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# ITERATION OF THE "LIN" OPERATION FOR CONVEX SETS

V. L. KLEE, JR.

For a subset $X$ of a real linear space $E$, $\operatorname{lin} X$ will denote the union of $X$ and the set of all endpoints of line segments in $X$. Equivalently, $\operatorname{lin} X$ is the set of all points $y \in E$ such that $y + ]0, 1]z \subset X$ for some $z \in E$. This operation (introduced in [1]) is of special interest for convex sets, for when $X$ is convex and $E$ finite-dimensional, $\operatorname{lin} X$ is the closure of $X$ in the "Euclidean" topology for $E$ (i.e., the unique topology making $E$ into a Hausdorff linear space). Iteration of the *lin* operation for convex sets was studied by Nikodym [2][3][4][5], whose results are quite definitive but whose proofs are lengthy. The present note supplies a more concise discussion of the subject. The proofs below of (1)–(3), (A), and (6) are abridgements of those of Nikodym [4] and others, included here only for the sake of completeness. However, the proofs of (4) and (5) are believed to be significantly simpler than those of Nikodym—especially in the case of (5), to which is devoted his entire paper [5]. Some unsolved problems are stated at the end of this note.

In the following paragraphs, $E$ will always denote a real linear space and $X$ a subset of $E$. Notation and terminology are fairly standard, with $\Phi$ denoting the neutral element of $E$, $\operatorname{conv} X$ the convex hull of $X$, $\Omega$ the first uncountable ordinal, $I$ the system of positive integers, $R$ the real number system, etc. All words of a topological nature will refer to the Euclidean topology mentioned above.

Let us write $\operatorname{lin}^0 X = X$; and having defined $\operatorname{lin}^\alpha X$ for all ordinals $\alpha < \beta$, let

$$\operatorname{lin}^\beta X = \begin{cases} \operatorname{lin}\operatorname{lin}^{\beta-1} X & \text{if} \quad \beta - 1 \text{ exists,} \\ \bigcup_{\alpha<\beta} \operatorname{lin}^\alpha X & \text{if} \quad \beta \text{ is a limit ordinal}. \end{cases}$$

We begin by recalling some basic properties of these operations and outlining their proofs.

(1) *If $X$ is convex, so is* $\operatorname{lin}^\alpha X$.

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PROOF. It suffices to treat the case $\alpha=1$. Consider an arbitrary pair of points $p, q \in \operatorname{lin} X$ and the point $v=tp+(1-t)q$, where $t\in[0, 1]$. There exist $y, z$ such that $p+]0, 1]y\subset X$ and $q+]0, 1]z\subset X$. Then

$$v + ]0, 1]\big(ty+(1-t)z\big) \subset X ,$$

whence $v\in\operatorname{lin} X$ and the proof is complete.

(2) *If $X$ is convex,* $\operatorname{lin}^{\Omega+1}X=\operatorname{lin}^{\Omega}X$.

PROOF. Consider $p\in\operatorname{lin}\operatorname{lin}^{\Omega}X$, with $p+]0, 1]y\subset\operatorname{lin}^{\Omega}X$. For each $n$ there exists $\alpha_n<\Omega$ such that $p+n^{-1}y\in\operatorname{lin}^{\alpha_n}X$. Then with $\beta=\sup\{\alpha_n\}_{n\in I}$, we have $\beta<\Omega$ and $p+]0, 1]y\subset\operatorname{lin}^{\beta}X$, since $\operatorname{lin}^{\beta}X$ is convex; whence

$$p \in \operatorname{lin}^{\beta+1}X \subset \operatorname{lin}^{\Omega}X$$

and the proof is complete.

(3) *If $X$ is convex and $E$ finite-dimensional,* $\operatorname{lin} X$ *is the closure of $X$ in the Euclidean topology. Thus* $\operatorname{lin}^2X=\operatorname{lin}^1X$.

PROOF. Observe first that $X$ must have an interior point $p$ relative to the smallest linear variety containing it, and then that $q$ in the closure of $X$ implies $]p, q]\subset X$.

Now define the *order* of a set $X$ as the smallest ordinal $\alpha$ for which $\operatorname{lin}^{\alpha}X=\operatorname{lin}^{\alpha+1}X$. Note that if $X$ is convex, then $\alpha\leqq\Omega$ and $\operatorname{lin}^{\alpha}X=\operatorname{lin}^{\Omega}X$. In [1], the author showed that $E$ is finite-dimensional if and only if every convex subset of $E$ is of order 0 or 1. Nikodym's principal theorems are as follows:

(4) *If $X$ is convex and* $\dim E\leqq\aleph_0$, *the order of $X$ is* $<\Omega$.
(5) *If* $\dim E\geqq\aleph_0$ *and* $\alpha<\Omega$, *$E$ contains a convex set of order* $\alpha$.
(6) *If* $\dim E>\aleph_0$, *$E$ contains a convex set of order* $\Omega$.

We begin by establishing the following result, of which (4) is an immediate corollary.

(4′) *If $X$ is a convex subset of $E$ and $L$ a (linear) subspace of $E$ with* $\dim L\leqq\aleph_0$, *there is an ordinal $\alpha<\Omega$ such that* $L\cap\operatorname{lin}^{\alpha}X=L\cap\operatorname{lin}^{\Omega}X$.

PROOF OF (4′). If the conclusion fails, a straightforward transfinite induction produces an uncountable set $U$ of nonlimit ordinals $<\Omega$ and a biunique function $f$ on $U$ to $L$ such that for each $\beta\in U$, $f\beta\in\operatorname{lin}^{\beta}X\sim\operatorname{lin}^{\beta-1}X$. Since $fU$ is uncountable and $L$ is the union of a countable family of finite-dimensional subspaces, there are an uncountable subset $V$ of $U$ and a finite-dimensional subspace $M$ of $L$ such that $fV\subset M$. Now since

$M$ is a separable metric space, $fV$ is separable and there is a countable subset $W$ of $V$ such that $fW$ is dense in $V$. Since $V$ is uncountable, then with $\delta = \sup W$ there exists $\gamma \in V$ with $\gamma > \delta + 1$. Since $f\gamma$ is in the closure of $fW$, it follows by (3) and (1) that

$$f\gamma \in \operatorname{lin}\operatorname{conv} fW \subset \operatorname{lin}\operatorname{lin}^{\delta} X .$$

Now $\gamma > \delta + 1$ and $f\gamma \notin \operatorname{lin}^{\gamma-1} X$; the resulting contradiction completes the proof.

The proof of (5) is based on three lemmas, the first of which is used also in deducing (6) from (5).

(A) *Suppose $E$ is the direct sum of a family $\{L_a \mid a \in A\}$ of its linear subspaces, and for each $a \in A$, $Y_a$ is a subset of $L_a$ with $\Phi \in Y_a$. Then for each ordinal $\beta$,*

$$\operatorname{lin}^{\beta} \Sigma_{a \in A} Y_a = \Sigma_{a \in A} \operatorname{lin}^{\beta} Y_a .$$

*The order of $\Sigma_{a \in A} Y_a$ is* $\sup\{\text{order of } Y_a \mid a \in A\}$.

PROOF. Each $x \in E$ has a unique expression in the form $x = \Sigma_{a \in A} x_a$, with always $x_a \in L_a$ and $x_a = \Phi$ for all but finitely many $a \in A$. It is convenient to omit the range and index of summation, thus writing $\Sigma x$ for $x$, $\Sigma Y$ for $\Sigma_{a \in A} Y_a$, etc. Let $P_\beta$ denote the first assertion of (A) for a given $\beta$; $P_1$ is easily checked, and it remains to show that if $P_\beta$ is true for all $\beta < \gamma$ with $\gamma > 1$, then $P_\gamma$ is true. If $\gamma - 1$ exists, a double use of the inductive hypothesis shows that

$$\operatorname{lin}\operatorname{lin}^{\gamma-1} \Sigma Y = \Sigma \operatorname{lin}\operatorname{lin}^{\gamma-1} Y ,$$

whence $P_\gamma$ holds. If $\gamma$ is a limit ordinal,

$$\operatorname{lin}^{\gamma} \Sigma Y = \bigcup_{\beta<\gamma} \operatorname{lin}^{\beta} \Sigma Y = \bigcup_{\beta<\gamma} \Sigma \operatorname{lin}^{\beta} Y \subset \Sigma \bigcup_{\beta<\gamma} \operatorname{lin}^{\beta} Y = \Sigma \operatorname{lin}^{\gamma} Y .$$

Consider an arbitrary $x \in \Sigma \operatorname{lin}^{\gamma} Y$ and let $F$ be the finite set of all $a \in A$ for which $x_a \neq \Phi$. For each $a \in F$, $x_a \in \operatorname{lin}^{\beta_a} Y_a$ for some $\beta_a < \gamma$. With $\beta = \sup\{\beta_a \mid a \in F\}$ we have $\beta < \gamma$ and

$$x \in \Sigma \operatorname{lin}^{\beta} Y = \operatorname{lin}^{\beta} \Sigma Y .$$

since $x_a = \Phi \in Y_a$ for each $a \in A \sim F$. It follows that $P_\beta$ is valid for all ordinals $\beta$.

Let $\gamma$ denote the order of $\Sigma Y$, $\delta_a$ the order of $Y_a$ for each $a \in A$, and $\delta = \sup\{\delta_a \mid a \in A\}$. Use of $P_\delta$ and $P_{\delta+1}$ shows that $\gamma \leqq \delta$. If $\gamma < \delta$, there exist (for some $a \in A$) $\delta_a > \gamma$ and $p \in \operatorname{lin}^{\delta_a} Y_a \sim \operatorname{lin}^{\gamma} Y_a$. With $x_a = p$ and $x_b = \Phi$ for $b \in A \sim \{a\}$, we have $x \in \operatorname{lin}^{\delta_a} \Sigma Y \sim \operatorname{lin}^{\gamma} \Sigma Y$, whence $\gamma < \gamma$ and the contradiction completes the proof of (A).

A subset $X$ of $E$ will be called *linearly bounded* provided for each line $L$ in $E$, the set $X \cap L$ is contained in some segment.

(B) *If $X$ is convex and linearly bounded, and $E$ is finite-dimensional, then $X$ has compact closure in the Euclidean topology of $E$.*

PROOF. Since $X$ must have nonempty interior relative to the smallest linear variety containing it, we may assume without loss of generality that $\Phi$ is interior to $X$ in $E$. Now if the conclusion fails, there is a sequence $\{x_n\}_{n \in I}$ with $\|x_n\| = 1$ for each $n$, such that always $[0, n]x_n \subset X$. With $y$ a cluster point of the sequence it follows that the closure of $X$ contains the ray $[0, \infty[y$, and then, since $\Phi$ is interior to $X$, that $[0, \infty[y \subset X$. This completes the proof, since it contradicts the assumption that $X$ is linearly bounded.

The third lemma is

(C) *Suppose $\{L_i\}$, $i = 0, 1, 2, \ldots$, is a sequence of linear subspaces of $E$ whose direct sum is $E$, $C_i$ is a linearly bounded nonempty convex subset of $L_i$ for each $i \in I$, and $I'$ is the set of all $i \in I$ such that $\Phi \in C_i$. Suppose $L_0$ is a line $Ru$, $\{a_i\}_{i \in I}$ is a sequence of positive numbers converging to zero with $a_i \neq a_j$ for $i \neq j$,*

$$J = \operatorname{conv} \bigcup_{i \in I} (a_i u + C_i) \quad \textit{and} \quad K = \operatorname{conv} \bigcup_{i \in I} (a_i u + \operatorname{lin} C_i) .$$

*Then*

$$\operatorname{lin} J = \begin{cases} K & \textit{if} \quad I' \textit{ is finite,} \\ [0, 1]K & \textit{if} \quad I' = I . \end{cases}$$

*If $I' = I$ and $\operatorname{lin} C_i = C_i$ for each $i \in I$, then* $\operatorname{lin}\operatorname{lin} J = \operatorname{lin} J$.

PROOF. Let us denote by $P$ the set of all sequences $\lambda = \{\lambda_i\}_{i \in I}$ in $[0, \infty[$ such that $\lambda_i = 0$ for all but finitely many $i \in I$. For unions and sums over $I$, the range and index of summation will often be omitted, so that $\operatorname{conv} \bigcup (au + C) = J$, $\Sigma \lambda = \Sigma_1^\infty \lambda_i$, etc. To commence the proof, we observe that clearly $K \subset \operatorname{lin} J$, and if $I' = I$ then $\Phi \in \operatorname{lin} J$. Since $\operatorname{lin} J$ is convex and $[0, 1]K \subset \operatorname{conv}(K \cup \{\Phi\})$, it follows that

$$[0, 1]K \subset \operatorname{lin} J \subset \operatorname{lin}[0, 1]J$$

when $I' = I$.

Now suppose either $I'$ is finite and $Q = \{1\}$ or $I' = I$ and $Q = [0, 1]$. Consider an arbitrary $y \in \operatorname{lin} QJ$, with $z \in E$ such that $y + ]0, 1]z \subset QJ$. For each $t \in ]0, 1]$ there exist $\lambda^t \in P$ with $\Sigma \lambda^t \in Q$ and $c_i^t \in C_i$ for each $i \in I$ such that

$$y + tz = \Sigma \lambda^t (au + c^t) .$$

There is a sequence $\{t_n\}_{n \in I}$ in $]0, 1]$ such that

$$\lim_{n\to\infty} t_n = 0, \qquad \lim_{n\to\infty} \Sigma \lambda^{t_n} = \sigma_0 \in Q, \qquad \lim_{n\to\infty} \Sigma \lambda^{t_n} a = m \in R\,,$$

and for each $i\in I$

$$\lim_{n\to\infty} \lambda_i^{t_n} = \lambda_i^0 \geqq 0\,.$$

Defining $y_i$ by the conditions that $y_i \in L_i$ for $i=0, 1, 2, \ldots$ and $y=\Sigma_0^\infty y_i$ we observe that for each $t\in ]0, 1]$

$$y_0 + tz_0 = (\Sigma \lambda^t a)u$$

while

$$y_i + tz_i = \lambda_i^t c_i^t$$

for each $i\in I$. Clearly $y_0 = mu$. If $i \in I$ and $\lambda_i^{t_n}=0$ for arbitrarily large values of $n$, it follows that $y_i=\Phi$ and $\lambda_i^0=0$, whence $y_i \in \lambda_i^0 \operatorname{lin} C_i$. Suppose, on the other hand, that $\lambda_i^{t_n}$ is different from 0 for all $n>m_i$ and let $M_i$ be the linear extension of $\{y_i, z_i\}$. Then $c_i^{t_n} \in C_i \cap M_i$ for $n>m_i$; since $C_i$ is linearly bounded it follows from (B) that $\{c_i^{t_n}\}_{n\in I}$ has a subsequence convergent to a point $c_i^0 \in M_i$. From (3) it follows that

$$c_i^0 \in \operatorname{lin} C_i \qquad \text{and thus} \qquad y_i = \lambda_i^0 c_i^0 \in \lambda_i^0 \operatorname{lin} C_i\,.$$

Thus we know that

$$y_i \in \lambda_i^0 \operatorname{lin} C_i \qquad \text{for each} \quad i\in I\,.$$

Now there is a finite $F \subset I$ such that $y_i = z_i = \Phi$ for all $i \in I \sim F$. For $i \in I \sim (I' \cup F)$ we have $\Phi = y_i + tz_i \in \lambda_i^t C_i$ and $\Phi \notin C_i$, whence $\lambda_i^t = 0$ for all $t \in ]0, 1]$.

When $I'$ is finite it follows that $\lambda_i^0 = 0$ for $i \in I \sim (I' \cup F)$, and that $\Sigma \lambda^0 \in Q = \{1\}$ and $\Sigma \lambda^0 au = mu = y_0$. Since, further, $y_i \in \lambda_i^0 \operatorname{lin} C_i$ for each $i \in I$, we have

$$y \in \Sigma \lambda^0 (au + \operatorname{lin} C) \subset K\,,$$

completing the proof when $I'$ is finite.

Suppose now that $I'=I$. To show that $y\in[0, 1]K$ it suffices to produce $\mu \in P$ such that $\Sigma \mu \leqq 1$, $\mu = \lambda^0$ on $F$, and $(\Sigma a\mu)u = mu = y_0$, for then

$$y \in \Sigma \mu (au + \operatorname{lin} C) \subset [0, 1]K\,.$$

(Recall that $\Phi \in C_i$ for $i \in I'$).

For each $\mu \in P$ let $h\mu = \Sigma \mu a$. Let $S$ be the set of all $\mu \in P$ such that $\Sigma \mu \leqq 1$ and $\mu = \lambda^0$ on $F$. We wish to obtain $\mu \in S$ with $h\mu = m$, and for this it suffices to show that $hS$ intersects both $]-\infty, m]$ and $[m, \infty[$. Now since always $a_i > 0$, and $\lim a_i = 0$ as $i \to \infty$, there exists $j \in I \sim F$ with $a_j = \sup\{a_i \mid i \in I \sim F\}$. Let the functions $\xi, \eta$ be defined as follows:

$$\xi = \begin{cases} \lambda_0 & \text{on} \quad F\,, \\ 0 & \text{on} \quad I\sim F\,, \end{cases}$$

$$\eta = \begin{cases} \lambda_0 & \text{on} \quad F\,, \\ 1-\Sigma_F\lambda_0 & \text{on} \quad \{j\}, \quad \text{that is, } \eta_j = 1-\Sigma_F\lambda_0\,, \\ 0 & \text{on} \quad I\sim(F\cup\{j\})\,. \end{cases}$$

It is easy to check that

$$\xi\in S, \quad h\xi \leqq m, \quad \eta\in S, \quad \text{and} \quad h\eta = \sup hS\,.$$

Thus it remains only to show that $\sup hS \geqq m$. Let

$$\zeta^n = \begin{cases} 0 & \text{for} \quad n\in F\,, \\ \lambda^{t_n} & \text{for} \quad n\in I\sim F\,. \end{cases}$$

If $\Sigma_F\lambda_0=\sigma_0$, then $\Sigma_I\zeta^n\to 0$ as $n\to\infty$, and since $\{a_i \mid i\in I\}$ is bounded it follows that $\Sigma_I\zeta^n a\to 0$ and $\Sigma_I\lambda^{t_n}a\to\Sigma_I\xi a$, whence $h\xi=m$ and $m\in hS$. It remains to consider the case $\Sigma_F\lambda^0<\sigma_0$. Note that

$$h(\xi+s\zeta^n) = \Sigma_I\lambda^{t_n}a + \Sigma_F(\lambda_0-\lambda^{t_n})a - (1-s)\Sigma_{I\sim F}\lambda^{t_n}a\,,$$

where the first term converges to $m$, the second to 0, and the third to $(1-s)(m-\Sigma_F\lambda^0 a)$ as $n\to\infty$. Thus the limit of $h(\xi+s\zeta^n)$ can be made arbitrarily close to $m$ by making $1-s$ close enough to 0, and to show that $\sup hS\geqq m$ it suffices to show that for each $s\in\,]0, 1[$,

$$\lim_{n\to\infty}\Sigma\,(\xi+s\zeta^n) < 1\,.$$

But

$$\Sigma\,(\xi+s\zeta^n) = \Sigma_I\lambda^{t_n}+\Sigma_F(\lambda^0-\lambda^{t_n})-(1-s)\Sigma_{I\sim F}\lambda^{t_n}$$

which converges to $\sigma_0-(1-s)(\sigma_0-\Sigma_F\lambda_0)<1$. It follows that $\sup hS\geqq m$, and hence that $\operatorname{lin}[0, 1]J\subset[0, 1]K$.

The above arguments establish that if $I'=I$, then $\operatorname{lin}J=\operatorname{lin}[0, 1]J=[0, 1]K$. But if furthermore $\operatorname{lin}C_i=C_i$ for each $i\in I$, then $J=K$ and it follows that

$$\operatorname{lin}\operatorname{lin}J = \operatorname{lin}[0, 1]K = \operatorname{lin}[0, 1]J = \operatorname{lin}J\,.$$

The proof of (C) is complete.

In order to carry through the necessary induction, we prove (5) in the slightly strengthened form

(5′) *If $\alpha<\Omega$, then every infinite dimensional linear space contains a convex set $X$ of order $\alpha$ such that $\operatorname{lin}^\Omega X$ is linearly bounded.*

PROOF. The theorem will be proved by transfinite induction, being evident when $\alpha \leqq 1$. Suppose it has been proved for all $\alpha < \gamma$, where $1 < \gamma < \Omega$, and consider an infinite-dimensional linear space $E$. Clearly there is a sequence $\{L_i\}$, $i = 0, 1, 2, \ldots$, of subspaces of $E$ whose direct sum is $E$ such that $L_0$ is a line $Ru$ and $L_i$ is infinite-dimensional for each $i \in I$. When $\gamma$ is a limit (resp. nonlimit) ordinal let $\{\alpha_i\}_{i \in I}$ be a nondecreasing sequence of nonlimit ordinals whose least upper bound is $\gamma$ (resp. $\gamma - 1$), and for each $i \in I$ let $D_i$ be a convex subset of $L_i$ such that $D_i$ is of order $\alpha_i$, $\mathrm{lin}^{\Omega} D_i$ is linearly bounded, and $\Phi \in D_i$ (resp. $\Phi \in \mathrm{lin}^{\alpha_i} D_i \sim \mathrm{lin}^{\alpha_i - 1} D_i$). The existence of such $D_i$ follows from the inductive hypothesis.

Now when $\gamma$ is a limit ordinal, let $D_0 = \{\Phi\}$ and let $X = \Sigma_0^{\infty} D_i$. It follows from (A) that $X$ is of order $\gamma$, and it is easy to check that $X$ is convex and $\mathrm{lin}^{\Omega} X$ is linearly bounded. It remains to treat the case of an ordinal $\gamma$ for which $\gamma - 1$ exists. For each $i \in I$, let $a_i = 1/i$ and for each $\alpha$ let

$$X^{\alpha} = \mathrm{conv} \bigcup_{i \in I} (a_i u + \mathrm{lin}^{\alpha} D_i) .$$

We will show that $X^0$ is of order $\gamma$ and $\mathrm{lin}^{\Omega} X^0$ is linearly bounded, whence by transfinite induction the proof is complete.

Let $N$ be the set of all ordinals $\alpha \leqq \gamma - 1$ such that $\mathrm{lin}^{\alpha} X^0 = X^{\alpha}$. Then clearly $0 \in N$ and we claim $\gamma - 1 \in N$. For suppose $\delta \leqq \gamma - 1$ and $N$ is known to include all ordinals $< \delta$. Then if $\delta - 1$ exists, we have $\delta - 1 < \gamma - 1$ and

$$\mathrm{lin}^{\delta} X^0 = \mathrm{lin}\,\mathrm{lin}^{\delta-1} X^0 = \mathrm{lin}\, X^{\delta-1} = X^{\delta} ,$$

where the second equality is justified by the inductive hypothesis and the third by lemma (C), for $\Phi \in \mathrm{lin}^{\delta-1} D_i$ for only finitely many $i$ since $\delta - 1 < \gamma - 1$. If $\delta$ is a limit ordinal,

$$\mathrm{lin}^{\delta} X^0 = \bigcup_{\alpha < \delta} \mathrm{lin}^{\alpha} X^0 = \bigcup_{\alpha < \delta} X^{\alpha} = X^{\delta} ,$$

where the first equality follows from the induction hypothesis, while the last equality follows readily from the definition of $\mathrm{lin}^{\delta}$ and $X^{\alpha}$. We have established that $\gamma - 1 \in N$.

From the preceding paragraph we conclude that $\mathrm{lin}^{\gamma-1} X^0 = X^{\gamma-1}$, whence $\Phi \notin \mathrm{lin}^{\gamma-1} X^0$, since always $a_i > 0$. On the other hand, since $\Phi \in \mathrm{lin}^{\gamma-1} D_i = \mathrm{lin}^{\gamma} D_i$ for all $D_i$, we conclude from lemma (C) that

$$\mathrm{lin}\,\mathrm{lin}\, X^{\gamma-1} = \mathrm{lin}\, X^{\gamma-1} = [0, 1] X^{\gamma-1} ,$$

whence $\Phi \in \mathrm{lin}^{\gamma+1} X^0 = \mathrm{lin}^{\gamma} X^0$. It follows that $X^0$ is of order $\gamma$. With $\mathrm{lin}^{\Omega} X^0 = [0, 1] X^{\gamma-1}$, it is easy to check that $\mathrm{lin}^{\Omega} X^0$ is linearly bounded and the proof of (5′) is complete.

Observe that (6) is an immediate consequence of (A) and (5), the argument being essentially the same as that used above when $\gamma$ was a limit ordinal.

We end the paper by stating some unsolved problems.

PROBLEM. What triples of ordinals $\alpha$, $\beta$, $\gamma$ can be realized as the orders of convex sets $A$, $B$, and $C$ respectively such that $A+B=C$?

PROBLEM. What triples of ordinals $\alpha$, $\beta$, $\gamma$ can be realized as the orders of convex sets $A$, $B$, and $C$ respectively such that $\mathrm{conv}(A \cup B)=C$?

PROBLEM. Let us say that a convex set $X$ is of *level* $\alpha$ provided there exists a convex set $C \subset X$ having $\mathrm{lin}^{\alpha} C = X$ and $\mathrm{lin}^{\beta} C \neq X$ for $\beta < \alpha$; denote by $\mathrm{lev} X$ the set of all such $\alpha$. Then $\mathrm{lev} X$ is a set of ordinals between 0 and $\Omega$. Which subsets of $[0, \Omega]$ can be attained as $\mathrm{lev} X$ for some convex $X$? If $E$ is finite-dimensional and $X$ a convex subset of $E$, then $\mathrm{lev} X = \{0, 1\}$ if $X$ is closed and not a linear variety, but otherwise $\mathrm{lev} X = \{0\}$. If, on the other hand, $X$ is an infinite-dimensional linear variety, then $1 \in \mathrm{lev} X$. (See the discussion in [1] of "ubiquitous" convex sets.) What is $\mathrm{lev} X$ when $X$ is a linear space of dimension $\aleph_0$?

## REFERENCES

1. V. L. Klee, Jr., *Convex sets in linear spaces*, Duke Math. J. 18 (1951), 443–466.
2. O. M. Nikodym, *Sur la clôture faible des ensembles convexes dans l'espace réel linéaire ou aucune topologie n'est admise*, C. R. Acad. Sci. Paris 234 (1952), 1727–1728.
3. O. M. Nikodym, *Sur les clôtures faible et forte des ensembles convexes dans les espaces linéaires réels abstraits*, C. R. Acad. Sci. Paris 234 (1952), 1831–1833.
4. O. M. Nikodym, *On transfinite iterations of the weak linear closure of convex sets in linear spaces.* Part A. *Two notions of linear closure*, Rend. Circ. Mat. Palermo (ser. 2) 2 (1953), 85–105.
5. O. M. Nikodym, *On transfinite iterations of the weak linear closure of convex sets in linear spaces.* Part B. *An existence theorem in weak linear closure*, Rend. Circ. Mat. Palermo (2) 3 (1954), 5–75.

UNIVERSITY OF WASHINGTON, SEATTLE, U.S.A.
AND
UNIVERSITY OF CALIFORNIA, LOS ANGELES, U.S.A.