\aleph_\alpha$ for every $\tau<\varkappa$, it follows that $\boldsymbol{L}_{\mathfrak{n}}\leqq\aleph_\alpha$. Letting $\boldsymbol{L}=\bigcup_{\mathfrak{n}<\aleph_\alpha}\boldsymbol{L}_{\mathfrak{n}}$ we therefore have ${}^\star\boldsymbol{L}\leqq\aleph_\alpha$. Hence we can arrange all the systems in $\boldsymbol{L}$ into an ω_α-termed sequence $\langle\mathfrak{B}_0, \mathfrak{B}_1, \ldots, \mathfrak{B}_\xi, \ldots\rangle$, and using II we can associate with each ordinal $\xi<\omega_\alpha$ a new extension $\mathfrak{B}'_\xi$ of $\mathfrak{A}_0$, which is equivalent to $\mathfrak{B}_\xi$ modulo A_0, in such a way that $\mathfrak{B}'_\xi\cap\mathfrak{B}'_\eta=\mathfrak{A}_0$ whenever $\xi<\eta<\omega_\alpha$. It follows by II, IV and V that we can associate with all the ordinals $\xi<\omega_\alpha$ systems $\mathfrak{C}_\xi\in\boldsymbol{K}$ such that $\mathfrak{B}'_\xi\prec\mathfrak{C}_\xi\prec\mathfrak{C}_\eta$ and $\mathfrak{C}_\xi\cap\mathfrak{B}'_\eta=\mathfrak{A}_0$ whenever $\xi<\eta<\omega_\alpha$. Letting $\mathfrak{C}=\bigcup_{\xi<\omega_\alpha}\mathfrak{C}_\xi$ we have $\mathfrak{B}'_\xi\prec\mathfrak{C}$ for every $\xi\prec\omega_\alpha$, and $\mathfrak{C}\in\boldsymbol{K}$ by V. By $\mathrm{VI}_{\alpha+1}$ there exists a subsystem $\mathfrak{A}\in\boldsymbol{K}$ of $\mathfrak{C}$ such that ${}^\star\mathfrak{A}\leqq\aleph_\alpha$ and $\mathfrak{B}'_\xi\prec\mathfrak{A}$ for every $\xi<\omega_\alpha$. Since every extension $\mathfrak{B}\in\boldsymbol{K}$ of $\mathfrak{A}_0$ with ${}^\star\mathfrak{B}<\aleph_\alpha$ is equivalent modulo $\mathfrak{A}_0$ to one of the subsystems $\mathfrak{B}'_\xi$ of $\mathfrak{A}$, we see that $\mathfrak{A}$ is an $(\aleph_\alpha, \boldsymbol{K})$ universal extension of $\mathfrak{A}_0$.

THEOREM 2.9. *Let α be an ordinal with the following two properties:*

(i) *If $\lambda<\omega_\alpha$ and if $\mathfrak{n}_\mu<\aleph_\alpha$ for every $\mu<\lambda$, then $\Sigma_{\mu<\lambda}\mathfrak{n}_\mu<\aleph_\alpha$.*

(ii) *If $\mathfrak{n}<\aleph_\alpha$, then $2^{\mathfrak{n}}\leqq\aleph_\alpha$.*

If $\boldsymbol{K}$ is a class of systems which satisfies the conditions I–V *and* VI_α, *then there exists an $(\aleph_\alpha, \boldsymbol{K})$ universal system.*

PROOF. From I and VI_α it follows that there exists $\mathfrak{A}_0\in\boldsymbol{K}$ such that

$^*\mathfrak{A}_0 < \aleph_\alpha$; we let $\mathfrak{A}_{0,\eta} = \mathfrak{A}_0$ for every $\eta < \omega_\alpha$. Now suppose $0 < \lambda < \omega_\alpha$, and assume that we have obtained systems $\mathfrak{A}_\xi\ \mathfrak{A}_{\xi,\eta} \in \boldsymbol{K}$, for all ordinals $\xi < \lambda$ and $\eta < \omega_\alpha$, satisfying the following conditions:

(1) $^*\mathfrak{A}_{\xi,\eta} < \aleph_\alpha$ for every $\xi < \lambda$ and $\eta < \omega_\alpha$.

(2) $\mathfrak{A}_{\xi,\eta} \prec \mathfrak{A}_{\xi',\eta'}$ whenever $\xi \leqq \xi' < \lambda$ and $\eta \leqq \eta' < \omega_\alpha$.

(3) $\mathfrak{A}_\xi = \bigcup_{\eta < \omega_\alpha} \mathfrak{A}_{\xi,\eta}$ for every $\xi < \lambda$.

(4) $\mathfrak{A}_{\xi+1}$ is an $(\aleph_\alpha, \boldsymbol{K})$ universal extension of $\mathfrak{A}_{\xi,\xi}$ whenever $\xi + 1 < \lambda$.

If λ is a limit ordinal, then we let $\mathfrak{A}_\lambda = \bigcup_{\xi < \lambda} \mathfrak{A}_\xi$ and $\mathfrak{A}_{\lambda,\eta} = \bigcup_{\xi < \lambda} \mathfrak{A}_{\xi,\eta}$ for every $\eta < \omega_\alpha$, and it is easy to see that $\mathfrak{A}_\lambda \in \boldsymbol{K}$ and $\mathfrak{A}_{\lambda,\eta} \in \boldsymbol{K}$ for every $\eta < \omega_\alpha$, and that the conditions (1)–(4) hold with λ replaced by $\lambda + 1$.

If λ is not a limit ordinal, then it is of the form $\lambda = \mu + 1$. By 2.8 there exists an $(\aleph_\alpha, \boldsymbol{K})$ universal extension $\mathfrak{A}'$ of $\mathfrak{A}_{\mu,\mu}$ with $^*\mathfrak{A}' \leqq \aleph_\alpha$. In view of II we may assume that $\mathfrak{A}_\mu \cap \mathfrak{A}' = \mathfrak{A}_{\mu,\mu}$, and it follows from IV, VI_α and 2.6 that there exists a common extension $\mathfrak{A}_\lambda \in \boldsymbol{K}$ of $\mathfrak{A}_\mu$ and $\mathfrak{A}'$ such that $^*\mathfrak{A}_\lambda \leqq \aleph_\alpha$. According to 2.5 we can find subsystems $\mathfrak{A}'_{\lambda,\eta} \in \boldsymbol{K}$ of $\mathfrak{A}_\lambda$, associated with all the ordinals $\eta < \omega_\alpha$, such that $^*\mathfrak{A}'_{\lambda,\eta} < \aleph_\alpha$ for every $\eta < \omega_\alpha$, $\mathfrak{A}'_{\lambda,\eta} \prec \mathfrak{A}'_{\lambda,\eta'}$ whenever $\eta < \eta' < \omega_\alpha$, and $\mathfrak{A}_\lambda = \bigcup_{\eta < \omega_\alpha} \mathfrak{A}'_{\lambda,\eta}$. Using VI_α and (i) we can associate subsystems $\mathfrak{A}_{\lambda,\eta} \in \boldsymbol{K}$ of $\mathfrak{A}_\lambda$ with all the ordinals $\eta < \omega_\alpha$ in such a way that $\mathfrak{A}_{\mu,\eta} \prec \mathfrak{A}_{\lambda,\eta}$, $\mathfrak{A}'_{\lambda,\eta} \prec \mathfrak{A}_{\lambda,\eta}$ and $^*\mathfrak{A}_{\lambda,\eta} < \aleph_\alpha$ for every $\eta < \omega_\alpha$, and $\mathfrak{A}_{\lambda,\eta} \prec \mathfrak{A}_{\lambda,\eta'}$ whenever $\eta < \eta' < \omega_\alpha$. It is now easy to see that, in this case also, the conditions (1)–(4) hold with λ replaced by $\lambda + 1$.

Having thus shown that we can always continue the process of picking out systems $\mathfrak{A}_\xi$ and $\mathfrak{A}_{\xi,\eta}$ subject to the conditions (1)–(4), we conclude that we can so choose $\mathfrak{A}_\xi$ and $\mathfrak{A}_{\xi,\eta}$ for every $\xi < \omega_\alpha$ and $\eta < \omega_\alpha$ that (1)–(4) hold with $\lambda = \omega_\alpha$. Letting

$$\mathfrak{A} = \bigcup_{\xi < \omega_\alpha} \mathfrak{A}_\xi \tag{5}$$

we infer from (1)–(3) and V that $\mathfrak{A} \in \boldsymbol{K}$ and $^*\mathfrak{A} \leqq \aleph_\alpha$. We are going to show that $\mathfrak{A}$ is an $(\aleph_\alpha, \boldsymbol{K})$ universal system.

Suppose $\mathfrak{B} \in \boldsymbol{K}$ and $^*\mathfrak{B} \leqq \aleph_\alpha$. Then it follows from 2.5 that there exist subsystems $\mathfrak{B}_\mu \in \boldsymbol{K}$ of $\mathfrak{B}$, associated with all the ordinals $\mu < \omega_\alpha$, such that $^*\mathfrak{B}_\mu < \aleph_\alpha$ for every $\mu < \omega_\alpha$, $\mathfrak{B}_\mu \prec \mathfrak{B}_\nu$ whenever $\mu < \nu < \omega_\alpha$, and $\mathfrak{B} = \bigcup_{\mu < \omega_\alpha} \mathfrak{B}_\mu$.

According to II, III and VI_α there exists an extension $\mathfrak{B}' \in \boldsymbol{K}$ of $\mathfrak{B}_0$ such that $^*\mathfrak{B}' < \aleph_\alpha$ and $\mathfrak{A}_{0,0}$ is isomorphic to a subsystem of $\mathfrak{B}'$. Since $\mathfrak{A}_1$ is an $(\aleph_\alpha, \boldsymbol{K})$ universal extension of $\mathfrak{A}_{0,0}$, it follows that $\mathfrak{B}'$ is isomorphic to a subsystem of $\mathfrak{A}_1$. Hence there exists a function φ_0 which maps $\mathfrak{B}_0$ isomorphically onto a subsystem $\mathfrak{C}_0$ of $\mathfrak{A}_1$.

Now suppose $0 < \lambda < \omega_\alpha$, and assume that we have associated with each ordinal $\mu<\lambda$ a function φ_μ mapping $\mathfrak{B}_\mu$ isomorphically onto a subsystem $\mathfrak{C}_\mu$ of $\mathfrak{A}$ in such a way that

(6) $\quad \varphi_\mu(x) = \varphi_\nu(x) \quad$ whenever $\quad \mu<\nu<\lambda \quad$ and $\quad x\in\mathfrak{B}_\mu$.

Letting

$$\mathfrak{B}'_\lambda = \bigcup_{\mu<\lambda}\mathfrak{B}_\mu \quad \text{and} \quad \mathfrak{C}'_\lambda = \bigcup_{\mu<\lambda}\mathfrak{C}_\mu ,$$

we infer that $\mathfrak{B}'_\lambda, \mathfrak{C}'_\lambda \in \boldsymbol{K}$, $\mathfrak{B}'_\lambda \prec \mathfrak{B}_\lambda$, $\mathfrak{C}'_\lambda \prec \mathfrak{A}$, and that there exists a unique function ψ mapping $\mathfrak{B}'_\lambda$ isomorphically onto $\mathfrak{C}'_\lambda$ in such a way that $\psi(x)=\varphi_\mu(x)$ whenever $\mu<\lambda$ and x is an element of $\mathfrak{B}_\mu$. Since ${}^\star\mathfrak{C}'_\lambda = {}^\star\mathfrak{B}'_\lambda \leqq {}^\star\mathfrak{B}_\lambda < \aleph_\alpha$, it follows from (i), (2) and (5) that there exists $\xi<\omega_\alpha$ such that $\mathfrak{C}'_\lambda \prec \mathfrak{A}_\xi$. Hence, by (i), (2) and (3), $\mathfrak{C}'_\lambda \prec \mathfrak{A}_{\xi,\eta}$ for some $\eta<\omega_\alpha$. Letting ζ be the larger of the two ordinals ξ and η, we infer by (2) that $\mathfrak{C}'_\lambda \prec \mathfrak{A}_{\zeta,\zeta}$. Now $\mathfrak{A}_{\zeta+1}$ is an $(\aleph_\alpha, \boldsymbol{K})$ universal extension of $\mathfrak{A}_{\zeta,\zeta}$, so that, by 2.7, $\mathfrak{A}_{\zeta+1}$ is an $(\aleph_\alpha, \boldsymbol{K})$ universal extension of $\mathfrak{C}'_\lambda$. We can therefore find a function φ_λ which maps $\mathfrak{B}_\lambda$ isomorphically onto a subsystem $\mathfrak{C}_\lambda$ of $\mathfrak{A}_{\zeta+1}$ in such a way that $\varphi_\lambda(x)=\psi(x)$ for every element x of $\mathfrak{B}'_\lambda$. It readily follows that (6) holds with λ replaced by $\lambda+1$.

We have shown that we can always continue the process of picking functions φ_μ mapping the systems $\mathfrak{B}_\mu$ isomorphically onto subsystems $\mathfrak{C}_\mu$ of $\mathfrak{A}$, subject to the condition (6), and we conclude that we can so choose φ_μ for every $\mu<\omega_\alpha$ that (6) holds with $\lambda=\omega_\alpha$. Consequently there exists a unique function φ mapping $\mathfrak{B}$ isomorphically onto the subsystem $\mathfrak{C}=\bigcup_{\mu<\omega_\alpha}\mathfrak{C}_\mu$ of $\mathfrak{A}$, such that $\varphi(x)=\varphi_\mu(x)$ whenever $\mu<\omega_\alpha$ and x is an element of $\mathfrak{B}_\mu$. Thus every system $\mathfrak{B}\in\boldsymbol{K}$ with ${}^\star\mathfrak{B}\leqq\aleph_\alpha$ is isomorphic to a subsystem of $\mathfrak{A}$. Since by 2.4 there exists $\mathfrak{B}\in\boldsymbol{K}$ with ${}^\star\mathfrak{B}=\aleph_\alpha$, we conclude that $\mathfrak{A}$ is an $(\aleph_\alpha, \boldsymbol{K})$ universal system.

THEOREM 2.10. *If $\boldsymbol{K}$ is a class of systems which satisfies the conditions* I–V *and* VI_0, *then there exists an* $(\aleph_0, \boldsymbol{K})$ *universal system.*

PROOF. By 2.9.

THEOREM 2.11. *If the Generalized Continuum Hypothesis holds, if $\boldsymbol{K}$ is a class of systems which satisfies the conditions* I–V *and* VI_1, *and if α is a positive ordinal, then there exists an* $(\aleph_\alpha, \boldsymbol{K})$ *universal system.*

PROOF. It follows from the hypothesis of the present theorem and from 2.6 that VI_α and 2.9 (ii) hold. If α is not a limit ordinal, then 2.9 (i) is also satisfied, and we conclude from 2.9 that there exists an $(\aleph_\alpha, \boldsymbol{K})$ universal system.

Now suppose α is a limit ordinal. Using 2.6, 2.8 and V we can associate systems $\mathfrak{A}_\mu\in\boldsymbol{K}$ with all the ordinals $\mu\leqq\alpha$ in such a way that the following conditions are satisfied:

(1) $\mathfrak{A}_0$ is an $(\aleph_1, \boldsymbol{K})$ universal system.

(2) $^\star\mathfrak{A}_\mu \leqq \aleph_{1+\mu}$ for every $\mu \leqq \alpha$.

(3) $\mathfrak{A}_{\mu+1}$ is an $(\aleph_{1+\mu+1}, \boldsymbol{K})$ universal extension of $\mathfrak{A}_\mu$ for every $\mu < \alpha$.

(4) $\mathfrak{A}_\mu = \bigcup_{\nu<\mu} \mathfrak{A}_\nu$ whenever $0 < \mu \leqq \alpha$ and μ is a limit ordinal.

We shall prove that $\mathfrak{A}_\alpha$ is an $(\aleph_\alpha, \boldsymbol{K})$ universal system.

Suppose $\mathfrak{B} \in \boldsymbol{K}$ and $^\star\mathfrak{B} = \aleph_\alpha$. Using 2.5, choose subsystems $\mathfrak{B}'_\xi \in \boldsymbol{K}$ of $\mathfrak{B}$, associated with all the ordinals $\xi < \omega_\alpha$, in such a way that $^\star\mathfrak{B}'_0 \leqq \aleph_0$, $^\star\mathfrak{B}'_\xi < \aleph_\alpha$ for every $\xi < \omega_\alpha$, $\mathfrak{B}'_\xi \prec \mathfrak{B}'_\eta$ whenever $\xi < \eta < \omega_\alpha$, and $\mathfrak{B} = \bigcup_{\xi<\omega_\alpha} \mathfrak{B}'_\xi$. Let $\mathfrak{B}_0 = \mathfrak{B}'_0$. If $0 < \mu < \alpha$, then there exists an ordinal $\xi < \omega_\alpha$ such that $^\star\mathfrak{B}'_\xi \geqq \aleph_\mu$. Letting ξ_μ be the smallest such ordinal, let $\mathfrak{B}_\mu = \bigcup_{\xi<\xi_\mu} \mathfrak{B}'_\xi$. Also let $\mathfrak{B}_\alpha = \mathfrak{B}$. Then $\mathfrak{B}_\mu \in \boldsymbol{K}$ and $^\star\mathfrak{B}_\mu \leqq \aleph_\mu$ for every $\mu \leqq \alpha$, $\mathfrak{B}_\mu \prec \mathfrak{B}_\nu$ whenever $\mu < \nu \leqq \alpha$, and $\mathfrak{B}_\mu = \bigcup_{\nu<\mu} \mathfrak{B}_\nu$ whenever $0 < \mu \leqq \alpha$ and μ is a limit ordinal. In order to verify the last assertion for the case when $\mu = \alpha$, we observe that if $\xi < \omega_\alpha$, then $^\star\mathfrak{B}'_\xi < \aleph_\alpha$ and hence $^\star\mathfrak{B}'_\xi < \aleph_\nu$ for some $\nu < \alpha$, but this implies that $\xi < \xi_\nu$ and hence $\mathfrak{B}'_\xi \prec \mathfrak{B}_\nu$.

By (1) there exists a function φ_0 which maps $\mathfrak{B}_0$ isomorphically onto a subsystem of $\mathfrak{A}_0$, and using (2), (3), (4) and 2.7 we can successively associate functions φ_μ with all the ordinals $\mu \leqq \alpha$ in such a way that φ_μ maps $\mathfrak{B}_\mu$ isomorphically onto a subsystem of $\mathfrak{A}_\mu$ for every $\mu \leqq \alpha$, and $\varphi_\mu(x) = \varphi_\nu(x)$ whenever $\mu < \nu < \alpha$ and x is an element of $\mathfrak{B}_\mu$. Consequently $\mathfrak{B} = \mathfrak{B}_\alpha$ is isomorphic to a subsystem of $\mathfrak{A}_\alpha$, and the proof is complete.

We do not know whether, in the above theorem, the assumption that VI_1 hold could be replaced by the weaker assumption that VI_α hold. In the present proof for the case when α is a limit ordinal, essential use is made of 2.8 with α replaced by smaller ordinals, and our proof would therefore not apply under the modified assumption. We could nevertheless weaken the hypothesis somewhat by assuming only that VI_β holds for some $\beta < \alpha$; this would require only a minor change in our reasoning. However, since we restrict ourselves to systems with finitely many relations of finite rank (this is used explicitly or implicitly several times in this section), most classes $\boldsymbol{K}$ which arise in a natural way, and to which our results can be applied, do in fact satisfy the condition VI_1, so that generalizations in this direction do not appear to be of any great interest.

3. Applications. We shall now apply the principal results of the preceding section, Theorems 2.10 and 2.11, to certain specific classes of relational systems. By an $\aleph_\alpha$ *universal group*, an $\aleph_\alpha$ *universal groupoid*, an $\aleph_\alpha$ *universal lattice*, etc., we shall of course mean an $(\aleph_\alpha, \boldsymbol{K})$ universal relational system where $\boldsymbol{K}$ is, respectively, the class of all groups, the class of all groupoids, the class of all lattices, etc.

THEOREM 3.1. *If the Generalized Continuum Hypothesis holds, and if α is any positive ordinal, then there exists an $\aleph_\alpha$ universal group*[1].

PROOF. Since, as was pointed out in the discussion preceding 2.4, the class $\boldsymbol{K}$ of all groups satisfies the conditions I–V and VI_1, the present theorem follows from 2.11.

By a *groupoid* we mean a system $\langle A, \cdot \rangle$ where A is a non-empty set, $\cdot$ is a binary operation, and A is closed under the operation $\cdot$.

THEOREM 3.2. *If the Generalized Continuum Hypothesis holds, and if α is any positive ordinal, then there exists an $\aleph_\alpha$ universal groupoid*[2].

PROOF. It is obvious that the class $\boldsymbol{K}$ of all groupoids satisfies the conditions I, II, III, V and VI_1. Our theorem will therefore follow from 2.11 if we show that the condition IV also holds for this class $\boldsymbol{K}$.

Consider two groupoids $\mathfrak{A}' = \langle A', \cdot' \rangle$ and $\mathfrak{A}'' = \langle A'', \cdot'' \rangle$, and assume that $\mathfrak{A}' \cap \mathfrak{A}''$ is also a groupoid. This implies in particular that $x \cdot' y = x \cdot'' y$ whenever $x, y \in A' \cap A''$. Letting $A = A' \cup A''$, and choosing an element $c \in A$, we can therefore define a binary operation $\cdot$ on A in such a way that

$$\begin{aligned} x \cdot y &= x \cdot' y && \text{for every} \quad x, y \in A', \\ x \cdot y &= x \cdot'' y && \text{for every} \quad x, y \in A'', \\ x \cdot y &= y \cdot x = c && \text{for every} \quad x \in A - A' \quad \text{and} \quad y \in A - A''. \end{aligned}$$

Hence the new groupoid $\mathfrak{A} = \langle A, \cdot \rangle$ is an extension of both $\mathfrak{A}'$ and $\mathfrak{A}''$. Thus IV holds, and the proof is complete.

By a *partially ordered system* we shall mean a relational system $\langle A, R \rangle$ where R is a binary relation which partially orders A. In the lemma which follows we make use of the notion of the *relative product*, $R;S$, of two binary relations R and S. By this we mean the new binary relation T consisting of all ordered pairs $\langle x, y \rangle$ such that, for some element z, $\langle x, z \rangle \in R$ and $\langle z, y \rangle \in S$.

LEMMA 3.3. *Suppose $\mathfrak{A} = \langle A, R \rangle$ and $\mathfrak{B} = \langle B, S \rangle$ are partially ordered systems such that $\mathfrak{A} | (A \cap B) = \mathfrak{B} | (A \cap B)$. If*

$$C = A \cup B \quad \text{and} \quad T = R \cup S \cup (R;S) \cup (S;R),$$

---

[1] In Neumann and Neumann [7] it is shown that there does not exist an $\aleph_0$ universal group.

[2] In Evans and Neumann [2] it is shown that there exist infinitely many independent identities in one operation and two variables. From this it readily follows that there does not exist an $\aleph_0$ universal groupoid.

then $\langle C, T\rangle$ *is a partially ordered system and an extension of* $\mathfrak{A}$ *and* $\mathfrak{B}$. *Furthermore*

$$\text{(i)} \qquad T\cap A^2 = R, \quad T\cap(A\times B) = R;S, \quad T\cap(B\times A) = S;R, \quad T\cap B^2 = S\,.$$

PROOF. It is obvious that $T\subseteq C^2$ and that $\langle x, x\rangle\in T$ for every $x\in C$. In view of the transitivity of R and S we have

$$T;T = R\cup S\cup(R;S)\cup(S;R)\cup(R;S;R)\cup(S;R;S) \cup(R;S;R;S)\cup(S;R;S;R)\,.$$

If $\langle x, y\rangle\in R;S;R$, then there exist $u,v\in C$ such that $\langle x, u\rangle\in R$, $\langle u, v\rangle\in S$ and $\langle v, y\rangle\in R$. It follows that $u,v\in A\cap B$, and hence that $\langle u, v\rangle\in R$, $\langle x, y\rangle\in R;R;R=R$. Thus $R;S;R\subseteq R$ and, similarly, $S;R;S\subseteq S$. With the aid of the last two formulae we find that $R;S;R;S\subseteq R;S$ and $S;R;S;R\subseteq S;R$. Consequently

$$T;T = R\cup S\cup(R;S)\cup(S;R) = T\,,$$

so that T is transitive.

We shall next prove the four formulae listed in (i). First suppose $\langle x, y\rangle\in T\cap A^2$. Then $x,y\in A$ and $\langle x, y\rangle\in T$. Hence one of the following four conditions holds:

$$\text{(1)} \qquad \begin{array}{l}\langle x, y\rangle \in R, \quad \langle x, u\rangle \in R \ \text{ and } \ \langle u, y\rangle \in S \ \text{ for some } u\,, \\ \langle x, u\rangle \in S \ \text{ and } \ \langle u, y\rangle \in R \ \text{ for some } u, \quad \langle x, y\rangle \in S\,.\end{array}$$

If the second condition in (1) holds, then $u, y\in A\cap B$ and hence $\langle u, y\rangle\in R$, so that $\langle x, y\rangle\in R;R=R$. Similarly, if the third condition in (1) holds, then $x,u\in A\cap B$, $\langle x, u\rangle\in R$, $\langle x, y\rangle\in R$. If the fourth condition in (1) holds, then $x,y\in A\cap B$ and hence $\langle x,y\rangle\in R$. Thus $T\cap A^2\subseteq R$. Since the inclusion in the opposite direction is obvious, we conclude that the first formula in (i) holds.

Next suppose $\langle x, y\rangle\in T\cap(A\times B)$. Then $x\in A$, $y\in B$ and $\langle x, y\rangle\in T$. Again one of the four conditions in (1) holds. If the first (fourth) condition holds, then we use the fact that $\langle y, y\rangle\in S$ ($\langle x, x\rangle\in R$) to infer that $\langle x, y\rangle\in R;S$. If the third condition in (1) holds, then $x,y,u\in A\cap B$ and hence $\langle x,u\rangle\in R$ and $\langle u,y\rangle\in S$, so that $\langle x,y\rangle\in R;S$. Thus $T\cap(A\times B)\subseteq R;S$. The inclusion in the opposite direction being obvious, we conclude that the second formula in (i) holds. The third and the fourth formulae can be proved similarly to the second and the first.

We next show that T is asymmetric. Suppose $\langle x, y\rangle\in T$ and $\langle y, x\rangle\in T$.

If $x, y \in A$, then $\langle x, y\rangle \in R$ and $\langle y, x\rangle \in R$ by the first formula in (i), so that $x=y$ by the asymmetry of R. If $x \in A$ and $y \in B$, then $\langle x, y\rangle \in R;S$ and $\langle y, x\rangle \in S;R$ by the second and third formulae in (i). Hence there exist elements u and v such that

$$\langle x, u\rangle \in R, \quad \langle u, y\rangle \in S, \quad \langle y, v\rangle \in S, \quad \langle v, x\rangle \in R .$$

Consequently $\langle v, u\rangle \in R;R=R$ and $\langle u, v\rangle \in S;S=S$. Since $u, v \in A \cap B$, it follows that $\langle u, v\rangle \in R$ and hence, by the asymmetry of R, that $u=v$. Thus both the ordered pairs $\langle x, u\rangle$ and $\langle u, x\rangle$ belong to R, whence $x=u$. Similarly $y=u$ so that, finally, $x=y$. If either $x \in B$ and $y \in A$ or else $x, y \in B$, then we procede as in the two cases already considered and conclude that, in these cases also, $x=y$. Thus T is asymmetric.

We have thus shown that $\mathfrak{C}=\langle C, T\rangle$ is a partially ordered system. Finally it follows from the first and last formulae in (i) that $\mathfrak{C}$ is an extension of both $\mathfrak{A}$ and $\mathfrak{B}$. This completes the proof.

THEOREM 3.4. *There exists an $\aleph_0$ universal partially ordered system. Furthermore, if the Generalized Continuum Hypothesis holds, and if α is any positive ordinal, then there exists an $\aleph_\alpha$ universal partially ordered system.*

PROOF. It is obvious that the class $\boldsymbol{K}$ of all partially ordered systems satisfies the conditions I, II, III and V. By **3.3** the condition IV is also satisfied. Finally, since every subsystem of a partially ordered system is again a partially ordered system, we see that the condition VI_α holds for every ordinal α. The present theorem therefore follows from 2.10 and 2.11.

THEOREM 3.5. *If the Generalized Continuum Hypothesis holds, and if α is any positive ordinal, then there exists an $\aleph_\alpha$ universal lattice.*

PROOF. Clearly the class $\boldsymbol{K}$ of all lattices satisfies the conditions I, II, III, V and VI_1. By 2.11 it is therefore sufficient to show that the condition IV holds.

Consider two lattices

$$\mathfrak{A} = \langle A, +', \cdot'\rangle \quad \text{and} \quad \mathfrak{B} = \langle B, +'', \cdot''\rangle ,$$

and assume that $\mathfrak{A} \cap \mathfrak{B}$ is also a lattice. This means that the set $A \cap B$ is non-empty and that

$$x+'y = x+''y \in A \cap B \quad \text{and} \quad x\cdot' y = x\cdot'' y \in A \cap B$$
$$\text{for every} \quad x, y \in A \cap B .$$

Let R and S be the inclusion relations of the two lattices $\mathfrak{A}$ and $\mathfrak{B}$. Then the systems

$$\mathfrak{A}_1 = \langle A, R\rangle \quad \text{and} \quad \mathfrak{B}_1 = \langle B, S\rangle$$

are partially ordered systems with

$$\mathfrak{A}_1 \cap \mathfrak{B}_1 = \mathfrak{A}_1|(A \cap B) = \mathfrak{B}_1|(A \cap B) .$$

Letting

$$C_1 = A \cup B \quad \text{and} \quad T = R \cup S \cup (R;S) \cup (S;R) ,$$

we therefore know by 3.3 that the system $\mathfrak{C}_1 = \langle C_1, T\rangle$ is a partially ordered system and an extension of both $\mathfrak{A}_1$ and $\mathfrak{B}_1$. We shall next show that this extension preserves the least upper bounds and the greatest lower bounds in $\mathfrak{A}_1$ and $\mathfrak{B}_1$.

Suppose $x, y \in A$ and let $z = x +' y$. Then $\langle x, z\rangle \in R$ and $\langle y, z\rangle \in R$, so that $\langle x, z\rangle \in T$ and $\langle y, z\rangle \in T$. Now consider any element $t \in C$ such that $\langle x, t\rangle \in T$ and $\langle y, t\rangle \in T$. If $t \in A$, then $\langle x, t\rangle \in R$ and $\langle y, t\rangle \in R$ by 3.3(i), so that $\langle z, t\rangle \in R$ and, consequently, $\langle z, t\rangle \in T$. If $t \in B$, then it follows from 3.3(i) that there exist $u, v \in C$ such that

$$\text{(1)} \qquad \langle x, u\rangle \in R, \quad \langle u, t\rangle \in S, \quad \langle y, v\rangle \in R, \quad \langle v, t\rangle \in S .$$

Observing that $u, v \in A \cap B$, we let $w = u +' v = u +'' v$ and infer that $\langle u, w\rangle \in R \cap S$ and $\langle v, w\rangle \in R \cap S$. Consequently $\langle x, w\rangle \in R$ and $\langle y, w\rangle \in R$, which implies that $\langle z, w\rangle \in R$. Since by the second and fourth formulae in (1), and by the definition of w, we have $\langle w, t\rangle \in S$, it follows that $\langle z, t\rangle \in R;S \subseteq T$. Thus $x +' y$ is the least upper bound of x and y with respect to T. Similarly we see that if $x, y \in A$, then $x \cdot' y$ is the greatest lower bound of x and y with respect to T, while for any $x, y \in B$ the elements $x +'' y$ and $x \cdot'' y$ are, respectively, the least upper bound and the greater lower bound of x and y with respect to T.

By a known theorem (cf. MacNeille [5, Theorem 11.9, p. 444] or Birkhoff [1, Theorem 12, p. 58]) there exists a lattice $\mathfrak{C} = \langle C, +, \cdot\rangle$ such that $C_1 \subseteq C$ and such that if the elements $x, y \in C_1$ have a least upper bound (greatest lower bound) u with respect to T, then $u = x + y$ ($u = x \cdot y$). Consequently $\mathfrak{C}$ is an extension of both $\mathfrak{A}$ and $\mathfrak{B}$, and the proof of the theorem is complete.

4. The condition IV. The classes $\boldsymbol{K}$ of relational systems considered in the preceding section were easily seen to satisfy the conditions I, II, III, V and VI_1, while the verification of the condition IV was less trivial. We shall now consider certain classes $\boldsymbol{K}$ of relational systems, for which it turns out that the condition IV fails.

By a *demi-group* we mean an associative groupoid. A demi-group $\mathfrak{A}=\langle A, \cdot\rangle$ is said to be freely generated by a subset X of A if $\mathfrak{A}$ is generated by X and if every mapping of X into another demi-group $\mathfrak{B}$ can be extended to a homomorphism mapping $\mathfrak{A}$ onto a sub-demi-group of $\mathfrak{B}$. We obtain a free demi-group $\mathfrak{A}=\langle A, \cdot\rangle$ generated by a given set X by letting A be the set of all finite non-void sequences whose terms belong to X and letting $\cdot$ be the operation of juxtaposition.

Consider two demi-groups

$$\mathfrak{A}' = \langle A', \cdot'\rangle \quad \text{and} \quad \mathfrak{A}'' = \langle A'', \cdot''\rangle$$

which are freely generated by the elements x, y, u and y, z, v, respectively. Let $\mathfrak{B}'$ and $\mathfrak{B}''$ be the sub-demi-groups generated by the elements $y, u, x\cdot' y$ and $y, v, y\cdot'' z$, respectively. It is easy to see that $\mathfrak{B}'$ and $\mathfrak{B}''$ are freely generated by these elements, whence there exists a function φ mapping $\mathfrak{B}'$ isomorphically onto $\mathfrak{B}''$ in such a way that

$$\varphi(y) = y, \quad \varphi(u) = y\cdot'' z \quad \text{and} \quad \varphi(x\cdot' y) = v\,.$$

We may therefore assume that

$$\mathfrak{A}'\cap\mathfrak{A}'' = \mathfrak{B}' = \mathfrak{B}'', \quad u = y\cdot'' z \quad \text{and} \quad x\cdot'' y = v\,.$$

Now suppose $\mathfrak{A}=\langle A, \cdot\rangle$ is a groupoid such that $\mathfrak{A}' \prec \mathfrak{A}$ and $\mathfrak{A}'' \prec \mathfrak{A}$. Then

$$x\cdot(y\cdot z) = x\cdot(y\cdot'' z) = x\cdot u = x\cdot' u\,,$$
$$(x\cdot y)\cdot z = (x\cdot' y)\cdot z = v\cdot z = v\cdot'' z\,.$$

However, $x\cdot' u$ is an element of $\mathfrak{A}'$ which is not an element of $\mathfrak{B}'$, while $v\cdot'' z$ belongs to $\mathfrak{A}''$ but not to $\mathfrak{B}''$. Since $\mathfrak{A}'\cap\mathfrak{A}''=\mathfrak{B}'=\mathfrak{B}''$, it follows that

$$x\cdot' u \neq v\cdot'' z, \quad \text{hence} \quad x\cdot(y\cdot z) \neq (x\cdot y)\cdot z\,.$$

Thus $\mathfrak{A}$ is not a demi-group. We therefore see that the class $\boldsymbol{K}$ of all demi-groups does not satisfy the condition IV. Incidentally, since $\mathfrak{A}'$ and $\mathfrak{A}''$ are semi-groups (demi-groups which satisfy the cancellation law), we can also infer that the condition IV fails if $\boldsymbol{K}$ is the class of all semi-groups.

Next consider the class $\boldsymbol{K}$ of all distributive lattices, and assuming that the elements x, y, z, u, v are all distinct, let

$$\mathfrak{A}' = \langle A', +', \cdot'\rangle \quad \text{and} \quad \mathfrak{A}'' = \langle A'', +'', \cdot''\rangle$$

be the four-element lattices whose elements are x, y, u, v and y, z, u, v, respectively, and which are characterized by the conditions

$$x+' y = u, \quad x\cdot' y = v, \quad y+'' z = u, \quad y\cdot'' z = v\,.$$

These two lattices are distributive, and the system $\mathfrak{A}' \cap \mathfrak{A}''$ is also a distributive lattice and hence a sublattice of both $\mathfrak{A}'$ and $\mathfrak{A}''$. Now consider any lattice $\mathfrak{A} = \langle A, +, \cdot \rangle$ such that $\mathfrak{A}' \prec \mathfrak{A}$ and $\mathfrak{A}'' \prec \mathfrak{A}$. Then

$$x+y = x+'y = u, \qquad y+z = y+''z = u,$$
$$x\cdot y = x\cdot' y = v, \qquad y\cdot z = y\cdot'' z = v.$$

Thus $x+y=z+y$ and $x\cdot y = z\cdot y$. Since $x \neq z$, this implies that $\mathfrak{A}$ is not distributive. Thus $\boldsymbol{K}$ does not satisfy the condition IV.

In proving that the class $\boldsymbol{K}$ of all groups satisfies the condition IV, we used the concept of a free product of two groups with an amalgamated subgroup. This notion can be generalized as follows: A relational system $\mathfrak{A} = \langle A, R_0, R_1, \ldots, R_{\varkappa-1} \rangle$ having the similarity type $\langle \mu_0, \mu_1, \ldots, \mu_{\varkappa-1} \rangle$ is called an *algebraic system* if, for each $\tau < \varkappa$ and for every

$$x_0, x_1, \ldots, x_{\mu_\tau - 2} \in A,$$

there exists a unique $x_{\mu_\tau - 1} \in A$ such that $\langle x_0, x_1, \cdots, x_{\mu_\tau - 1} \rangle \in R_\tau$; that is if all the relations $R_0, R_1, \ldots, R_{\varkappa-1}$ are operations under which the set A is closed. The notion of a homomorphism of one algebraic system into another is defined in an obvious manner. An algebraic system $\mathfrak{A} = \langle A, R_0, R_1, \ldots, R_{\varkappa-1} \rangle$ is said to be generated by the subset X of A if there exists no algebraic system $\mathfrak{B}$ such that $\mathfrak{A} | X \prec \mathfrak{B} \prec \mathfrak{A}$ and $\mathfrak{B} \neq \mathfrak{A}$. Now suppose $\boldsymbol{K}$ is a class of algebraic systems, and assume that

$$\mathfrak{A} = \langle A, R_0, R_1, \cdots, R_{\varkappa-1} \rangle \quad \text{and} \quad \mathfrak{B} = \langle B, S_0, S_1, \cdots, S_{\varkappa-1} \rangle$$

are two systems belonging to $\boldsymbol{K}$, such that $(\mathfrak{A} \cap \mathfrak{B}) \in \boldsymbol{K}$. A system $\mathfrak{C} \in \boldsymbol{K}$ is called an *amalgam* of $\mathfrak{A}$ and $\mathfrak{B}$ with respect to $\boldsymbol{K}$ if the following two conditions are satisfied:

(A) $\mathfrak{A} \prec \mathfrak{C}$, $\mathfrak{B} \prec \mathfrak{C}$, and $\mathfrak{C}$ is generated by the set $A \cup B$.

(B) For any system $\mathfrak{C}' \in \boldsymbol{K}$, if $\mathfrak{A} \prec \mathfrak{C}'$ and $\mathfrak{B} \prec \mathfrak{C}'$, then there exists a function φ mapping $\mathfrak{C}$ homomorphically into $\mathfrak{C}'$ in such a way that $\varphi(x) = x$ for every element $x \in A \cup B$.

Given a class $\boldsymbol{K}$ of algebraic systems, consider the following condition on $\boldsymbol{K}$:

IV$'$. *For any* $\mathfrak{A}, \mathfrak{B} \in \boldsymbol{K}$, *if* $(\mathfrak{A} \cap \mathfrak{B}) \in \boldsymbol{K}$, *then there exists* $\mathfrak{C} \in \boldsymbol{K}$ *such that* $\mathfrak{C}$ *is an amalgam of* $\mathfrak{A}$ *and* $\mathfrak{B}$ *with respect to* $\boldsymbol{K}$.

It is clear that IV$'$ implies IV. Hence the condition IV$'$ fails if $\boldsymbol{K}$ is the class of all demi-groups, the class of all semi-groups, or the class of all distributive lattices. In general IV does not imply IV$'$. However, it can be shown that if the class $\boldsymbol{K}$ of algebraic systems is closed under the operation of taking algebraic subsystems and under the operation of taking direct products with arbitrarily many factors, and if the condition IV holds, then the condition IV′ is also satisfied. Thus in particular, IV′ holds if $\boldsymbol{K}$ is the class of all groupoids or the class of all lattices.

Added in proofs: R. Fraïssé in his note, *Sur certaines relations qui généralisent l'ordre des nombres rationnels*, C. R. Acad. Sci. Paris 237 (1953), 540–542, considers classes $\boldsymbol{K}$ satisfying I–V and also the following condition which is stronger than VI_0:

$$\text{If } \mathfrak{A} \in \boldsymbol{K} \text{ and } \mathfrak{B} \prec \mathfrak{A}, \text{ then } \mathfrak{B} \in \boldsymbol{K}.$$

For such classes $\boldsymbol{K}$ he announces the existence of an $(\aleph_0, \boldsymbol{K})$ universal system $\mathfrak{A}$ with the property that any isomorphism between finite subsystems of $\mathfrak{A}$ can be extended to an automorphism of $\mathfrak{A}$. This additional property makes $\mathfrak{A}$ unique up to isomorphisms.

BIBLIOGRAPHY

1. G. Birkhoff, *Lattice theory*, Revised edition (Amer. Math. Soc. Colloquium Publications 25), New York, 1948.
2. T. Evans and B. H. Neumann, *On varieties of groupoids and loops*, J. London Math. Soc. 28 (1953) 342–350.
3. F. Hausdorff, *Grundzüge der Mengenlehre*, Leipzig 1914.
4. J. B. Johnston, *Universal infinite partially ordered systems*, Proc. Amer. Math. Soc. 7 (1956), 507–514.
5. H. MacNeille, *Partially ordered sets*, Trans. Amer. Math. Soc. 42 (1937), 416–460.
6. A. Mostowski, *Über gewisse universelle Relationen*, Ann. Soc. Polon. Math. 17 (1938), 117–118.
7. B. H. Neumann and H. Neumann, *A remark on generalized free products*, J. London Math. Soc. 25 (1950), 202–204.
8. O. Schreier, *Die Untergruppen der freien Gruppen*, Abh. Math. Sem. Hamburg 5 (1927), 161–183.

BROWN UNIVERSITY, PROVIDENCE, R.I., U.S.A.,
UNIVERSITY OF ICELAND, REYKJAVÍK,
AND
UNIVERSITY OF CALIFORNIA AT BERKELY, CALIF., U.S.A.