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THE ITERATION OF REGULAR MATRIX METHODS OF SUMMATION

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1. In this note we wish to discuss the iteration of regular matrix methods of summation. The iteration of a matrix $B=(b_{mn})$ with a matrix $A=(a_{mn})$ is defined by

$$\sigma_k = \sum_{m=1}^{\infty} b_{km} t_m , \qquad \text{where} \qquad t_m = \sum_{n=1}^{\infty} a_{mn} s_n .$$

In short the iteration of B with A consists of applying the matrix (b_{mn}) to the sequence of $\{t_n\}$ of A transforms of $\{s_n\}$. Different methods have been studied by Agnew, see [1] where an extensive bibliography is given.

Two methods of summation are said to be b-equivalent if every bounded sequence summable by one is also summable by the other.

A matrix is said to be a submatrix of a matrix $A=(a_{mn})$ if it is formed by extracting an infinite sequence of rows from the original matrix $A=(a_{mn})$. It is clear for example that the method of summation defined by any submatrix of $A=(a_{mn})$ sums all A summable sequences. Further theorems on submatrices are treated in a paper by Casper Goffman and the author [4]. We shall now prove a theorem.

Theorem 1. *Given two regular matrices, $B=(b_{mn})$ and $A=(a_{mn})$, there is a matrix $C=(c_{mn})$ that is b-equivalent to $A=(a_{mn})$ and such that the iteration of B with C is b-equivalent to A.*

Proof. For the matrix $B=(b_{mn})$ we can define two non-decreasing functions $\alpha(m)$ and $\beta(m)$ and a sequence $\{m_k\}$, $k=1, 2, \ldots$, so that

$$\sum_{n=1}^{\alpha(m)} |b_{mn}| = \varepsilon_m \qquad \text{and} \qquad \sum_{n=\beta(m)}^{\infty} |b_{mn}| = \varepsilon_m$$

where $\lim_{m\to\infty} \varepsilon_m = 0$, and $\alpha(m_{k+1}) = \beta(m_k)$. We now form our matrix by choosing $c_{\mu n} = a_{kn}$ for all $\alpha(m_k) \leqq \mu < \beta(m_k)$, $k=1, 2, \ldots$, so that the matrix $C=(c_{\mu n})$ is formed by repeating the row

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$$t_k = \sum_{n=1}^{\infty} a_{kn} s_n$$

$\beta(m_k) - \alpha(m_k)$ times. Clearly C is equivalent to A.

Since B is regular, the iteration of B with C sums all C summable, i.e. all A summable, sequences. On the other hand, for the iteration of B with C,

$$\tau_{m_k} = \sum_{\mu=1}^{\infty} b_{m_k \mu} \sigma_\mu , \qquad \text{where} \qquad \sigma_\mu = \sum_{n=1}^{\infty} c_{\mu n} s_n ,$$

or

$$\tau_{m_k} = \sum_{\mu=1}^{\alpha(m_k)} b_{m_k \mu} \sigma_\mu + t_k \sum_{\alpha(m_k)}^{\beta(m_k)-1} b_{m_k \mu} + \sum_{\beta(m_k)}^{\infty} b_{m_k \mu} \sigma_\mu .$$

Hence $\lim_{k \to \infty} t_k = \lim_{k \to \infty} \tau_{m_k}$ for all bounded sequences $\{s_n\}$, so that summability of a bounded sequence by the iteration of B with C implies the A summability of the sequence. This proves our theorem.

Regular summation methods $B = (b_{mn})$ that have the additional property $\lim_{m \to \infty} \max_n |b_{mn}| = 0$ have been studied by G. G. Lorentz [3]. One of their distinguishing characteristics is the existence of a counting function $\Omega(n)$, such that any bounded sequence $\{s_n\}$ is B summable to 0, provided the number of non zero s_n for $n \leqq N$ does not exceed $\Omega(N)$ for all N. In our next theorem we shall merely ask for the existence of a single sequence $\{n_\mu\}$ such that any bounded sequence $\{s_n\}$ for which $s_m = 0$, $m \neq n_\mu$, is B summable to zero. We shall now prove our theorem.

THEOREM 2. *If $B = (b_{mn})$ and $A = (a_{mn})$ are two regular matrices and there exists a sequence $\{n_\mu\}$ such that any bounded sequence $\{s'_n\}$ for which $s'_n = 0$, $n \neq n_\mu$, is B summable to zero, then corresponding to any bounded sequence $\{s_n\}$ there exists a matrix $C = (c_{mn})$ such that C is b-equivalent to A and such that the iteration of B with C sums the sequence $\{s_n\}$.*

PROOF. If A sums $\{s_n\}$ then it is clear that the iteration of B with A sums $\{s_n\}$ and our assertion is correct.

If A does not sum $\{s_n\}$, it is evident that there exists a submethod of A that does sum $\{s_n\}$. For example, if

$$\limsup_{m \to \infty} \sum_{n=1}^{\infty} a_{mn} s_n = M ,$$

we can choose a sequence $\{n_k\}$ so that

$$\lim_{k \to \infty} \sum_{n=1}^{\infty} a_{n_k n} s_n = M .$$

The submethod $A' = (a'_{kn})$ defined by $a'_{kn} = a_{n_k n}$ for all n and $k = 1, 2, \ldots$ will sum the sequence $\{s_n\}$ to M.

Let

$$\sigma_k = \sum_{n=1}^{\infty} a'_{kn} s_n \qquad \text{and} \qquad t_n = \sum_{n=1}^{\infty} a_{mn} s_n .$$

We define the matrix method $C=(c_{mn})$ by $c_{mn}=a'_{mn}$, $m \neq n_\mu$, and $c_{mn}= a_{\mu n}$, $m=n_\mu$, where the sequence $\{n_\mu\}$ is defined as in the hypothesis. So that if

$$\tau_m = \sum_{n=1}^{\infty} c_{mn} s_n ,$$

then $\tau_m=\sigma_m$, $m \neq n_\mu$, and $\tau_m=t_\mu$, $m=n_\mu$.

Since A is a submethod of C, C can not be stronger than A; on the other hand the rows of C consist merely of repetitions of the rows of A, hence, C is equivalent to A. It is now clear that the sequence $\{\tau_m\}$ converges to M, for the sequence $\{s_n\}$, everywhere but on the subsequence $\{\tau_{n_\mu}\}$. The iteration of B with C will sum $\{\tau_m\}$ and hence $\{s_n\}$ to M. This proves our assertion.

There are matrices for which the results of Theorem 2 are not true. For example, it is evident that the iteration of the identity matrix and any matrix $A=(a_{mn})$ will be equivalent to A and can not sum any non A summable sequence. A question proposed to the author by Casper Goffman [5] is whether there exist two regular consistent matrices such that no matrix includes both of them.

THEOREM 3. *Let $B=(b_{mn})$ be a regular matrix for which there is a sequence $\{n_\mu\}$ such that any bounded sequence $\{s_n\}$ with $s_n=0$, $n \neq n_\mu$, is B summable to 0. If there is a matrix $C=(c_{mn})$ that sums all B and $A=(a_{mn})$ bounded summable sequences, then there is a matrix $A'=(a'_{mn})$ b-equivalent to A, such that the iteration of B with A' sums all bounded A and B summable sequences.*

PROOF. If

$$t_m = \sum_{n=1}^{\infty} c_{mn} s_n \qquad \text{and} \qquad \tau_m = \sum_{n=1}^{\infty} a_{mn} s_n ,$$

define a matrix A' by $a'_{n_\mu n}=a_{\mu n}$ and $a'_{mn}=c_{mn}$ if $m \neq n_\mu$. Since A' is a submethod of A, A sums all sequences that are A' summable. Since C sums all A summable sequences, it is evident that A' is b-equivalent to A. The iteration of B with A' will converge whenever C converges and this proves our assertion.

2. We now turn to a different topic. Brudno [2] has defined the norm of a matrix $A=(a_{mn})$ as $\sup_m \sum_{n=1}^{\infty} |a_{mn}|$. He also defines the norm, $\|\mathfrak{A}\|$, of a method $\mathfrak{A}$ by

$$\inf_m \sup \Sigma_{n=1}^{\infty} |a_{mn}| ,$$

where the inf is taken over all the matrices equivalent to $\mathfrak{A}$ for bounded sequences. Finally he has shown that if the method $\mathfrak{B}$ sums all bounded sequences that are $\mathfrak{A}$ summable, then $||\mathfrak{B}|| \geqq ||\mathfrak{A}||$.

Several other properties of these norms can be observed. For example, we can choose a submatrix $(a'_{m_k n})$ such that

$$\lim_{k\to\infty} \sum_{n=1}^{\infty} |a'_{m_k n}| = \liminf_{m\to\infty} \sum_{n=1}^{\infty} |a_{mn}| .$$

Since this submatrix sums all A summable sequences, it is clear that $||\mathfrak{A}||$ is given by

$$\inf \liminf_{m\to\infty} \sum_{n=1}^{\infty} |a_{mn}| ,$$

where the inf is again taken over all matrices equivalent to $\mathfrak{A}$ for bounded sequences.

It does not seem to be known whether the norm of a method is always attained, that is, if there exists a matrix b-equivalent to $\mathfrak{A}$ whose norm is $||\mathfrak{A}||$. Our previous remark shows that if the norm is attained by a matrix $A=(a_{mn})$, then

$$\lim_{m\to\infty} \sum_{n=1}^{\infty} |a_{mn}| = ||\mathfrak{A}|| .$$

To return to iterations, it is clear that if the matrices $B=(b_{mn})$ and $A=(a_{mn})$ have the norms M and M' respectively, then the norm of the iteration of B with A does exceed MM'. However, if B represents the method $\mathfrak{B}$ and A the method $\mathfrak{A}$, and the iteration of B with A represents a method $\mathfrak{C}$, then it may turn out that

$$||\mathfrak{C}|| \geqq ||\mathfrak{A}|| \cdot ||\mathfrak{B}||$$

though of course $||\mathfrak{C}|| \leqq MM'$. We shall illustrate this with an example. We shall choose for our first matrix the Cesàro mean, C,

$$t_n = \frac{1}{n+1}(s_0+s_1+\ldots+s_n);$$

it is evident that the norm of this matrix and the norm of the method it represents are both 1. For our second matrix, let

$$t_{2^p} = \tfrac{1}{2}(s_{2p}+s_{2p+1}) \quad \text{and} \quad t_\mu = 2s_{2p}-s_{2p+1}, \qquad 2^p < \mu < 2^{p+1} ,$$

$p=0, 1, 2, \ldots$. From our previous remarks we can conclude that the norm of the method represented by this second matrix is 1. On the other hand, it transforms the sequence given by $s_n=(-1)^n$ into a sequence $\{s'_n\}$ where $s'_{2^p}=0$, $s'_n=3$, $n \neq 2^p$.

The sequence $\{s'_n\}$ is summable to 3 by the Cesàro matrix, that is to say the iteration of the Cesàro matrix with our second matrix produces a matrix that sums $s_n = (-1)^n$ to 3. The method represented by this iteration must have norm 3, as no matrix whose norm is less than 3 could sum $s_n = (-1)^n$ to 3. However, the product of the norms of the methods represented by the two matrices is 1.

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