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RANDOM SEQUENCES AND ADDITIVE NUMBER THEORY

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1. The fundamental problem in additive number theory is to decide if a given sequence of positive integers $\{a_v\}_0^\infty$ constitutes a *basis*, i.e. if every positive integer n can be represented in the form

$$n = \sum_1^p \xi_i, \qquad \xi_i \in \{a_v\}_0^\infty,$$

with p independent of n. In the vast literature on this subject, there seems to be no investigation concerning the 'average' behaviour of a sequence $\{a_v\}_0^\infty$ in this respect. For example, if $a_v = v \cdot p_v{}^k$, where p_v is the vth prime and $a_0 = 1$, is it 'likely' or 'unlikely' that $\{a_v\}_0^\infty$ is a basis? The object of this note is to prove a result which makes it possible to answer a question of this type. It should be stressed that the randomization of the problem makes the proof comparatively easy and that the classical tools of analytic number theory are quite sufficient for our purpose. Our result can, no doubt, also be proved quite elementary.

2. Let $\{\lambda_n\}_1^\infty$ be a given sequence of positive integers such that $\log \lambda_n$ is an increasing convex function of $\log n$. We now consider the set of all sequences $\{t_v\}_0^\infty$ such that

$$(1) \qquad t_0 = 1, \quad 0 \leqq t_v < \lambda_v, \quad v = 1, 2, \ldots .$$

With the sequence $\{t_v\}_0^\infty$ we associate the real number

$$t = \sum_1^\infty \frac{t_v}{\lambda_1 \lambda_2 \ldots \lambda_v},$$

so that $0 \leqq t \leqq 1$ and Lebesgue measure on this interval is interpreted as probability of the corresponding set of sequences [1]. The theorem to be proved is the following:

THEOREM. *Under the conditions stated above, a sequence* $\{t_v\}_0^\infty$, *satisfying conditions* (1), *is a basis with probability* 1 *if*

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(2) $$\lim_{n=\infty} \frac{\log \lambda_n}{\log n} < \infty .$$

If (2) *does not hold, the above mentioned probability is zero.*

The last part of the theorem is easily proved.—Suppose that $n^k/\lambda_n \to 0$ for every k. Then $n^k/t_n \to 0$ with probability 1, since the probability of the inequality $t_n < n^k$ is n^k/λ_n and the series $\Sigma_1^\infty n^k \lambda_n^{-1}$ converges. If now $n^k/t_n \to 0$, choose m so that $t_n > t_m$ for $n > m$. The number of integers $\leqq t_m$ which can be represented by a sum of p numbers t_v is $< (m+1)^p < t_m$ for m sufficiently large. Hence $\{t_v\}_0^\infty$ is no basis, which was to be proved.

3. We now turn to the direct part of the theorem. We first observe that *we may assume* $\lambda_n = n^k$. Namely, if a is the limit (2), choose $k > 2a$. For every n, define m by the inequality

$$\lambda_m < n^k \leqq \lambda_{m+1} .$$

Since $\log \lambda_n$ is a convex function of $\log n$ and since $m > n^2$ for n sufficiently large, it follows that

$$\frac{n^k - \lambda_m}{n^k} < \text{Const.} \frac{\lambda_m}{m\, n^k} < \text{Const.} \frac{1}{n^2} .$$

The series $\Sigma\,(1 - \lambda_m/n^k)$ being convergent, a sequence $\{t_v\}_0^\infty$ chosen at random under the conditions $0 \leqq t_n < n^k$, with probability 1 also satisfies the inequalities $0 \leqq t_n < \lambda_m$ for n sufficiently large. But, assuming the theorem proved for the sequence $\{n^k\}$, $\{t_v\}_0^\infty$ is also a basis with probability 1. *A fortiori*, the theorem then holds for the sequence $\{\lambda_n\}$; we need only use the subsequence of λ_m's defined above.

In the following we shall use the notation $\lambda_n = n^k$ and $\alpha = 1/k$. We set

(3) $$s_n(\theta; t) = s_n(\theta) = \sum_{t_v \leqq n} e^{it_v\theta}, \qquad -\pi < \theta \leqq \pi ,$$

and choose a positive integer $p > k$ and define

$$T_n(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} |s_n(\theta)|^{2p} d\theta .$$

The above definitions have a sense for almost every t and the following lemma holds.

LEMMA. *If*

(4) $$m_n = \int_0^1 T_n(t) dt$$

and

(5) $$\sigma_n^2 = \int_0^1 \{T_n(t) - m_n\}^2\, dt$$

are the meanvalue and standard deviation of $T_n(t)$, then

(6) $$m_n = O(n^{2p\alpha-1})$$

and

(7) $$\sigma_n = O(n^{2p\alpha-1-\frac{1}{2}\alpha}) .$$

PROOF. Define the function $\psi_n(\theta)$ by the relation

(8) $$\psi_n(\theta) = \frac{1}{2\pi}\int_0^1 |s_n(\theta)|^{2p}\, dt$$
$$= \sum_{\alpha_j=0}^{\infty} \frac{1}{2\pi} \int_{t_{\alpha_j} \leq n} \exp\{i\theta\,(t_{\alpha_1} + \ldots + t_{\alpha_p} - t_{\alpha_{p+1}} - \ldots - t_{\alpha_{2p}}\}\, dt .$$

Let us first consider the terms where all α_j are unequal. For such a term the integral is easily evaluated and its value is

$$\prod_{v=1}^{2p} \frac{e^{i\mu_v\theta} - 1}{e^{i\theta} - 1} \cdot \frac{1}{\lambda_{\alpha_v}}, \qquad \mu_v = \min\,(\lambda_{\alpha_v}, n+1) .$$

The sum of all these terms is thus majorized by

$$\text{Const.} \frac{1}{|\theta|^{2p}} \left(\sum_1^{\infty} \frac{1}{\lambda_v}\right)^{2p} .$$

Let us now consider the terms for which, for example, $\alpha_1 = \alpha_{p+1}$ while the other α_j's are unequal and $\neq \alpha_1$. In the same way as before, we find that the corresponding sum is less than

$$\text{Const.} \frac{1}{|\theta|^{2p-2}} \left(\sum_1^{\infty} \frac{1}{\lambda_v}\right)^{2p-2} \left(n^{\alpha} + n \sum_{\lambda_v > n} \frac{1}{\lambda_v}\right)$$

which, by Hölder's inequality, is dominated by

(9) $$\text{Const.} \left\{\frac{1}{|\theta|^{2p}} \left(\sum_1^{\infty} \frac{1}{\lambda_v}\right)^{2p} + n^{p\alpha}\right\} .$$

It is easy to see that the sum of the terms corresponding to the remaining cases all have the same majorant (9). We get a better estimate if we observe that for every positive integer h

$$|s_n(\theta)|^{2p} \leqq \text{Const.}\left\{h^{2p} + \left|\sum_{\substack{t_v \leqq n \\ v \geqq h}} e^{it_v\theta}\right|^{2p}\right\}.$$

This yields

$$\psi_n(\theta) \leqq \text{Const.}\left\{h^{2p} + \frac{1}{|\theta|^{2p}}\left(\sum_{v=h}^{\infty}\frac{1}{\lambda_v}\right)^{2p} + n^{p\alpha}\right\}.$$

If we choose h in the most favourable way, $h \sim |\theta|^{-\alpha}$, we obtain

$$\psi_n(\theta) \leqq \text{Const.}\,\{|\theta|^{-2p\alpha} + n^{p\alpha}\}. \tag{10}$$

A trivial upper bound of $\psi_n(\theta)$ is Const. $n^{2p\alpha}$ obtained by replacing $e^{it_v\theta}$ by 1 in the definition of $s_n(\theta)$. Using this estimate in $(-\delta, \delta)$ and (10) for $|\theta| > \delta$, we find

$$m_n = \int_{-\pi}^{\pi} \psi_n(\theta)\,d\theta < \text{Const.}\,\{n^{2p\alpha}\delta + \delta^{-2p\alpha+1} + n^{p\alpha}\}.$$

Here we choose $\delta = n^{-1}$ and obtain (6) since $2p\alpha - 1 > p\alpha$.

For the proof of (7), which proceeds along the same lines as above, we introduce the function $\eta(x)$, defined by

$$\eta(x) = \begin{cases} 1, & x = 0 \\ 0, & x \neq 0. \end{cases}$$

T_n can be written as a sum of statistical variables

$$\tau_j = \eta(t_{\alpha_1} + t_{\alpha_2} + \ldots - t_{\alpha_{2p}})$$

for a suitable numbering of τ_j. If τ_j has meanvalue $\mu_j = M_n(\tau_j)$, then

$$\sigma_n^2 = \sum_{i,j}\{M_n(\tau_i\tau_j) - \mu_i\mu_j\}.$$

Here M_n indicates that we take the meanvalue under the conditions $t_{\alpha_j} \leqq n$. In this sum, all terms (i, j) disappear with the exception for those for which τ_i and τ_j have one or more t_{α_v} in common. The estimation above then easily shows that the sum of $M_n(\tau_i\tau_j)$ over those pairs (i, j) is less than

$$O(n^{4p\alpha-2-\alpha}).$$

4. The proof of our theorem is now completed by well-known arguments (see e.g. [2]). Let $f(z; t)$ be the function defined formally by, $z = re^{i\theta}$, $r < 1$,

$$f(z; t) = \sum_0^{\infty} z^{t_v} = (1-r)\sum_0^{\infty} s_n(\theta) r^n$$

and set

$$f(z;\,t)^p = \sum_0^\infty a_v(t) z^v .$$

We have the following set of relations:

$$(11)\quad \Big(\sum_0^\infty r^{2\lambda_v}\Big)^{2p} \leqq f(r^2;\,t)^{2p} = \Big(\sum_0^\infty a_v(t)\, r^{2v}\Big)^2 \leqq \Big(\sum_{a_v\neq 0} r^{2v}\Big)\Big(\sum_0^\infty |a_v|^2 r^{2v}\Big) .$$

To get an estimate of the last factor we observe that

$$|f(z;\,t)|^{2p} \leqq (1-r)^{2p}\Big(\sum_0^\infty |s_n(\theta)|\, r^n\Big)^{2p} \leqq \text{Const.}\ (1-r)\sum_0^\infty |s_n(\theta)|^{2p} r^n$$

and find

$$(12)\quad \sum_0^\infty |a_v|^2 r^{2v} = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(z;\,t)|^{2p}\, d\theta \leqq \text{Const.}\ (1-r)\sum_0^\infty r^n T_n(t) .$$

Whenever the last series converges, all estimates (11) and (12) are legitimate. Let $K>0$ be a (large) constant and choose t so that

$$(13)\qquad R_n(t) = \sum_0^{2^n} T_v(t) \leqq K\, 2^{2p\alpha n}, \qquad n = 1, 2, \ldots .$$

Since

$$\varrho_n = \int_0^1 R_n(t)\, dt = O(2^{2p\alpha n})$$

and

$$\Big\{\int_0^1 (R_n(t)-\varrho_n)^2\, dt\Big\}^{\frac12} = O(2^{2p\alpha n - \frac12 \alpha n}) ,$$

the measure of the set where $|R_n(t)-\varrho_n| > A\, 2^{2p\alpha n}$ is $< A^{-1} O(2^{-\frac12 \alpha n})$. Hence (13) holds except on a set whose measure tends to zero as $K \to \infty$. Now if (13) holds, a partial summation shows that the right hand side of (12) is $O\big((1-r)^{1-2p\alpha}\big)$. If we combine this with the fact that the left hand side of (11) is $>\text{const.}(1-r)^{-2p\alpha}$, we find that

$$(1-r)\sum_{a_v\neq 0} r^{2v} > \text{const.} > 0, \qquad r \to 1 ,$$

and this is equivalent to the fact that the set of integers representable by p numbers t_v has positive density. Our theorem now follows from the famous Schnirelmann density theorem [2, p. 4].

REFERENCES

1. É. Borel, *Les probabilités dénombrables et leur applications arithmétiques,* Rend. Circ. Mat. Palermo 27 (1909), 247–271.
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