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ON THE PRODUCT OF TWO LEGENDRE POLYNOMIALS

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In 1878 F. Neumann [1, p. 86] and J. C. Adams [2] found the formula expressing the product of two Legendre polynomials in terms of Legendre polynomials,

$$(1)\qquad P_p(x)P_q(x) = \sum_{r=0}^{q} \frac{A_r A_{p-r} A_{q-r}}{A_{p+q-r}} \cdot \frac{2p+2q-4r+1}{2p+2q-2r+1} P_{p+q-2r}(x)\,,$$

where $p \geqq q$,

$$A_r = \frac{(\frac{1}{2})_r}{r!}, \qquad (a)_r = a(a+1)(a+2)\ldots(a+r-1), \quad \text{and} \quad (a)_0 = 1\,.$$

Later, W. N. Bailey gave in [3] and [4] respectively the following formulae for the product of two associated Legendre polynomials in terms of associated Legendre polynomials:

$$(2)\qquad (x^2-1)^{\frac{1}{2}m} P_p^m(x) P_q^m(x) = \sum_{s=0}^{q+m} C_s(p,q,m)\, P_{p+q+m-2s}^m(x)\,,$$

$$(3)\qquad (x^2-1)^{-\frac{1}{2}m} P_p^m(x) P_q^m(x) = \sum_{s=0}^{q-m} D_s(p,q,m)\, P_{p+q-m-2s}^m(x)\,,$$

where $q \geqq m$, $p \geqq q+2m$, and

$$C_s(p,q,m) = \frac{(p+m)!\,(q+m)!\,A_{s,m} A_{p-s}^m A_{q-s}^m}{2^m (p-m)!\,(q-m)!\,A_{p+q+m-s}^{-m}} \cdot \frac{2p+2q+2m-4s+1}{2p+2q+2m-2s+1}\,,$$

$$D_s(p,q,m) = \frac{2^m A_{s,-m} A_{p-s}^{-m} A_{q-s}^{-m} (p+q-2m-2s)!}{A_{p+q-m-s}^m (p+q-2s)!} \cdot \frac{2p+2q-2m-4s+1}{2p+2q-2m-2s+1}\,,$$

$$A_{s,m} = \frac{(\frac{1}{2}-m)_s}{s!}, \qquad A_s^m = \frac{(\frac{1}{2})_s}{(s+m)!}\,.$$

If s is negative, $(\frac{1}{2})_s$ is to be replaced by $(-1)^s/(\frac{1}{2})_{-s}$.

It is easily seen that both of (2) and (3) reduce to (1) when $m=0$. It is also true that (2) and (3) reduce to each other when m is replaced by $-m$ and the well known formula

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$$P_n^{-m}(x) = \frac{(n-m)!}{(n+m)!} P_n^m(x)$$

is applied.

The purpose of this note is to furnish inverse formulae corresponding to (1) and (3). We shall assume that m, p, and q are non-negative integers such that $q \geqq m$ and $p \geqq q+2m$.

We first prove the formula

$$(4) \qquad \frac{A_p A_q}{A_{p+q}} P_{p+q}(x) = \sum_{k=0}^{q} \frac{(-\frac{1}{2})_k A_{p+q-k}}{k!\, A_{p+q}} \cdot \frac{p+q-2k+1}{p+q-k+1} P_{p-k}(x) P_{q-k}(x) .$$

To prove (4) apply the Neumann Adams formula (1) to the RHS (right-hand side) to obtain

$$(5) \qquad \text{RHS of } (4) =$$

$$\sum_{k=0}^{q} \sum_{s=0}^{q-k} \frac{(-\frac{1}{2})_k A_{p+q-k}}{k!\, A_{p+q}} \cdot \frac{p+q-2k+1}{p+q-k+1} \cdot \frac{A_s A_{p-k-s} A_{q-k-s}}{A_{p+q-2k-s}} \cdot \frac{2p+2q-4k-4s+1}{2p+2q-4k-2s+1} P_{p+q-2k-2s}(x).$$

Now it is obvious that the first term in this double sum is equal to the LHS (left-hand side) of (4). Hence the proof will be complete if we show that all the other terms in (5) cancel each other. In fact we prove that the coefficient of $P_{p+q-2j}(x)$ is zero. This coefficient is indeed

$$S_j = \sum_{k=0}^{j} \frac{(-\frac{1}{2})_k A_{p+q-k} (p+q-2k+1) A_{j-k} A_{p-j} A_{q-j} (2p+2q-4j+1)}{k!\, A_{p+q} (p+q-k+1) A_{p+q-j-k} (2p+2q-2j-2k+1)}$$

$$= \frac{(2p+2q-4j+1) A_{p-j} A_{q-j}}{A_{p+q}} \sum_{k=0}^{j} \frac{(-\frac{1}{2})_k (\frac{1}{2})_{p+q-k}}{k!\,(p+q-k)!\,(j-k)!} \cdot$$

$$\cdot \frac{(p+q-j-k)!\,(\frac{1}{2})_{j-k} (p+q-2k+1)}{(\frac{1}{2})_{p+q-j-k} (p+q-k+1)(2p+2q-2j-2k+1)} .$$

Now since

$$(n-k)! = (-1)^k \frac{n!}{(-n)_k} \qquad \text{and} \qquad (a)_{j-k} = \frac{(-1)^k (a)_j}{(1-a-j)_k}$$

we have

$$S_j = K \sum_{k=0}^{j} \frac{(-p-q-1)_k (-\frac{1}{2}p - \frac{1}{2}q + \frac{1}{2})_k (-\frac{1}{2})_k (-\frac{1}{2}-p-q+j)_k (-j)_k}{k!\,(-\frac{1}{2}p-\frac{1}{2}q-\frac{1}{2})_k (\frac{1}{2}-p-q)_k (\frac{1}{2}-j)_k (-p-q+j)_k}$$

$$= K\, {}_5F_4 \left[\begin{matrix} -p-q-1, & -\frac{1}{2}p-\frac{1}{2}q+\frac{1}{2}, & -\frac{1}{2}, & -\frac{1}{2}-p-q+j, & -j; \\ & -\frac{1}{2}p-\frac{1}{2}q-\frac{1}{2}, & \frac{1}{2}-p-q, & \frac{1}{2}-j, & -p-q+j \end{matrix} \right],$$

where K is some constant. This is a well-poised series of the form

$${}_5F_4 \left[\begin{matrix} a, & 1+\frac{1}{2}a, & c, & d, & -j; \\ & \frac{1}{2}a, & 1-a-c, & 1+a-d, & 1+a+j \end{matrix} \right]$$

whose sum is

$$\frac{(1+a)_j\,(1+a-c-d)_j}{(1+a-c)_j\,(1+a-d)_j}$$

(see [5, p. 25]). In our case $(1+a-c-d)_j=(1-j)_j=0$ for all $j \geqq 1$. Hence the proof is complete.

Now the inverse formula for (3) is

$$\frac{A_p^{-m} A_q^{-m} (p+q-2m)!}{2^{-m} A_{p+q-m}^m (p+q)!} P_{p+q-m}^m(x) = (x^2-1)^{-\frac{1}{2}m} \sum_{k=0}^{q-m} E_k P_{p-k}^m(x) P_{q-k}^m(x), \tag{6}$$

where

$$E_k = \frac{(-\frac{1}{2}-m)_k A_{p+q-m-k}^m}{k!\,A_{p+q-m}^m} \cdot \frac{p+q-2k+1}{p+q-k+1},$$

which obviously reduces to (4) when $m=0$.

The proof of this formula follows in exactly the same way as the proof of (4). Applying (3) to the right hand side of (6), we get

$$\text{RHS of (6)} = \sum_{k=0}^{q-m} \sum_{s=0}^{q-m-k} E_k D_s(p-k, q-k, m) P_{p+q-m-2k-2s}^m(x)$$

whose first term is the same as the LHS of (6) and in which the coefficient of $P_{p+q-m-2j}^m(x)$ is

$$S_j = \sum_{k=0}^{j} E_k D_{j-k}(p-k, q-k, m), \qquad q-m \geqq j \geqq 1\,.$$

When we substitute the values of E_k and $D_{j-k}(p-k, q-k, m)$ in this sum we find

$$S_j = \sum_{k=0}^{j} \frac{(p+q-2k+1) A_{p+q-m-k}^m (-\frac{1}{2}-m)_k 2^m A_{j-k,\,-m} A_{p-j}^{-m} A_{q-j}^{-m}}{k!\,(p+q-k+1) A_{p+q-m}^m A_{p+q-m-k-j}^m} \cdot$$

$$\cdot \frac{(p+q-2m-2j)!\,(2p+2q-2m-4j+1)}{(p+q-2j)!\,(2p+2q-2m-2j+1)}$$

$$= K \sum_{k=0}^{j} \frac{(-m-\frac{1}{2})_k (-\frac{1}{2}p-\frac{1}{2}q+\frac{1}{2})_k (-p-q-1)_k (-j)_k (-\frac{1}{2}-p-q+m+j)_k}{k!\,(\frac{1}{2}-p-q+m)_k (-\frac{1}{2}p-\frac{1}{2}q-\frac{1}{2})_k (\frac{1}{2}-j-m)_k (-p-q+j)_k}$$

$$= K\,{}_5F_4\left[\begin{matrix} -p-q-1, & \frac{1}{2}-\frac{1}{2}p-\frac{1}{2}q, & -\frac{1}{2}-p-q+m+j, & -m-\frac{1}{2}, & -j; \\ & -\frac{1}{2}-\frac{1}{2}p-\frac{1}{2}q, & \frac{1}{2}-j-m, & \frac{1}{2}-p-q+m, & -p-q+j \end{matrix}\right],$$

where K is a constant. Hence $S_j=0$ as before.

We point out here that (6) does not lead (by putting $-m$ for m) to an inverse formula for (2). In fact, I believe that there is not an inverse formula to (2) of the type we obtained in this note.

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