MATH. SCAND. 4 (1956), 247–258

ON ONE-SIDED APPROXIMATION BY TRIGONOMETRICAL POLYNOMIALS

TORD GANELIUS

The problem of one-sided approximation by trigonometrical polynomials to periodic functions with some given regularity properties is interesting for several applications.

Let S_n denote a trigonometrical polynomial of order $n-1$ at most, and suppose that the deviation from the approximated function f is measured in some norm $\|S_n - f\|$. We then look for the minimum of $\|S_n - f\|$ if the polynomials S_n are restricted by the condition $S_n - f \geqq 0$. In the case of uniform approximation, where the norm is given by $\sup |S_n - f|$, it is trivial that the best one-sided approximation is less than twice the best approximation without this restriction, but if we consider the L_p-norm more interesting problems arise. In this paper the problem of the best one-sided trigonometrical approximation in the L_1-norm will be solved for functions belonging to a class H_r $(r = 0, 1, \ldots)$ which can be roughly described as consisting of those functions which have an rth derivative of bounded variation. This result corresponds to a result given by Freud [2] for the approximation by rational polynomials.

Let $\overline{B}_m$ denote the function of period 1 which in the interval $(0,1)$ coincides with the Bernoulli polynomial B_m. In section 1 we first solve our approximation problem in the special case $f(\theta) = b_m(\theta) = \overline{B}_m(\theta/(2\pi))$. (For the solution of this problem without the restriction $S_n \geqq b_m$, see [1, p. 195–199]).

With the aid of this preliminary investigation it is easy to treat the case $f \in H_r$. From our result on one-sided approximation to b_m, we finally derive an inequality for periodic functions which is related to a variant of Bohr's inequality given by Hörmander [4].

1. The best approximation to b_m. Let b_1 be the periodic function which in the interval $0 < \theta < 2\pi$ is given by

$$b_1(\theta) = \theta - \pi .$$

Received December 1, 1956.

Then the function b_m is uniquely determined for any integer $m>1$ by the conditions

$$b_m'(\theta) = m b_{m-1}(\theta), \qquad \int_0^{2\pi} b_m(\theta)d\theta = 0 .$$

For $m=2, 3$ we find the expressions

$$b_2(\theta) = (\theta-\pi)^2 - \tfrac{1}{3}\pi^2 ,$$
$$b_3(\theta) = (\theta-\pi)^3 - \pi^2(\theta-\pi) ,$$

in the interval $0 \leqq \theta \leqq 2\pi$, and it is easily seen that

$$b_m(\theta) = -2\, m! \sum_{k=1}^{\infty} k^{-m} \cos(k\theta - \tfrac{1}{2}m\pi) , \tag{1.1}$$

The most important property of these functions for our purposes is the following. If g is a periodic function which is $m-2$ times continuously differentiable, and if $g^{(m-2)}$ is the integral of a function g_{m-1} of bounded variation, then

$$2\pi g(\theta) - \int_0^{2\pi} g(\vartheta)d\vartheta = -(m!)^{-1} \int_0^{2\pi} b_m(\theta-\vartheta)\, dg_{m-1}(\vartheta) \tag{1.2}$$

as follows by successive integrations by parts.

We shall also make use of the well-known formula

$$\sum_{k=0}^{n-1} b_m(a + 2\pi k/n) = n^{-(m-1)} b_m(na) , \tag{1.3}$$

which easily follows from (1.1).

From now on we shall use the notation $\|h\|$ defined by

$$\|h\| = (2\pi)^{-1} \int_0^{2\pi} |h(\vartheta)|\, d\vartheta .$$

LEMMA. *Let b_m be one of the functions defined above. Then to every integer n there are trigonometrical polynomials $T_{m,n}$ and $t_{m,n}$ of order $n-1$ such that*

$$T_{m,n} \geqq b_m \geqq t_{m,n} \tag{1.4}$$

and

$$\|T_{m,n} - b_m\| = n^{-m} \sup b_m , \tag{1.5}$$

$$\|b_m - t_{m,n}\| = -n^{-m} \inf b_m . \tag{1.6}$$

These polynomials give the best possible one-sided approximation to b_m.

In order to prove the lemma we shall construct trigonometrical polynomials with certain interpolatory properties. Afterwards we show that these are the extremal polynomials.

Let us first suppose that $m > 2$ so that the function b_m has a well-defined derivative at each point. For our construction we consider the n points

$$\theta_k = \theta_k(m, n) = A_{m,n} + 2\pi k/n, \qquad k = 0, 1, \ldots, n-1 .$$

We look for a trigonometrical polynomial of order $n-1$ having the same value and the same derivative as b_m at the points θ_k. Since there are $2n$ conditions and a trigonometrical polynomial of order $n-1$ has $2n-1$ coefficients, this construction is possible only after a proper choice of $A_{m,n}$. Now it is well-known (cf. Zygmund [6, p. 49] and section 3 below) that the necessary and sufficient condition for the existence of a polynomial of order $n-1$ with the desired interpolatory property is

$$\sum_{k=0}^{n-1} b_m'(\theta_k) = 0 .$$

By aid of (1.3) we get

$$\sum_{k=0}^{n-1} b_m'(\theta_k) = m \sum_{k=0}^{n-1} b_{m-1}(A_{m,n} + 2\pi k/n)$$
$$= mn^{-(m-2)} b_{m-1}(nA_{m,n}) = n^{-(m-2)} b_m'(nA_{m,n}) .$$

If A_m satisfies $b_n(A_m) = \max b_m$ (in fact, by this condition A_m is uniquely determined), then $b_m'(A_m) = 0$ so that a suitable choice of $A_{m,n}$ is

$$A_{m,n} = n^{-1} A_m .$$

The interpolating polynomial constructed with this value of $A_{m,n}$ is denoted $T_{m,n}$ and we shall prove that this polynomial has the properties required in the lemma.

We consider the function

$$H_{m,n}(\theta) = T_{m,n}(\theta) - b_m(\theta) ,$$

and we shall first prove that this function is positive for all θ. We know that there are double zeros at the points $\theta_k = A_{m,n} + 2\pi k/n$, thus at least $2n$ zeros in a period. If the function $H_{m,n}$ should change its sign, then there would be more than $2n$ zeros and since the number of zeros is even, this assumption implies the existence of at least $2n+2$ zeros. But then it follows from Rolle's theorem that $H_{m,n}{}^{(m-2)}$ has at least $2n+1$ zeros in the open interval $(0, 2\pi)$. We conclude that there must be at least $2n-1$ zeros of $H_{m,n}{}^{(m)}$ in this interval and this is impossible, since

$H_{m,n}{}^{(m)}$ is a trigonometrical polynomial of order $n-1$ with non-vanishing constant term,

$$H_{m,n}{}^{(m)}(\theta) = T_{m,n}{}^{(m)}(\theta) - m! \qquad \text{if} \quad \theta \neq 2k\pi .$$

Hence we have proved that $H_{m,n}$ does not change its sign. That $H_{m,n} \geqq 0$ follows by calculation of

$$\int_0^{2\pi} H_{m,n}(\vartheta)\,d\vartheta = \int_0^{2\pi} T_{m,n}(\vartheta)\,d\vartheta .$$

For this purpose we observe that if S_n is a trigonometrical polynomial of order $n-1$ with constant term s_0, then it holds for every c that

$$(1.7) \qquad (2\pi)^{-1} \int_0^{2\pi} S_n(\vartheta)\,d\vartheta = s_0 = n^{-1} \sum_{k=0}^{n-1} S_n(c + 2\pi k/n) .$$

Thus we find

$$(1.8) \qquad (2\pi)^{-1} \int_0^{2\pi} T_{m,n}(\vartheta)\,d\vartheta = n^{-1} \sum_{k=0}^{n-1} T_{m,n}(\theta_k) = n^{-1} \sum_{k=0}^{n-1} b_m(\theta_k)$$
$$= n^{-m} b_m(nA_{m,n}) = n^{-m} \max b_m > 0 .$$

Hence $H_{m,n}(\theta) = T_{m,n}(\theta) - b_m(\theta) \geqq 0$. At the same time we have also proved formula (1.5). That our polynomial gives the best approximation is also evident from (1.7) and (1.8). For let S_n be another polynomial of order $n-1$ and satisfying $S_n - b_m \geqq 0$, then

$$\|S_n - b_m\| = (2\pi)^{-1} \int_0^{2\pi} S_n(\vartheta)\,d\vartheta$$
$$= n^{-1} \sum_{k=0}^{n-1} S_n(\theta_k) \geqq n^{-1} \sum_{k=0}^{n-1} b_m(\theta_k) = n^{-m} \max b_m .$$

That part of the lemma which concerns $T_{m,n}$ for $m \geqq 3$ is thus proved. If we take $\theta_k = n^{-1}a_m + 2\pi k/n$ where $b_m(a_m) = \min b_m$, the interpolating polynomial will be $t_{m,n}$ and the proof follows in the same way as above if $m \geqq 3$.

Also for $t_{2,n}$ this construction and proof are valid, and $T_{2,n}$ can be obtained in the same way if we make the formal agreement to put $b_2'(0) = 0$.

The only remaining case is $m = 1$. Then $T_{1,n}$ is defined by the following conditions if $b_1(\theta) = \theta - \pi$, $0 \leqq \theta \leqq 2\pi$,

$$T_{1,n}(2\pi k/n) = b_1(2\pi k/n), \qquad k = 1, 2, \ldots, n ,$$
$$T_{1,n}'(2\pi k/n) = b_1'(2\pi k/n) = 1, \qquad k = 1, 2, \ldots, n-1 .$$

The existence of a polynomial with these properties is evident, and the application of Rolle's theorem and the other parts of the proof are easier than above. The polynomial $t_{1,n}$ is defined by

$$t_{1,n}(2\pi k/n) = b_1(2\pi k/n), \qquad k = 0, 1, \ldots, n-1 ,$$
$$t_{1,n}'(2\pi k/n) = b_1'(2\pi k/n) = 1, \qquad k = 1, 2, \ldots, n-1 ,$$

or more simply by the observation that

$$t_{1,n}(\theta) = -T_{1,n}(2\pi-\theta)$$

must hold.

For future reference we remark that it follows from the definitions of $T_{1,n}$ and $t_{1,n}$ that

$$T_{1,n}(\theta)-t_{1,n}(\theta) = 2\pi F_n(\theta) = 2\pi n^{-2}(\sin\tfrac{1}{2}\theta)^{-2}(\sin\tfrac{1}{2}n\theta)^2 . \tag{1.9}$$

This is evident since $(T_{1,n}-t_{1,n})/2\pi$ has double zeros at the points $2\pi k/n$, $k=1, 2, \ldots, n-1$, and equals 1 for $\theta=0$, and these properties characterize the Fejér kernel F_n among the trigonometrical polynomials of order $n-1$.

It can be proved that the polynomials $T_{m,n}$ satisfy the inequality

$$\operatorname{var}(T_{m,n}-b_m) \leqq K_m n^{-(m-1)} ,$$

where var denotes the variation over a period and K_m is a constant depending on m but not on n. There is a corresponding result for $t_{m,n}$.

Easier to prove is the following somewhat weaker result.

THEOREM I. *Let b_m be the function defined in section* 1. *Then to every natural number n there are trigonometrical polynomials $S_{m,n}$ and $s_{m,n}$ of order $n-1$ such that*

$$S_{m,n} \geqq b_m \geqq s_{m,n} \tag{1.10}$$

$$\|S_{m,n}-b_m\| \leqq C_m n^{-m}, \qquad \|b_m-s_{m,n}\| \leqq C_m n^{-m} \tag{1.11}$$

$$\operatorname{var}(S_{m,n}-b_m) \leqq C_m n^{-(m-1)}, \qquad \operatorname{var}(b_m-s_{m,n}) \leqq C_m n^{-(m-1)} , \tag{1.12}$$

where C_m is a constant depending on m but independent of n.

We shall show that the trigonometrical polynomial of order $n-1$

$$S_{m,n} = m(T_{1,n} * T_{m-1,n} - T_{m-1,n} * b_1 - T_{1,n} * b_{m-1})$$

fulfils the requirements of theorem I. We use the notation $f*g$ for the convolution,

$$f * g = (2\pi)^{-1}\int_0^{2\pi} f(\theta-\vartheta)g(\vartheta)d\vartheta .$$

Our basic observation is that

$$(1.13) \qquad S_{m,n}-b_m = m\,(T_{1,n}-b_1) * (T_{m-1,n}-b_{m-1})$$

since it follows from (1.2) that $mb_{m-1}*b_1 = -b_m$. The formula (1.11) now follows since

$$\|S_{m,n}-b_m\| = m\,\|T_{1,n}-b_1\|\,\|T_{m-1,n}-b_{m-1}\| \leqq C_m n^{-m} .$$

For the variation we obtain

$$\operatorname{var}\,(S_{m,n}-b_m) \leqq m\,\|T_{m-1,n}-b_{m-1}\| \operatorname{var}\,(T_{1,n}-b_1) ,$$

and (1.12) is proved if we show that $\operatorname{var}(T_{1,n}-b_1)$ is bounded independently of n. Now (1.9) shows that

$$H_{1,n} = T_{1,n}-b_1 \leqq (T_{1,n}-b_1)+(b_1-t_{1,n}) = 2\pi F_n .$$

Hence the graph of $H_{1,n} \geqq 0$ is situated below the graph of $2\pi F_n$. The proof of the positivity of $H_{1,n}$ given above shows that this function can not have more than one maximum point between two adjacent zeros (plus one extra maximum in each period on account of the discontinuity of b_1). It follows that $\operatorname{var} H_{1,n}$ is bounded if $\operatorname{var} F_n$ is bounded. However, this is well-known and follows for instance by application of the L_1-formulation of Bernstein's inequality [1, p. 144]. Then it follows from $\|F_n\| = n^{-1}$ that $\operatorname{var} F_n = 2\pi\|F_n'\| \leqq 2\pi$.

Thus the part of theorem I concerning $S_{m,n}$ is proved. The result for $s_{m,n}$ follows in the same way by considering

$$b_m - s_{m,n} = m(T_{1,n}-b_1) * (b_{m-1}-t_{m-1,n}) .$$

2. The approximation to functions in H_r. A function h of period 2π is said to belong to H_0 if h has bounded variation over a period. The notation $h \in H_r$ where $r \geqq 1$ is an integer, means that h is $r-1$ times continuously differentiable and $h^{(r-1)}$ is the integral of a function h_r of bounded variation. If $h \in H_r$ we put

$$(2.0) \qquad V_r = V_r(h) = \int_0^{2\pi} |dh_r| .$$

THEOREM II. *If $h \in H_r$ then to every positive integer n there is a trigonometrical polynomial U_n of order $n-1$ such that*

$$(2.1) \qquad U_n \geqq h$$

and

$$(2.2) \qquad \|U_n - h\| \leqq n^{-(r+1)} C_r V_r ,$$

where
$$C_r = \big(4\pi(r+1)!\big)^{-1}\{\sup b_{r+1} - \inf b_{r+1}\} \leqq \tfrac{1}{2},$$
and this statement is not true for any smaller value of C_r.

In the sequel we write $f * dg$ for the convolution

$$(2\pi)^{-1}\int_0^{2\pi} f(\theta-\vartheta)\,dg(\vartheta).$$

Let h_r^+ and h_r^- be the positive and negative variations of h_r. Application of formula (1.2) yields

$$\begin{aligned}(r+1)!\left\{h(\theta)-(2\pi)^{-1}\int_0^{2\pi} h(\vartheta)\,d\vartheta\right\} &= -b_{r+1} * dh_r \\ &= -b_{r+1} * dh_r^+ + b_{r+1} * dh_r^-.\end{aligned}$$

We define U_n by

$$U_n(\theta) = (2\pi)^{-1}\int_0^{2\pi} h(\vartheta)\,d\vartheta + \big((r+1)!\big)^{-1}\{-t_{r+1,n} * dh_r^+ + T_{r+1,n} * dh_r^-\},$$

where $T_{r+1,n}$ and $t_{r+1,n}$ are the polynomials in the lemma. It is easily seen that U_n is a trigonometrical polynomial of order $n-1$. We now consider the difference

$$(2.3)\quad (r+1)!\{U_n(\theta)-h(\theta)\} = (b_{r+1}-t_{r+1,n}) * dh_r^+ + (T_{r+1,n}-b_{r+1}) * dh_r^-.$$

The right side is evidently non-negative and hence (2.1) is satisfied. Since

$$\int_0^{2\pi} dh_r^+ = \int_0^{2\pi} dh_r^- = \tfrac{1}{2}V_r$$

we obtain by integration of (2.3) with respect to θ and application of the lemma that

$$4\pi^2(r+1)!\,\|U_n-h\| = \pi n^{-(r+1)} V_r(\sup b_{r+1} - \inf b_{r+1}),$$

and (2.2) is proved. The inequality $C_r \leqq C_0 = \frac{1}{2}$ follows easily from the properties of b_m given in section 1.

To see that the result is best possible we consider the difference

$$h(\theta;\alpha,\beta) = b_{r+1}(\beta+\theta) - b_{r+1}(\alpha+\theta).$$

Since we can choose $h_r(\theta;\alpha,\beta) = (r+1)!\{b_1(\beta+\theta)-b_1(\alpha+\theta)\}$, it is evident that $h \in H_r$ and that $V_r(h) = 4\pi(r+1)!$. Suppose now that the trigonometrical polynomial S_n of order $n-1$ satisfies $S_n(\theta) \geqq h(\theta;\alpha,\beta)$. Then we find as before that

$$\|S_n - h\| = (2\pi)^{-1} \int_0^{2\pi} S_n(\vartheta)\,d\vartheta = n^{-1} \sum_{k=0}^{n-1} S_n(2\pi k/n) \geqq n^{-1} \sum_{k=0}^{n-1} h(2\pi k/n; \alpha, \beta).$$

Application of formula (1.3) gives

$$\|S_n - h\| \geqq n^{-(r+1)} \{b_{r+1}(n\beta) - b_{r+1}(n\alpha)\},$$

and we can choose α and β so that

$$\|S_n - h\| \geqq n^{-(r+1)} C_r V_r,$$

where C_r is the constant in theorem II.

By well-known methods (e.g. Jackson [5, p. 13]) this result can be transferred to the case of approximation by rational polynomials. For instance, if f is of bounded variation V on the closed interval $[-1, 1]$, we consider $h(\theta) = f(\cos\theta)$ and find that

$$\operatorname*{var}_{0 \leqq \theta \leqq 2\pi} h(\theta) = 2V.$$

Hence $h \in H_0$ and theorem II can be applied. The approximation polynomials may be supposed to be cosine polynomials and since a cosine polynomial of order $n-1$ is a rational polynomial of degree $n-1$ in $\cos\theta$, we obtain the following corollary of theorem II.

If $\mathrm{var}_{[-1,1]} f = V$ *then to every natural number* n *there is a polynomial* P_n *of degree at most* $n-1$ *such that* $P_n(x) \geqq f(x)$ *for* $-1 \leqq x \leqq 1$, *and*

$$\int_{-1}^{+1} \{P_n(x) - f(x)\}(1-x^2)^{-\frac{1}{2}}\,dx \leqq \pi V n^{-1}.$$

This is a special case of a theorem given by Freud [2, p. 13], but with the best possible constant on the right side of the inequality. It is also possible to derive the general case of Freud's theorem from theorem II, but since we cannot add anything to Freud's result in the general case, we omit the calculations.

The polynomial U_n in theorem II was constructed with the help of the polynomials given in our lemma. If we use theorem I instead of the lemma, that is, if we define a polynomial W_n by (cf. (2.3))

$$(r+1)!\{W_n(\theta) - h(\theta)\} = (b_{r+1} - s_{r+1,n}) * dh_r^+ + (S_{r+1,n} - b_{r+1}) * dh_r^-,$$

then the methods we have used in the proof of theorems I and II immediately give

THEOREM III. *If* $h \in H_r$, *then to every positive integer* n *there is a trigonometrical polynomial* W_n *of order* $n-1$ *such that*

$$W_n \geqq h\ ,$$
$$\|W_n - h\| \leqq D_r V_r n^{-(r+1)}\ ,$$
$$\operatorname{var}(W_n - h) \leqq D_r V_r n^{-r}\ ,$$

where D_r is a constant depending on r but independent of n, and $V_r = V_r(h)$ is the number defined by (2.0).

The corresponding theorem for approximation by rational polynomials is of importance in the proofs of certain tauberian theorems [3].

3. On the coefficients of the extremal polynomials. In the application in section 4 of the lemma it will be of some interest to have estimates for the coefficients of the polynomials.

Let us consider $T_{m,n}$ and put

$$T_{m,n}(\theta) = A_{m,n}^{(0)} + \sum_{k=1}^{n-1} (A_{m,n}^{(k)} \cos k\theta + B_{m,n}^{(k)} \sin k\theta)\ . \tag{3.1}$$

The constant term has already been calculated; formula (1.5) implies that

$$A_{m,n}^{(0)} = n^{-m} \sup b_m\ .$$

If $1 \leqq k < n$ it follows from (3.1) and the Fourier series (1.1) for b_m that

$$A_{m,n}^{(k)} + 2(m!)k^{-m} \cos \tfrac{1}{2}m\pi = \pi^{-1} \int_0^{2\pi} \cos k\theta \, \{T_{m,n}(\theta) - b_m(\theta)\}\, d\theta\ ,$$

and we get

$$|A_{m,n}^{(k)}| - 2(m!)k^{-m} \leqq 2\|T_{m,n} - b_m\| = 2n^{-m} \sup b_m\ .$$

Now our introductory remarks concerning the functions b_m show that

$$\sup b_m \leqq \tfrac{1}{2} b_2(0)\, m!\ .$$

This gives

$$k^m |A_{m,n}^{(k)}| \leqq 9\, m!\ . \tag{3.2}$$

In exactly the same way we find the same bound for $B_{m,n}^{(k)}$ and also for the coefficients $a_{m,n}^{(k)}$ and $b_{m,n}^{(k)}$ of

$$t_{m,n}(\theta) = a_{m,n}^{(0)} + \sum_{k=1}^{n-1} (a_{m,n}^{(k)} \cos k\theta + b_{m,n}^{(k)} \sin k\theta)\ . \tag{3.3}$$

We shall not use any explicit expressions for the extremal polynomials, but we make the following remarks.

The trigonometrical polynomial S_n which at the points $\theta_k = \theta_0 + 2\pi k/n$, $k = 0, 1, \ldots, n-1$, satisfies

$$S_n(\theta_k) = y_k, \quad S_n'(\theta_k) = y_k', \qquad \text{where} \quad \sum_{k=0}^{n-1} y_k' = 0 ,$$

is given (Zygmund [6, p. 49]) by

$$S_n(\theta) = \sum_{k=0}^{n-1} \{y_k F_n(\theta-\theta_k) + y_k' D_n(\theta-\theta_k)\} . \tag{3.4}$$

Here we have used the notations

$$F_n(\theta) = n^{-2}(\sin\tfrac{1}{2}\theta)^{-2}(\sin\tfrac{1}{2}n\theta)^2 = n^{-2}\left\{ n + 2\sum_{r=1}^{n-1}(n-r)\cos r\theta \right\}$$

and

$$D_n(\theta) = 2n^{-2}(\sin\tfrac{1}{2}n\theta)^2 \cot\tfrac{1}{2}\theta = 2n^{-2}\sum_{r=1}^{n-1}\sin r\theta + n^{-2}\sin n\theta .$$

If this formula is applied to our construction of $T_{m,n}$ and $t_{m,n}$ for $m \geqq 2$, one obtains by a tedious calculation after insertion of the Fourier series (1.1) for b_m that

$$T_{m,n}(\theta) = \Phi_{m,n}(\theta, A_{m,n}) \qquad \text{and} \qquad t_{m,n}(\theta) = \Phi_{m,n}(\theta, a_{m,n}) ,$$

where $\Phi_{m,n}$ is defined by

$$\Phi_{m,n}(\theta, x)$$
$$= n^{-m} b_m(nx) - 2(m!)\sum_{r=1}^{n-1}\sum_{p=-\infty}^{\infty}(p+1)(r+pn)^{-m}\cos(pnx - \tfrac{1}{2}m\pi + r\theta) .$$

(If $m=2$ the infinite sum is interpreted as $\lim_{p\to\infty}\Sigma_{-p}^{p}$).

In the case $m=1$ the calculation is easier. Using the relations

$$\left.\begin{aligned} \sum_{k=1}^{n-1} k\sin(2\pi kr/n) &= -\tfrac{1}{2}n\cot(\pi r/n) , \\ \sum_{k=1}^{n-1} k\cos(2\pi kr/n) &= -\tfrac{1}{2}n , \end{aligned}\right\} \quad r = 1, 2, \ldots, n-1 ,$$

we obtain from (3.4), with $y_k = \theta_k - \pi$ for $k=0, 1, \ldots, n-1$, with $y_k' = 1$ for $k=1, 2, \cdots, n-1$, and $y_0' = -(n-1)$, that the coefficients in $t_{1,n}$ are given by

$$\left.\begin{aligned} &(3.5\text{a}) \qquad -a_{1,n}^{(k)} = 2\pi n^{-2}(n-k) , \\ &(3.5\text{b}) \qquad -b_{1,n}^{(k)} = 2n^{-1} + 2\pi n^{-2}(n-k)\cot(\pi k/n) , \end{aligned}\right\} \quad 1 \leqq k \leqq n-1 .$$

Elementary inequalities give the bound

$$|kb_{1,n}^{(k)}| \leqq 5n^{-1}(n-k) . \tag{3.5c}$$

On account of the relation $T_{1,n}(\theta) = -t_{1,n}(2\pi-\theta)$, the coefficients of $T_{m,n}$

have the same absolute values as those of $t_{m,n}$. We also observe that (1.9) follows from (3.5).

4. An inequality for periodic functions. Let us say that a real function g with the period 2π belongs to $C_1(K)$ if $g(\theta)-K\theta$ is non-increasing. If m is an integer >1, the class $C_m(K)$ consists of the periodic functions that are integrals of order $m-1$ of functions in $C_1(K)$. For instance if $g^{(m)}$ exists and $g^{(m)} \leqq K$, then $g \in C_m(K)$. A function g belonging to any of these classes is evidently of bounded variation, and we shall use the notation

$$\gamma_k = G_k + i g_k = \pi^{-1} \int_0^{2\pi} g(\theta) e^{-ik\theta} d\theta$$

for the Fourier coefficients.

If $g \in C_m(K)$ and $\gamma_k = 0$ for $|k| < n$, we say that $g \in C_m{}^n(K)$.

In [4] Hörmander has given an interesting generalization of an inequality of Bohr. He obtained his theorem by extending a corresponding result for periodic functions. A result in the periodic case which was proved by Hörmander may be stated as follows [4, p. 38].

If $g \in C_m{}^n(K)$, then

$$Kn^{-m} \inf b_m \leqq m!\, g(\theta) \leqq Kn^{-m} \sup b_m , \tag{4.1}$$

where b_m is the function defined in section 1.

The bounds are best possible as the function $K(m!)^{-1} n^{-m} b_m(n\theta)$ shows.

As an application of our lemma we shall prove the following theorem.

THEOREM IV. *If $g \in C_m{}^1(K)$, then for every integer $n \geqq 1$,*

$$g(\theta) \leqq 9 \sum_{k=1}^{n-1}{}' \{|G_k| + |g_k|\} + (m!)^{-1} K n^{-m} \sup b_m , \tag{4.2}$$

and

$$g(\theta) \geqq -9 \sum_{k=1}^{n-1}{}' \{|G_k| + |g_k|\} + (m!)^{-1} K n^{-m} \inf b_m . \tag{4.3}$$

In particular, if $g \in C_m{}^n(K)$ we obtain (4.1).

To prove theorem IV we apply formula (1.2). We put

$$h(\theta) = g^{(m-1)}(\theta) - K\theta$$

and obtain in our previous notations

$$\begin{aligned} m!\, g(\theta) &= -b_m * dg^{(m-1)} \\ &= (T_{m,n} - b_m) * dh + K \|T_{m,n} - b_m\| - T_{m,n} * dg^{(m-1)} . \end{aligned} \tag{4.4}$$

The first term on the right is not positive and the second equals $Kn^{-m} \sup b_m$ by the lemma. It remains to estimate $|T_{m,n} * dg^{(m-1)}|$.

But $T_{m,n} * dg^{(m-1)}$ is obviously a trigonometrical polynomial of order $n-1$, and its absolute value is less than the sum of the moduli of the coefficients. If these coefficients are expressed in terms of the coefficients $A_{m,n}^{(k)}$ and $B_{m,n}^{(k)}$ of $T_{m,n}$ and the Fourier coefficients of g, we easily obtain

$$(4.5) \qquad |T_{m,n} * dg^{(m-1)}| \leqq \tfrac{1}{2} \sum_{k=1}^{n-1} k^m (|A_{m,n}^{(k)}| + |B_{m,n}^{(k)}|)(|G_k| + |g_k|) .$$

Inserting the bound given in (3.2) and combining (4.4) and (4.5) we get (4.2). The inequality (4.3) is obtained in the same way by using $t_{m,n}$ instead of $T_{m,n}$.

In the special case $m = 1$ we apply (3.5) and find that *if* $g \in C_1^1(K)$ *then*

$$(4.6) \qquad \sup |g| \leqq 2\pi \sum_{k=1}^{n-1} (1 - k/n)(|G_k| + |g_k|) + \pi K n^{-1} .$$

This formula is a suitable tool in the proofs of several results concerning uniform distribution and logarithmic potentials as will be shown in a forthcoming note.

REFERENCES

1. N. J. Achieser, *Vorlesungen über Approximationstheorie*, Berlin 1953.
2. G. Freud, *Über einseitige Approximation durch Polynome* I, Acta Scient. Math. Szeged **16** (1955), 12–28.
3. T. Ganelius, *Un théorème taubérien pour la transformation de Laplace*, C. R. Acad. Sci. Paris 242 (1956), 719–721.
4. L. Hörmander, *A new proof and a generalization of an inequality of Bohr*, Math. Scand. 2 (1954), 33–45.
5. D. Jackson, *The theory of approximation*, New York, 1930.
6. A. Zygmund, *Trigonometric interpolation*, Lecture Notes, Chicago, 1950.

UNIVERSITY OF LUND, SWEDEN