MATH. SCAND. 4 (1956), 209–230

# A GENERALIZATION OF THE IDEAL THEORY OF COMMUTATIVE RINGS WITHOUT FINITENESS ASSUMPTIONS

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**Introduction.** The object of the present paper is to give a lattice generalization of Krull's ideal theory of commutative rings without finiteness assumptions (see [7]), which in particular includes rings, distributive lattices and semi-groups as special cases. (Some of our results have been stated without proofs in [1]; for the non-commutative case see [2] and [4].) We shall develop the theory in a complete Boolean algebra over which are defined an operation of multiplication and an operation of subtraction both subject to a number of conditions. The axiom system to be given looks at first sight somewhat complicated, but is understandable if we keep the subset calculus of a commutative ring in mind. In short, what we shall do amounts to the same thing as to develop the Krull theory entirely in terms of subsets, avoiding any reference to the elements of the original algebraic system. Using the language of Boolean algebra this means that we do not rely on atomic assumptions. The fact that we are able to exhibit examples of non-atomic Boolean algebras satisfying all our conditions promises that the theory may also have applications essentially different from the standard applications to the atomic Boolean algebra of all subsets of a given algebraic system.

V. S. Krishnan [6] has given an entirely different lattice generalization of Krull's theory. Krishnan's theory is rather complicated and it does not seem to have any essential applications which may not be subsumed under the present theory. A really simple lattice-theoretic approach to the Krull theory is still lacking.

In the present paper we have of course not tried to give any exhaustive account of what results of ideal theory carry over to this general setting. We believe, however, that the results and proofs of this note should sufficiently illustrate the general method to the extent that it should be clear how other similar results could be derived.

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Received April 30, 1956.

**1. Multiplicative lattices over which an operation of subtraction is defined.** We shall not develop any theory of multiplicative lattices over which a subtraction is defined. We shall content ourselves with giving some basic definitions concerning such lattices as will be needed in the sequel.

By a (commutative and associative) multiplicative lattice we mean a lattice $L$ over which a binary commutative and associative multiplication is defined such that

$$a(b\cup c) = ab\cup ac\,.$$

In addition, in the following we shall suppose that $L$ is complete and satisfies the unrestricted distributive law

$$a\bigcup_{i\in I} b_i = \bigcup_{i\in I} ab_i\,.$$

Further we shall assume that there is defined a binary *subtraction* in $L$ which satisfies the unrestricted distributive laws

$$a-\bigcup_{i\in I} b_i = \bigcup_{i\in I}(a-b_i) \qquad \text{and} \qquad \Bigl(\bigcup_{i\in I} b_i\Bigr)-a = \bigcup_{i\in I}(b_i-a)\,.$$

$L$ is said to have a greatest element $u$ if $x\subseteq u$ for all $x\in L$. Dually, $z$ is a least element of $L$ if $z\subseteq x$ for all $x\in L$. These two elements will always be denoted by $u$ and $z$, respectively. An element $x\neq z$ will be called *regular*. An element $a\in L$ is an *s-ideal element* or shortly an *s-ideal* if $xa\subseteq a$ for all $x\in L$. If $a-a\subseteq a$ we shall call $a$ a *group element* or shortly a *g-element*. An element which is at the same time an $s$-ideal and a *g-element* will be called a *d-ideal element* or shortly a *d-ideal*. An element $a$ having the property that $a\cdot a\subseteq a$ is called a multiplicatively closed element or shortly an *m-element*. By the *radical* of an element $a$ we understand the element $r$ which represents the union of all elements $x\in L$ such that $x^n\subseteq a$ for some positive integer $n$. An element which coincides with its radical is said to be *half-prime*. If $a$ is an arbitrary element from $L$ we shall denote by $(a)_s$, $(a)_d$ and $(a)_m$ the $s$-ideal, $d$-ideal and $m$-element, respectively, generated by $a$. Thus $(a)_d$ is equal to the intersection of all the $d$-ideals containing $a$.

**2. The notion of local closure.** The notion of local closure which we shall now introduce is perhaps the basic tool in the following considerations. In fact, this notion will take care of the non-atomic case by acting together with the axiom III below as a substitute for atomicity. As usual we understand by a closure operation $C$ on $L$ a mapping $a\to(a)_c$ of $L$ into itself such that

$$a \subseteq (a)_c, \qquad (a)_c = ((a)_c)_c \qquad \text{and} \qquad a \subseteq b$$

implies

$$(a)_c \subseteq (b)_c .$$

The element $a$ is called *C-closed* if $(a)_c = a$. Further we shall say that $a \in L$ is *locally C-closed* if for any regular element $b \subseteq a$ there exists a regular element $b_1 \subseteq b$ such that $(b_1)_c \subseteq a$. The mappings $a \to (a)_s$, $a \to (a)_d$ and $a \to (a)_m$ are all closure operations, and we shall in these cases speak of (local) $s$-closure, (local) $d$-closure and (local) $m$-closure, respectively. For instance, in usual ideal theory where $L$ is the Boolean algebra consisting of all subsets of a commutative ring with an identity element, any $s$-closed set is locally $d$-closed. Further the complement of any half-prime set is locally $m$-closed. These examples illustrate why we use the term "local". For instance, the complement $A'$ of a half-prime subset $A$ of the ring $R$ need not be $m$-closed "in the large" in the sense that $A' \cdot A' \subseteq A'$. (As usual, a product $A \cdot B$ of two subsets $A, B$ of $R$ means the set of all products $ab$ with $a \in A$ and $b \in B$.) But any subset $B$ of $A'$ contains a sufficiently small non-void subset $B_1$ such that $B_1 \cdot B_1 \subseteq A'$. In fact, we may always take a subset of $B$ consisting of a single element $b$, that is, $B_1 = \{b\}$. Then $(B_1)_m = \{b, b^2, \ldots, b^n, \ldots\} \subseteq A'$ since $A$ is supposed to be half-prime.

**3. The axioms defining a Boolean $d$-algebra.** We shall now give the complete set of axioms on which the generalized Krull theory will be based.

We suppose that the basic lattice $L$ is a complete Boolean algebra over which are defined an operation of multiplication and an operation of subtraction both being binary and univalued. The multiplication is assumed to be commutative and associative and we shall also, for the sake of simplicity, assume the existence of an identity element $e$ satisfying $ex = x$ for all $x \in L$. (This assumption is not essential; it has been included only in order to avoid some few extra complications occurring in cases where this identity element is absent.) Further this operation of multiplication shall satisfy the unrestricted distributive law

I $$a \bigcup_{i \in I} b_i = \bigcup_{i \in I} ab_i$$

(the index set $I$ being arbitrary) as well as the following three axioms:

II The set of regular elements of $L$ is closed under multiplication.

III If $ab \cap c$ is a regular element of $L$ then there exist regular elements $a_1 \subseteq a$ and $b_1 \subseteq b$ such that $a_1 b_1 \subseteq c$.

IV The complement of a half-prime $d$-ideal of $L$ is locally $m$-closed.

The operation of subtraction shall satisfy the following two unrestricted distributive laws

$$\text{V} \qquad a - \bigcup_{i \in I} b_i = \bigcup_{i \in I} (a - b_i) \qquad \text{and} \qquad \Big(\bigcup_{i \in I} b_i\Big) - a = \bigcup_{i \in I} (b_i - a) .$$

Finally, multiplication and subtraction are interrelated by the following one-sided distributive law

$$\text{VI} \qquad a(b-c) \subseteq ab - ac .$$

Remarks. From condition III it follows that $z$ is an $s$-ideal. For supposing $az = b \neq z$, we get $az \cap b = b \neq z$ and by condition III there exist regular elements $a_1$ and $z_1$ such that $a_1 \subseteq a$, $z_1 \subseteq z$ and $a_1 z_1 \subseteq b$, which is clearly impossible. In the notes [1] and [2] we supposed that the elements $a$ and $b$ occurring in condition III should be regular. This is unnecessary since this follows from condition III in its present form. For suppose for instance that $b = z$; then, since $z$ is an $s$-ideal, $az \cap c = z$ contrary to the assumption that $ab \cap c$ is regular. We note also that since $z$ is an $s$-ideal it is in particular an $m$-element. In the case of rings one makes the convention that the void set shall be considered as a multiplicatively closed set. The above remarks show that within the present axiomatic setting the corresponding fact may be proved.

A Boolean algebra verifying all the above conditions we shall call a *Boolean d-algebra.* If we do not take the operation of subtraction into account and hence omit all the axioms involving this operation we get a *Boolean s-algebra.* We note the following obvious proposition which reduces the construction of Boolean $d$-algebras to that of Boolean $s$-algebras and which is basic for the application of the present theory to semi-groups.

Proposition 1. *Any Boolean s-algebra becomes a Boolean d-algebra by defining $a - b$ as $a \cup b$.*

If $\mathfrak{A}$ is an algebraic system with a multiplication (and possibly other operations), then $A \cdot B$ denotes as usual the set of all products $ab$ with $a \in A \subseteq \mathfrak{A}$ and $b \in B \subseteq \mathfrak{A}$. If the given multiplication is not everywhere defined in $\mathfrak{A}$ it is natural to put $A \cdot B = \Phi$ whenever none of the products $ab$ are defined.

The following proposition will now clarify the meaning of the axioms II and III.

Proposition 2. *If $L$ is the Boolean algebra of all subsets of an algebraic system $\mathfrak{A}$ with a multiplication (and possibly other operations), then axiom*

II *is equivalent to the fact that the multiplication is everywhere defined in* $\mathfrak{A}$, *and* III *is equivalent to the fact that the multiplication is single-valued.*

Proof: The first part obviously follows from the above convention. Further, the assumption that $A$, $B$ and $C$ are subsets of $\mathfrak{A}$ such that $A \cdot B \cap C \neq \Phi$ means that if the multiplication is single-valued there exist $a \in A$ and $b \in B$ such that $ab \in C$ and we may put $A_1 = \{a\}$, $B_1 = \{b\}$ in order to get $A_1 \cdot B_1 \subseteq C$. Conversely, if multiplication is not single-valued there exist two elements $a, b \in \mathfrak{A}$ such that $ab = \{c\} \cup C$ where $C$ is a non-void subset of $\mathfrak{A}$. Then $\{a\} \cdot \{b\} \cap \{c\} \neq \Phi$, but there exist no non-void subsets $A_1 \subseteq \{a\}$ and $B_1 \subseteq \{b\}$ such that $A_1 \cdot B_1 \subseteq \{c\}$.

**4. The existence of non-atomic Boolean *d*-algebras.** It is of course essential for the interest of the present theory that we are able to exhibit examples of non-atomic Boolean $d$-algebras (both in the commutative and in the non-commutative case; for the definition of atomicity see Section 5). For if a Boolean $d$-algebra were atomic our development avoiding reference to atoms would be unnecessarily complicated. The use of atoms would have made possible practically the same treatment as in the case of rings. In particular, our technique of using the concept of "local closure" could have been dispensed with. On the other hand, the existence of non-atomic Boolean $d$-algebras also promises that the theory may have applications different from the standard application to the Boolean algebra of all subsets of a given algebraic system — like ring, distributive lattice or semi-group.

Let $S$ denote the multiplicative semi-group of strictly positive real numbers $<1$, that is, $S = ]0, 1[$. Under the usual topology of the real line, $S$ is a topological semi-group. It is a well-known fact that the family of all open sets of $S$ having the property that they are equal to the interior of their closure, forms a complete Boolean algebra $\mathfrak{B}$ under set-inclusion. These are the so-called *regular* open sets. This Boolean algebra $\mathfrak{B}$ is obviously non-atomic. In the finite case, the operation of intersection in $\mathfrak{B}$ is the usual set-theoretic intersection while the union $A \vee B$ of two sets in $\mathfrak{B}$ is the set $(A \cup B)^\star$ representing the smallest regular open set containing the set-theoretic union $A \cup B$. Denoting the usual complex multiplication of two subsets $A$ and $B$ of $S$ by $A \cdot B$ we define a multiplication in $\mathfrak{B}$ by putting $A \circ B = (A \cdot B)^\star$. We shall now prove that this makes $\mathfrak{B}$ a (non-atomic) Boolean $s$-algebra, and hence also a Boolean $d$-algebra, by using the union operation as the operation of subtraction (Proposition 1).

We first show that the mapping $A \to A^\star$ is related to complex multiplication in $S$ by the following

LEMMA. *The closure operation $A \to A^\star$ defined within the lattice of open sets in $S$ satisfies $A \cdot B^\star \subseteq (A \cdot B)^\star$.*

PROOF. Denoting the topological closure of $A$ by $\bar{A}$ we have, by the continuity of multiplication, $A \cdot \bar{B} \subseteq \overline{A \cdot B}$. Noting that $A^\star \subseteq \bar{A}$ we get $A \cdot B^\star \subseteq A \cdot \bar{B} \subseteq \overline{A \cdot B}$. Since the complex product of an open set with an arbitrary set in a topological group is open, $A \cdot B^\star$ will be open, and since, by definition, $(A \cdot B)^\star$ is the maximal open set in $\overline{A \cdot B}$ we must have $A \cdot B^\star \subseteq (A \cdot B)^\star$. (Here we consider $S$ as imbedded in the multiplicative topological group of all strictly positive real numbers.)

PROPOSITION 3. *$\mathfrak{B}$ is a commutative and non-atomic Boolean s-algebra.*

PROOF. In order to show that the multiplication defined in $\mathfrak{B}$ is completely distributive with respect to the union operation in $\mathfrak{B}$ we use the above lemma remarking first that the inclusion

$$\bigvee_{i \in I} A \circ B_i \subseteq A \circ \bigvee_{i \in I} B_i$$

is obvious. The reverse inclusion is obtained as follows,

$$A \circ \bigvee_{i \in I} B_i = \Big(A\Big(\bigcup_{i \in I} B_i\Big)^\star\Big)^\star \subseteq \Big(\Big(A \cdot \bigcup_{i \in I} B_i\Big)^\star\Big)^\star = \Big(\bigcup_{i \in I} AB_i\Big)^\star \subseteq \bigvee_{i \in I} A \circ B_i .$$

Axiom II is trivially verified. In order to show that III is satisfied we may argue as follows. Let $A$, $B$ and $C$ be sets in $\mathfrak{B}$ such that $A \circ B \cap C \neq \Phi$. Then also $A \cdot B \cap C \neq \Phi$; for otherwise $A \cdot B \subseteq C'$ ($C'$ denoting the complement of $C$ in $S$) and thus also $A \circ B = (A \cdot B)^\star \subseteq C'$ since $C'$ is closed. Hence there are two elements $a \in A$ and $b \in B$ such that $ab \in C$. Since $C$ is open it follows from the continuity of the multiplication in $S$ that there exist two sufficiently small open (non-void) intervals $A_1 \subseteq A$ and $B_1 \subseteq B$ such that $A_1 \cdot B_1 \subseteq C$; and since $C \in \mathfrak{B}$ we also have $A_1 \circ B_1 \subseteq C$. The fact that $A_1$ and $B_1$ are open intervals in $S$ and thus belong to $\mathfrak{B}$ completes the proof of III. That IV is also verified is clear since the $s$-ideals in $S$ are just the open intervals $]0, a[$ with $0 < a \leqq 1$ and $]0, 1[$ is the only one among these $s$-ideals which is half-prime.

In the case of a non-commutative multiplication we may construct an example of a non-atomic Boolean $d$-algebra in a trivial fashion. Take any non-atomic Boolean algebra and define multiplication as follows: $ab = b$ whenever $a \neq z$ and $zb = z$. We omit the routine check that this really defines a Boolean $s$-algebra. We have thus proved the following

THEOREM 1. *There exist both commutative and non-commutative non-atomic Boolean d-algebras.*

In the non-commutative case we were able to define a multiplication intrinsically using only the Boolean operations of a given abstract Boolean algebra. (The above non-commutative multiplication is essentially identical with one of the 16 possible binary Boolean operations.) This cannot be done in the commutative case, due to the following

PROPOSITION 4. *In a Boolean algebra $B$ there exists no commutative binary Boolean operation which makes $B$ a Boolean s-algebra by taking this operation as multiplication.*

This is easily checked by going through the various possible commutative, binary Boolean operations. Some of these operations, like intersection and union, are distributive with respect to union but fail to satisfy II or III.

**5. On the independence of the axioms.** Before entering our subject matter proper we shall make some remarks on the logical independence of the system of axioms given in the preceding section. Since in any case we shall deal with a Boolean algebra satisfying the distributive laws I, V and VI, we shall here principally be interested in the independence of the conditions II, III and IV relative to the remaining set of axioms. We first remark that condition III is satisfied in every ideal Boolean algebra ($a_2$ multiplicative lattice $L$ is said to be an *ideal* lattice if $a \cdot b \subseteq a \cap b$ is always fulfilled in $L$) since an ideal Boolean algebra can be residuated in one way only, namely by taking the operation of intersection as multiplication. For in this case $ab \cap c \neq z$ ($z$ always denotes the least element of $L$) is equivalent to $a \cap b \cap c \neq z$ and we may simply put $a_1 = b_1 = a \cap b \cap c$ in order to satisfy III. But condition II is obviously not satisfied by this interpretation of the multiplication. Since the rest of the axioms are satisfied by identifying the operation of subtraction with the union operation in $L$, we have thus shown the independence of condition II relative to the remaining set of axioms. In order to show that III cannot be deduced from the other postulates we may identify both the operation of subtraction and the operation of multiplication with the union operation in $L$. In this case all the conditions except III are satisfied. III is not satisfied; for instance, for a regular element $c$ different from the greatest element of $L$ the complement $c'$ of $c$ as well as $(c' \cup c) \cap c$ are regular; but there exist no regular elements $c_1 \subseteq c'$ and $c_2 \subseteq c$ such that $c_1 \cup c_2 \subseteq c$.

A Boolean algebra is called *atomic* if any of its elements except $z$ may be written as a union of atoms. A multiplicative Boolean algebra is called *regular* if condition II is satisfied. Finally $L$ is called *a-closed* if the set of atoms of $L$ is closed under the operation of multiplication, i.e., if the product of two atoms is an atom. In connection with the question of the logical independence of conditions I–VI we shall establish some simple connections between the three concepts just defined and conditions II, III and IV. The proof of the following propositions is obvious and may be omitted.

PROPOSITION 5. *A multiplicative Boolean algebra which is atomic and a-closed is regular.*

But conversely, atomicity and regularity do not imply that $L$ is $a$-closed (take for instance the Boolean algebra $L$ where the product of any two elements is the greatest element of $L$). It is when one tries to add a convenient condition in order to obtain this converse result that III enters. Specifically we have the following

PROPOSITION 6. *If a multiplicative Boolean algebra is atomic, regular and satisfies condition* III, *then it is a-closed.*

PROOF. Let $a$ and $b$ be atoms and put

$$ab = \bigcup_{i \in I} c_i ,$$

the $c_i$ being atoms. Now, taking one of these atoms, say $c_1$, we have $ab \cap c_1 = c_1 \neq z$. Hence, according to III there exist regular elements $a_1$ and $b_1$ such that $a_1 b_1 \subseteq c_1$, $a_1 \subseteq a$ and $b_1 \subseteq b$. But as $a$, $b$ and $c_1$ are atoms we must have $a = a_1$, $b = b_1$ and $ab = c_1$.

PROPOSITION 7. *If a multiplicative Boolean algebra is atomic and a-closed then the conditions* III *and* IV *are satisfied.*

PROOF. Suppose that III were not satisfied. If $a$, $b$ and $ab \cap c$ are suitable regular elements of $L$, then $a_1 b_1 \nsubseteq c$ for all regular $a_1$ and $b_1$ such that $a_1 \subseteq a$ and $b_1 \subseteq b$. In particular we may put $a_1 = a^\star$ and $b_1 = b^\star$ where $a^\star$ and $b^\star$ are atoms. But in this case $a^\star b^\star \nsubseteq c$ is equivalent to $a^\star b^\star \subseteq c'$ ($c'$ denoting the complement of $c$). Now, writing $a$ and $b$ as unions of atoms,

$$a = \bigcup_{a^\star \subseteq a} a^\star \quad \text{and} \quad b = \bigcup_{b^\star \subseteq b} b^\star ,$$

we get

$$ab = \bigcup_{\substack{a^\star \subseteq a \\ b^\star \subseteq b}} a^\star b^\star \subseteq c' .$$

(See Lemma 4 of the next section.) Thus $ab \cap c = z$ contradicting the assumption that $ab \cap c$ is regular.

Let further $r$ be a half prime element and $a$ be a regular element contained in $r'$. We then have $a^n \nsubseteq r$ for all $n$. By atomicity let $x$ be an atom contained in $a$. Since by $a$-closure the powers $x^n$ are all atoms, $x^n \nsubseteq r$ means that $x^n \subseteq r'$ for all $n$, that is

$$(x)_m = \bigcup_{n \geqq 1} x^n \subseteq r' ,$$

which proves the local $m$-closure of $r'$.

COROLLARY. *In an atomic multiplicative Boolean algebra condition* IV *is always verified. More precisely* IV *is implied by atomicity*, I, II *and* III.

PROOF. By Proposition 6 atomicity together with II and III imply that $L$ is $a$-closed. Then the corollary follows immediately from Proposition 7.

**6. Lemmas.** In this section we shall prove some simple lemmas which will be constantly used in the following. We suppose that all conditions listed in Section 3 are satisfied.

LEMMA 1. *$a \subseteq b$ and $c \subseteq d$ implies $ac \subseteq bd$.*

LEMMA 2. *$a \cap b = z$ is equivalent to $a \subseteq b'$ and to $b \subseteq a'$.*

The proofs of these two lemmas are obvious.

LEMMA 3. *Any complete Boolean algebra satisfies the infinite distributive laws*

$$a \cap \bigcup_{i \in I} b_i = \bigcup_{i \in I} (a \cap b_i) \qquad \text{and} \qquad \left(\bigcup_{i \in I} a_i\right) \cap \left(\bigcup_{j \in J} b_j\right) = \bigcup_{(i,j) \in I \times J} a_i \cap b_j .$$

For proof see [5, p. 165].

LEMMA 4. *If the operation $\circ$ satisfies the infinite distributive laws*

$$a \circ \bigcup_{i \in I} b_i = \bigcup_{i \in I} a \circ b_i \qquad \text{and} \qquad \left(\bigcup_{j \in J} b_j\right) \circ a = \bigcup_{j \in J} b_j \circ a ,$$

*then also*

$$\left(\bigcup_{i \in I} a_i\right) \circ \left(\bigcup_{j \in J} b_j\right) = \bigcup_{(i,j) \in I \times J} a_i \circ b_j .$$

PROOF.

$$\left(\bigcup_{i \in I} a_i\right) \circ \left(\bigcup_{j \in J} b_j\right) = \bigcup_{i \in I} \left(a_i \circ \bigcup_{j \in J} b_j\right) = \bigcup_{i \in I} \bigcup_{j \in J} a_i \circ b_j = \bigcup_{(i,j) \in I \times J} a_i \circ b_j .$$

The expression $x_1 a - x_2 a - \dots - x_n a$ may be made meaningful, i.e. yield an element in $L$, in a certain number of ways by putting parentheses.

For a given $n$-tuple $(x_1, x_2, \ldots, x_n) \in L^n$ this will give $\varphi(n)$ (not necessarily distinct) elements of $L$. With respect to a certain ordering of these $\varphi(n)$ elements we shall denote the $i$th element by

$$\sum^i x_j a\,.$$

The following lemma now gives the explicit form of $(a)_d$.

LEMMA 5. *The $d$-ideal $(a)_d$ has the explicit form*

$$(a)_d = \bigcup \sum^i x_j a$$

*where the union is taken over all finite $n$-tuples $(x_1, \ldots, x_n) \in L^n$ with $n \geqq 1$ and over all $i$.*

PROOF. When $(a)_d$ is defined in this way we obviously have that $a \subseteq (a)_d$ and also that $(a)_d$ is contained in every $d$-ideal containing $a$. We therefore only have to prove that so defined $(a)_d$ really forms a $d$-ideal. Putting

$$\bigcup \sum^i x_j a = b$$

we have for arbitrary $c \in L$

$$cb = c \bigcup \sum^i x_j a = \bigcup c \sum^i x_j a \subseteq \bigcup \sum^i c x_j a \subseteq b\,.$$

Thus $b$ is an $s$-ideal. That $b$ is a $g$-element follows from

$$b - b = \Big(\bigcup \sum^i x_j a\Big) - \Big(\bigcup \sum^k x_l a\Big) = \bigcup \Big(\sum^i x_j a - \sum^k x_l a\Big)\,,$$

due to Lemma 4.

LEMMA 6. $a(b)_d \subseteq (ab)_d$.

PROOF.

$$a(b)_d = a \bigcup \sum^i x_j b = \bigcup a \sum^i x_j b \subseteq \bigcup \sum^i x_j ab = (ab)_d\,.$$

LEMMA 7. $(a)_d (b)_d \subseteq (ab)_d$.

PROOF. This is an immediate corollary of Lemma 6.

LEMMA 8. *The $m$-element generated by $a$ is equal to the union*

$$m = \bigcup_{n \geqq 1} a^n\,.$$

PROOF. Obviously $a \subseteq m$ and $m \subseteq (a)_m$. We therefore only have to show that $m$ is $m$-closed. This follows from

$$m \cdot m = \bigcup_{n \geqq 1} a^n \cdot \bigcup_{p \geqq 1} a^p = \bigcup_{q \geqq 2} a^q \subseteq m\,.$$

LEMMA 9. *If the operation* $\circ$ *defined in* $L$ *satisfies the distributive law*

$$a \circ \bigcup_{i \in I} b_i = \bigcup_{i \in I} a \circ b_i ,$$

*then it also satisfies the one-sided distributive law*

$$a \circ \bigcap_{i \in I} b_i \subseteq \bigcap_{i \in I} a \circ b_i .$$

PROOF. In fact, it is enough to assume the homogeneity condition $a \subseteq b \rightarrow c \circ a \subseteq c \circ b$; for then

$$a \circ \bigcap_{i \in I} b_i \subseteq a \circ b_i$$

for all $i$ and therefore also

$$a \circ \bigcap_{i \in I} b_i \subseteq \bigcap_{i \in I} a \circ b_i .$$

LEMMA 10. *Any intersection of d-ideals is a d-ideal.*

PROOF. If $a = \bigcap_{i \in I} a_i$ is the given intersection then $a$ is an $s$-ideal, since

$$ba = b \bigcap_{i \in I} a_i \subseteq \bigcap_{i \in I} ba_i \subseteq a$$

(by Lemma 9). Further, since $a - a \subseteq a_i$ for all $i$ because of $a \subseteq a_i$, we have $a - a \subseteq \bigcap_{i \in I} a_i = a$ which shows that $a$ is also a $g$-element.

LEMMA 11. *If* $\{a_i\}_{i \in I}$ *is a family of d-ideals being simply ordered with respect to inclusion in* $L$, *then the union*

$$a = \bigcup_{i \in I} a_i$$

*is again a d-ideal.*

PROOF. Follows immediately from

$$b \bigcup_{i \in I} a_i = \bigcup_{i \in I} ba_i \subseteq \bigcup_{i \in I} a_i = a$$

and

$$a - a = \bigcup_{(i,j) \in I \times I} (a_i - a_j) \subseteq a ,$$

since $a_i - a_j \subseteq a_i$ or $\subseteq a_j$.

**7. The notion of a prime ideal element in a Boolean $s$-algebra.** Apparently, there is a lot of different ways of generalizing the usual ideal-theoretic notions of a prime ideal and a primary ideal to a Boolean $s$-algebra. In the present section we shall only treat the notion of primeness. We shall show that in a Boolean $s$-algebra essentially one concept of primeness may be defined.

The different possibilities of generalization of the usual notion of a prime ideal in ring theory arise from the fact that the relations $\alpha \in \mathfrak{a}$ and $\alpha \notin \mathfrak{a}$ each have (at least) two possible generalizations to a multiplicative Boolean algebra. $\beta \in \mathfrak{a}$ may be translated into $b \subseteq a$ or the weaker condition that $a \cap b$ be regular. Similarly $\beta \notin \mathfrak{a}$ may be translated into $b \nsubseteq a$ or $b \subseteq a'$. We shall say that $a$ is in the relation $P_1$, $P_2$, $P_3$ or $P_4$ to $p$ and write $aP_i p$ $(i=1, 2, 3, 4)$ if $a \subseteq p$, $a \cap p \neq z$, $a \nsubseteq p$ and $a \subseteq p'$, respectively. Making the convention that $i$, $j$ and $k$ can only take values from the index sets $\{1, 2\}$, $\{3, 4\}$ and $\{1, 2\}$, respectively, we may give the general

DEFINITION 1. *The s-ideal element $p$ is said to be $P_{i,j,k}$-prime if for regular $a$ and $b$, $abP_i p$ and $aP_j p$ imply $bP_k p$.*

This definition includes 8 apparently different notions of primeness. That most of these notions are only apparently different is obvious. But we shall here prove the following strong result.

PROPOSITION 8. *If $L$ is a Boolean s-algebra all these eight notions of primeness are equivalent apart from $P_{2,3,1}$-primeness, $P_{2,3,2}$-primeness and $P_{2,4,1}$-primeness which we shall exclude from consideration because these properties may only be verified by the greatest element of $L$.*

PROOF. $P_{1,4,2} \to P_{1,4,1}$: If $a \subseteq p'$ and $ab \subseteq p$ we cannot have $b \nsubseteq p$. For from $b \cap p' \neq z$ we should get $a(b \cap p') \subseteq ab \subseteq p$ which is impossible because of $P_{1,4,2}$.

$P_{1,4,1} \to P_{1\ 3,2}$: Suppose $a \nsubseteq p$ and $ab \subseteq p$. We then have $b \cap p \neq z$, for $b \subseteq p'$ would according to $P_{1,4,1}$ give $a \subseteq p$ in contradiction with the assumption.

$P_{1,3,2} \to P_{1,3,1}$: Let $a \nsubseteq p$ and $ab \subseteq p$. If $b \cap p' \neq z$ we should have $a(b \cap p') \subseteq ab \subseteq p$ contradicting $P_{1,3,2}$.

$P_{1,3,1} \to P_{2,4,2}$: If $a \subseteq p'$ and $ab \cap p \neq z$ for regular $a$ and $b$, then according to condition III there exist regular elements $a_1$, $b_1$ such that $a_1 \subseteq a$, $b_1 \subseteq b$ and $a_1 b_1 \subseteq p$. $P_{1,3,1}$ then gives $b_1 \subseteq p$, that is $b \cap p \neq z$.

$P_{2,4,2} \to P_{1,4,2}$: Obvious.

These five implications prove the equivalence of $P_{1,4,2}$, $P_{1,4,1}$, $P_{1,3,2}$, $P_{1,3,1}$ and $P_{2,4,2}$-primeness. That $P_{2,3,1}$-primeness and $P_{2,4,1}$-primeness are impossible except for the greatest element of $L$ is evident. In order to see that $P_{2,3,2}$-primeness is also impossible except for the greatest element of $L$ we remark that $p$ is supposed (by definition) to be an $s$-ideal. If now $p \neq z$ is not the greatest element of $L$ we have $p \cap (p \cup p')p' \neq z$ and $p \cup p' \nsubseteq p$ but $p' \cap p = z$ which proves that $p$ is not $P_{2,3,2}$-prime.

In the following we shall use the formulation given by $P_{1,3,1}$-primeness which coincides with the usual definition of a prime ideal element in a general multiplicative lattice. In this case we shall use the term $s$-prime (element) for short. If in addition $p$ is a $g$-element we shall call $p$ a $d$-prime (element).

As in the usual ideal theory of rings we may also here give a useful characterization of $d$-ideals which are not $d$-primes.

LEMMA 12. *The d-ideal a is non-d-prime if and only if there exist d-ideals b and c both containing a properly and for which* $bc \subseteq a$.

PROOF. The "if"-part is obvious. If conversely $a$ is not a $d$-prime there exist two elements $b_1$ and $c_1$ such that $b_1 \nsubseteq a$, $c_1 \nsubseteq a$ and $b_1c_1 \subseteq a$. Then $b_2 = a \cup b_1$ and $c_2 = a \cup c_1$ both contain $a$ properly, and $b_2c_2 \subseteq a$. By Lemma 7,

$$(b_2)_d(c_2)_d \subseteq (b_2c_2)_d \subseteq (a)_d = a\,.$$

The lemma is therefore proved by choosing $b = (b_2)_d$ and $c = (c_2)_d$.

## 8. $d$-primes and $m$-elements.

In this and the following sections we shall show that the methods used by Krull carry over to the type of Boolean algebras defined in Section 3.

PROPOSITION 9. *The d-ideal p is a d-prime if and only if the complement of p is an m-element.*

PROOF. Suppose that $p$ is a $d$-prime different from $z$ and $u$. (That the proposition is verified for $p = z$ or $p = u$ follows immediately from axiom II and the fact that $z$ is an $m$-element.) If now $p'$ were not an $m$-element we should have $p'p' \nsubseteq p'$ or $p'p' \cap p \neq z$. Condition III then implies the existence of regular elements $a$, $b$ such that $a \subseteq p'$, $b \subseteq p'$ and $ab \subseteq p$. This contradicts the fact that $p$ is a $d$-prime. Conversely, let $p'$ be an $m$-element. If there exist elements $a$ and $b$ which are not contained in $p$ and such that $ab \subseteq p$, then $a \cap p'$ and $b \cap p'$ are regular and $(a \cap p')(b \cap p') \subseteq p'p' \subseteq p'$. Since according to II the product $(a \cap p')(b \cap p')$ is a regular element this contradicts $(a \cap p')(b \cap p') \subseteq ab \subseteq p$.

PROPOSITION 10. *If m is an m-element then a maximal d-ideal contained in m′ is a d-prime.*

PROOF. Let $p$ be a maximal $d$-ideal contained in $m'$ (the existence proofs are gathered in Section 11). If $p$ were not a $d$-prime, then according to Lemma 12 there would exist $d$-ideals $a$ and $b$ properly containing $p$ such that $ab \subseteq p$. But $p$ is maximal in $m'$, and hence $a \cap m$ and $b \cap m$ will be regular elements for which

$$(1) \qquad (a \cap m)(b \cap m) \subseteq m$$

since $m$ is an $m$-element. On the other hand,

$$(2) \qquad (a \cap m)(b \cap m) \subseteq ab \subseteq p \subseteq m' .$$

(1) and (2) contradict the regularity of $(a \cap m)(b \cap m)$ (Condition II).

PROPOSITION 11. *Let $a$ be a $d$-ideal and $m$ be a maximal $m$-element contained in the complement of $a$. If $p$ is a maximal $d$-ideal containing $a$ and being contained in $m'$, then $p$ is a minimal $d$-prime containing $a$.*

PROOF. That $p$ is a $d$-prime follows from Proposition 10 and that $p$ is minimal follows from Proposition 9. In fact it follows that $p = m'$; for if $p$ were properly contained in $m'$, then $p'$ would be an $m$-element contained in $a'$ and properly containing $m$, against the maximality of $m$.

**9. Radicals and weak primary $d$-ideals.** By the definition in Section 1 the radical of a $d$-ideal $a$ is the element $r$ which is the union of all elements for which a finite power is contained in $a$. Since $L$ is supposed to be complete, the radical always exists and is uniquely determined by $a$. But the radical itself need not have the property that a finite power of it is contained in $a$.

PROPOSITION 12. *The radical of the $d$-ideal $a$ is a $d$-ideal which contains $a$ and which is equal to the union of all $d$-ideals for which a finite power is contained in $a$.*

PROOF. The last assertion follows from the fact that if $x^n \subseteq a$ then also $(x)_d{}^n \subseteq a$. For $(x)_d{}^n \subseteq (x^n)_d \subseteq (a)_d = a$ follows by repeated applications of Lemma 7. We may therefore write the radical thus,

$$r = \bigcup_{x^n \subseteq a} (x)_d .$$

That $r$ is an $s$-ideal follows from

$$br = \bigcup_{x^n \subseteq a} b(x)_d \subseteq \bigcup_{x^n \subseteq a} (x)_d = r .$$

Further,

$$r - r = \bigcup_{\substack{x^n \subseteq a \\ y^m \subseteq a}} \big((x)_d - (y)_d\big) .$$

Here $(x)_d - (y)_d \subseteq (x \cup y)_d$, and if $x^n \subseteq a$ and $y^m \subseteq a$ then

$$[(x)_d - (y)_d]^{n+m-1} \subseteq \big((x \cup y)_d\big)^{n+m-1} \subseteq [(x \cup y)^{n+m-1}]_d .$$

Expanding the last expression we get

$$((x \cup y)^{n+m-1})_d = \Big(\bigcup_{\substack{i+j=\\n+m-1}} x^i y^j\Big)_d \subseteq a\,.$$

This obviously proves that $r - r \subseteq r$

LEMMA 13. *Let $c$ be a regular element contained in the radical $r$ of the $d$-ideal $a$. Then there exists a regular element $c_1 \subseteq c$ such that $c_1{}^n \subseteq a$ for some $n$.*

PROOF. Since

$$c \cap \bigcup_{x^n \subseteq a} x = \bigcup_{x^n \subseteq a} c \cap x = c \mp z\,,$$

the element $c_1 = x \cap c$ must be regular for at least one $x$. From $c_1 \subseteq x$ and $x^n \subseteq a$ we get $c_1{}^n \subseteq a$ as desired.

The following lemma, similar to the preceding one, will be needed later in connection with the isolated weak primary components.

LEMMA 14. *Let $p$ be a minimal $d$-prime containing the regular $d$-ideal $a$. Further, let the element $q$ be defined as the union of all elements $x$ such that $xp_1 \subseteq a$ for a suitable regular element $p_1 \subseteq p'$. Then, if $c \subseteq q$ and $c$ is regular, $c$ will contain a regular element $x$ having this property.*

PROOF. Since $c \subseteq q$, $c$ must have a regular intersection with at least one $x$. If $c_1 = c \cap x$ is regular, $c_1$ will have the desired property.

DEFINITION 2. *The $d$-ideal $q$ is said to be a weak primary $d$-ideal if for regular elements $a$, $b$ such that $ab \subseteq q$ and $a \nsubseteq q$ we always have a regular $b_1 \subseteq b$ such that $b_1{}^n \subseteq q$ for some $n$.*

In usual commutative ideal theory there is a distinction between *strong* and *weak* primary ideals. The ideal $\mathfrak{q}$ is weak primary if for *elements* $a, b \in R$ such that $ab \in \mathfrak{q}$ and $a \notin \mathfrak{q}$ we have $b^n \in \mathfrak{q}$ for some $n$. On the other hand, $\mathfrak{q}$ is strong primary if for ideals $\mathfrak{a}, \mathfrak{b} \subseteq R$ such that $\mathfrak{a}\mathfrak{b} \subseteq \mathfrak{q}$ and $\mathfrak{a} \notin \mathfrak{q}$ we have $\mathfrak{b}^n \subseteq \mathfrak{q}$ for some $n$. A strong primary ideal is always weak primary, but the converse statement is not generally valid. If $R$ satisfies the ascending chain condition the two concepts will coincide. It is the concept of a strong primary ideal that lends itself most naturally to lattice translation, since this concept is defined entirely in terms of ideals without using the ring elements. For the purpose of the Krull theory, where no chain condition is assumed, the above definition which constitutes an analogue to the definition of a weak primary ideal seems, however, to be more useful.

THEOREM 2. *The radical of a weak primary $d$-ideal is a $d$-prime.*

Proof. Let $q$ be a weak primary $d$-ideal and $r$ its radical. Suppose that $r$ were not a $d$-prime. Then there exist elements $a$, $b$ such that $ab \subseteq r$, $a \nsubseteq r$ and $b \nsubseteq r$, that is, $a_1 = a \cap r'$ and $b_1 = b \cap r'$ are regular elements such that $a_1 b_1 \subseteq r$. According to Lemma 13 there exists a regular $c \subseteq a_1 b_1$ such that $c^n \subseteq q$ for some $n$. By condition III we further have regular elements $a_2 \subseteq a_1$ and $b_2 \subseteq b_1$ such that $a_2 b_2 \subseteq c$. Thus $a_2{}^n \cdot b_2{}^n \subseteq c^n \subseteq q$. Let us suppose that $n$ is the least exponent for which $(a_2 b_2)^n$ is contained in $q$. Then $a_2{}^{n-1} \cdot b_2{}^{n-1} \nsubseteq q$, and therefore also $b_2 a_2{}^{n-1} \cdot b_2{}^{n-1} \nsubseteq q$ since $q$ is weak primary and no regular $b_3 \subseteq b_2$ can satisfy $b_3{}^m \subseteq q$ for any $m$ since $b_3 \subseteq b_2 \subseteq b_1 \subseteq r'$. The same argument applied to the product

$$a_2 (b_2 a_2{}^{n-1} \cdot b_2{}^{n-1}) = a_2{}^n \cdot b_2{}^n \subseteq q$$

gives the desired contradiction since $b_2 a_2{}^{n-1} \cdot b_2{}^{n-1} \nsubseteq q$ and no regular element $a_3 \subseteq a_2$ can satisfy $a_3{}^m \subseteq q$ for any $m$ since $a_3 \subseteq a_2 \subseteq a_1 \subseteq r'$.

Theorem 3. *The radical $r$ of the $d$-ideal $a$ is identical with the intersection of all the (minimal) $d$-primes containing $a$.*

Proof. According to Lemma 13,

$$r \subseteq \bigcap_{a \subseteq p} p = h .$$

Let us suppose that $r$ were properly contained in $h$, that is, that $h \cap r'$ were regular. By condition IV, $r'$ is locally $m$-closed and hence there exists a regular element $c \subseteq h \cap r'$ such that the $m$-element

$$m = \bigcup_{n \geq 1} c^n$$

generated by $c$ is contained in $r'$. According to Proposition 10, a maximal $d$-ideal containing $a$ and being contained in $m'$ is a $d$-prime which does not contain $h$. This establishes the desired contradiction.

As the notions of primeness and $m$-closure correspond to each other in the sense of Proposition 9, we shall now see that the notions of half-primeness and local $m$-closure correspond to each other in the sense of the following

Proposition 13. *A $d$-ideal $p$ is half-prime if and only if its complement is locally $m$-closed.*

Proof. The "only if"-part of the proposition coincides with condition IV, hence there is nothing to prove in this case. Conversely, if $p'$ is locally $m$-closed then $p$ must be half-prime. For if $c \nsubseteq p$, then $c \cap p'$ is regular and contains an element $c_1$ such that $c_1{}^n \subseteq p'$ for all $n$. Therefore, since $c_1{}^n \subseteq c^n$ and $c_1{}^n \cap p'$ is regular for all $n$, the element $c^n \cap p'$ is also regular for all $n$, and hence $c^n$ cannot be contained in $p$ for any integer $n \geqq 1$.

We shall now prove the main theorem concerning the isolated weak primary components.

THEOREM 4. *To every minimal d-prime $p$ containing the d-ideal $a$ there corresponds a uniquely determined minimal weak primary d-ideal $q$ which contains $a$ and has $p$ as its radical. This element $q$ is called the isolated weak primary component of $a$ which belongs to $p$.*

PROOF. Though somewhat more complicated, the proof of this theorem proceeds along the same lines as the corresponding theorem in the usual Krull theory. In fact, we shall see that the weak primary $d$-ideal in question is equal to the union of all elements $b_i$ for which there exists an element $s_i \nsubseteq p$ such that $b_i s_i \subseteq a$, that is,

$$q = \bigcup_{\substack{b_i s_i \subseteq a \\ s_i \cap p' \neq z}} b_i . \tag{3}$$

(In case $p$ is equal to the greatest element of $L$ we put $q = a$.) Because of

$$(b_i)_d (s_i)_d \subseteq (b_i s_i)_d \subseteq (a)_d = a$$

we may, as in the definition of the radical, give the definition of $q$ entirely in terms of $d$-ideals. If convenient we shall therefore assume that all the elements $b_i$ occurring in the union (3) are $d$-ideals. That $q$ will be a $d$-ideal follows from

$$tq = \bigcup tb_i \subseteq \bigcup b_i = q$$

and

$$q - q = \bigcup_{\substack{b_i s_i \subseteq a \\ s_i \cap p' = z}} b_i - \bigcup_{\substack{b_i s_j \subseteq a \\ s_j \cap p' = z}} b_i = \bigcup_{i,j} (b_i - b_j) \subseteq \bigcup_{\substack{b_k s_k \subseteq a \\ s_k \cap p' = z}} b_k = q .$$

The last inclusion follows from

$$(b_i - b_j) s_i s_j \subseteq b_i s_i s_j - b_j s_i s_j \subseteq a - a \subseteq a .$$

Since $s_i s_j \nsubseteq p$ this shows that $b_i - b_j$ is one of the elements $b_k$ entering in the definition of $q$. We then prove that $p$ is actually the radical $r$ of $q$, which is the union of all $c$ such that $c^\varrho \subseteq q$ for some positive integer $\varrho$. Obviously $r \subseteq p$. Let us assume that $r$ is properly contained in $p$. Then the element $r' \cap p$ will be regular, and according to the condition of local $m$-closure this element will contain a regular element $c$ such that $c^\varrho \subseteq r' \cap p$ for all $\varrho \geqq 1$. Now let us consider the element

$$m = \bigcup_{\sigma \geqq 1} (c \cup p')^\sigma ,$$

which is obviously $m$-closed. We shall prove that $m$ is contained in $r'$. If $m \cap r \neq z$ we have $(c \cup p')^{\sigma} \cap r \neq z$ for a certain $\sigma$, that is,

$$(c^{\sigma} \cup c^{\sigma-1} p' \cup \ldots \cup p'^{\sigma}) \cap r \neq z;$$

hence

$$c^{\sigma-\tau} p'^{\tau} \cap r \neq z$$

for a certain $\tau$ with $1 \leqq \tau \leqq \sigma - 1$. We remark that we may here suppose $p'$ to be regular; for if $p'$ were not regular, then $a = q$ according to the convention made above. Therefore, in this case it is immediate, according to Theorem 3, that $p$ is the radical of $q$ since $p$ is the only $d$-prime containing $q$. Thus, by condition III we have a regular $c_1 \subseteq c^{\sigma-\tau}$ and a regular $p_1 \subseteq p'^{\tau}$ such that

$$c_1 p_1 \subseteq r .$$

Lemma 13 now proves the existence of a regular element $d \subseteq c_1 p_1$ such that $d^{\varrho} \subseteq q$ for some $\varrho$. But then, again according to III, we have a regular $c_2 \subseteq c_1$ and a regular $p_2 \subseteq p_1$ such that $c_2 p_2 \subseteq d$ and therefore

$$c_2^{\varrho} p_2^{\varrho} \subseteq d^{\varrho} \subseteq q . \tag{4}$$

By Lemma 14, (4) implies the existence of a regular $e \subseteq c_2^{\varrho} p_2^{\varrho}$ and a regular $s \subseteq p'$ such that

$$es \subseteq a . \tag{5}$$

Again III implies the existence of a regular $c_3 \subseteq c_2^{\varrho}$ and a regular $p_3 \subseteq p_2^{\varrho}$ such that $c_3 p_3 \subseteq e$. This together with (5) gives $c_3 p_3 s \subseteq es \subseteq a$. Since $p_3 s \subseteq p'$ we shall have $c_3 \subseteq q \subseteq r$ contradicting $c_3 \subseteq c_2^{\varrho} \subseteq c_1^{\varrho} \subseteq (c^{\sigma-\tau})^{\varrho} \subseteq r'$. This shows that $m$ properly contains $p'$ and is contained in $r'$. Proposition 10 (together with an existence result to be proved later) therefore proves that there exists a $d$-prime containing $r$ (and therefore $q$) and which is properly contained in $p$. This contradicts the assumption that $p$ is a minimal $d$-prime containing $a$. The hypothesis that $r$ is properly contained in $p$ must therefore be false and $p$ is the radical of $q$.

We next prove that $q$ is weak primary, i.e., that if for regular $b$ and $c$ we have $bc \subseteq q$ and $b \nsubseteq q$, then $c \cap p \neq z$. Condition III together with Lemma 14 imply that there exist regular elements $b_1 \subseteq b \cap q'$ and $c_1 \subseteq c$ such that

$$b_1 c_1 p_1 \subseteq a \tag{6}$$

for a suitable regular $p_1 \subseteq p'$. (If $p$ is the greatest element of $L$, in which case $p'$ is not regular, then $c \cap p$ is of course always regular and there is nothing to prove in this case.) If now $c \cap p$ were not regular we should have $c_1 \subseteq c \subseteq p'$ and thus $c_1 p_1 \subseteq p'$, since $p'$ is an $m$-element. By the defining property of $q$, (6) therefore gives $b_1 \subseteq q$ contradicting $b_1 \subseteq b \cap q' \subseteq q'$, $b_1$ being regular.

We finally prove that $q$ is contained in any weak primary $d$-ideal containing $a$ and having $p$ as its radical. Suppose there exists another weak primary $d$-ideal $q^\star$ containing $a$, with radical $p$ and such that $q \nsubseteq q^\star$. Then there exists by Lemma 14 a regular $q_1 \subseteq q^{\star\prime} \cap q$ such that $q_1 p_1 \subseteq a \subseteq q^\star$ for a suitable regular $p_1 \subseteq p'$. (As before, there is nothing to prove if $p' = z$.) Since $q$ and $q^\star$ are supposed to have the same radical this is in contradiction with the fact that $q^\star$ is weak primary.

Theorem 4 is now completely proved.

**10. Further results. Another definition of the radical.** The results of the preceding section correspond to that part of the Krull theory which is formulated as the Struktursatz in [8, p. 9]. We give no exposition of the further results of the Krull theory here; we only remark that in our setting most of them may be obtained by use of the same techniques as in the preceding sections. We might for instance use the following definition: An element $b$ is said to be *related* to the $d$-ideal $a$ if there exists an element $s \subseteq a'$ such that $bs \subseteq a$. Here we should *not* define *unrelated* as an element which is not related to $a$, but by the stronger property that none of its regular 'subelements' is related to $a$. Then one may prove that any minimal $d$-prime containing $a$ is related to $a$. We omit further proofs because they are often rather lengthy and offer no particular new interest. We also remark that a few of the other results of the Krull theory seem to be rather difficult to obtain in the present setting without introducing further axioms. For instance, it seems difficult to prove a simple result like this: Any maximal $d$-ideal related to the $d$-ideal $a$ contains $a$. But we shall not enter into these questions, especially because there exists another very satisfactory generalization of all these results by means of $x$-ideals. See [3] and a forthcoming paper by the author. In Section 5 we did not say anything about the independence of condition IV with respect to the remaining body of axioms. In fact we have not been able to construct any example showing this independence. The corollary of Proposition 7 shows that such an example must be non-atomic. In any case axiom IV is necessary for the derivation of Theorem 3 when the present definition of the radical is used. But we can easily change the definition of the radical to the effect that by this new definition condition IV will be fulfilled by itself, whence it may be discarded. This definition runs as follows: An element $b$ shall belong to the *big radical* of $a$ if for any regular $b_1 \subseteq b$ we have $b_1^n \cap a \neq z$ for a certain $n$. It follows that $a$ is identical with its big radical if and only if the complement of $a$ is locally $m$-closed so that IV need no longer be postulated. The two radical notions coincide in the atomic case.

**11. Existence proofs.** We first give a direct proof (without using $m$-elements) of the fact that any $d$-prime $p^\star$ containing the $d$-ideal $a$ contains a minimal $d$-prime containing $a$. We formulate this as

PROPOSITION 14. *Let $p^\star$ be a d-prime containing the d-ideal a. Then the partially ordered set P consisting of all d-primes p satisfying $a \subseteq p \subseteq p^\star$ has at least one minimal element.*

PROOF. The proof is carried out in the standard fashion by means of Zorn's lemma. We only have to show that $P$ is inductive with respect to the relation $\supseteq$, i.e., that if $S$ is a simply ordered subset of $P$ then $S$ has a lower bound in $P$. Indeed, a lower bound for $S$ is given by the $d$-ideal

$$p_0 = \bigcap_{p \in S} p \, .$$

Obviously $p_0 \supseteq a$. That $p_0$ is a $d$-ideal follows from Lemma 10. If $p_0$ were not a $d$-prime we should have, by Lemma 12, two $d$-ideals $a, b$ which contain $p_0$ properly, and such that $ab \subseteq p_0$. Then $a \nsubseteq p_1$ and $b \nsubseteq p_2$ for some $p_1, p_2 \in S$. Since $S$ is simply ordered we may for instance suppose $p_1 \subseteq p_2$. We then have $a \nsubseteq p_1$, $b \nsubseteq p_1$ and $ab \subseteq p_0 \subseteq p_1$ contradicting the fact that $p_1$ is a $d$-prime.

PROPOSITION 15. *If a is a d-ideal and m is an m-element contained in $a'$, then there exists a maximal d-ideal containing a and being contained in $m'$.*

PROOF. Zorn's lemma may here be applied to the partially ordered set $Q$ consisting of all $d$-ideals $b$ satisfying $a \subseteq b \subseteq m'$. Lemma 11 shows that $Q$ is inductive.

Finally, we prove the following existence statement which was also used.

PROPOSITION 16. *If a is a d-ideal and $m^\star$ an m-element contained in $a'$, then there exists a maximal m-element contained in $a'$ and containing $m^\star$.*

PROOF. We here consider the partially ordered set $R$ consisting of all $m$-elements $m$ satisfying $m^\star \subseteq m \subseteq a'$. If $S = \{m_i\}_{i \in I}$ is a simply ordered subset of $R$ we only need show that the element

$$m_0 = \bigcup_{i \in I} m_i$$

belongs to $R$. Obviously $m_0 \subseteq a'$ and $m_0 m_0 \subseteq m_0$ follows from

$$m_0 m_0 = \bigcup_{i \in I} m_i \cdot \bigcup_{i \in I} m_j = \bigcup_{(i,j) \in I \times I} m_i m_j \subseteq \bigcup_{k \in I} m_k = m_0$$

($S$ being simply ordered we have $m_i m_j \subseteq m_i$ or $\subseteq m_j$). In particular, choosing $m^\star = z$ we have that in any case maximal $m$-elements are contained in $a'$.

**12. Applications.** Finally we make some remarks on the range of applications of the present theory.

1. *Rings.* It is clear that the Boolean algebra consisting of all subsets of a commutative ring $R$ will satisfy the conditions I–VII when the product $A \cdot B$ and the difference $A - B$ of two subsets $A$, $B$ of $R$ are defined as the subsets consisting respectively of all the products $a \cdot b$ and all the differences $a - b$ with $a \in A$ and $b \in B$. Then the usual ideal theory of Krull [7] will be subsumed under the present theory.

2. *Ringoids.* More generally $R$ may be taken to be a commutative and associative ringoid (in the sense of [5, p. 203]. A ringoid is an algebraic system with one multiplicative operation being distributive with respect to each member of a family of additive operations. The additive operations are subject to no restriction at all.

3. *Distributive lattices.* Apart from rings the most important example of a ringoid is perhaps that of a distributive lattice. In this case we get as a corollary of Theorem 3 the well-known result that any lattice ideal in a distributive lattice may be written as an intersection of prime ideals. This result is basic in the representation theory for distributive lattices and Boolean algebras.

4. *Semi-groups.* It is interesting to note that all the conditions I–VI are satisfied if we identify the subtraction operation with the union operation in $L$. Due to this fact the preceding theory will also automatically apply to the ideal theory of commutative semi-groups. For by this identification any subset of the given semi-group will be closed under subtraction since $a - a = a \cup a = a$ for all $a \in L$. In fact, this procedure has the same effect as letting the operation of subtraction disappear.

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