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ON SPECTRAL ANALYSIS IN THE NARROW TOPOLOGY

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The following concept of closure was introduced by A. Beurling in [1]:

Let D be a class of bounded continuous complex-valued functions on $(-\infty, \infty)$. We say that a function $\psi(x)$ is included in the narrow closure of D if, for every $\varepsilon > 0$ and every compact interval $(-a, a)$, there exists a function $\varphi_0(x) \in D$ such that

$$\max_{-a \leqq x_0 \leqq a} |\psi(x_0) - \varphi_0(x_0)| + \big| \|\psi\| - \|\varphi_0\| \big| < \varepsilon ,$$

where $\|.\|$ denotes the uniform norm, i.e.

$$\|\varphi\| = \sup_{-\infty < x < \infty} |\varphi(x)| .$$

Using the theory of analytic functions Beurling then proved the following theorem:

THEOREM 1. *If the bounded and uniformly continuous function $\varphi(x)$ is $\not\equiv 0$, then there is at least one function $e^{i\lambda x}$ (with real λ) included in the narrow closure of the class of all linear combinations of translates*

$$\sum_1^n c_\nu \varphi(x + x_\nu) .$$

(Here c_ν denote arbitrary complex numbers and x_ν arbitrary real numbers.)

We shall here give an alternative proof of Beurling's theorem, only using pure Fourier analysis arguments. As for the methods employed, one advantage connected with this limitation is that the proof can be easily carried over to the case where the real line is replaced by an arbitrary Abelian locally compact group. That the theorem is true even in that case was obtained from a more general theory in [2]. The proof in this paper is, however, more direct and does not use the theory of normed rings.

Since the function $\varphi(x)$ in the theorem was assumed to be uniformly continuous, any function of the form

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$$\varphi * f = \int_{-\infty}^{\infty} \varphi(x - x_0) f(x_0)\, dx_0 ,$$

where $f \in L^1$, can be approximated uniformly arbitrarily closely by functions

$$\sum_1^n c_\nu \varphi(x + x_\nu) .$$

Hence Theorem 1 is a corollary of the following theorem:

THEOREM 2. *If the bounded Lebesgue measurable function $\varphi(x)$ is $\not\equiv 0$, then there is at least one function $e^{i\lambda x}$ included in the narrow closure of the class of functions $\varphi * f$, where $f \in L^1$.*

We shall prove Theorem 2, and for that purpose we need a simple lemma.

LEMMA. *Given $\varepsilon > 0$, there exists a function*

$$h(\theta) = \sum_{-\infty}^{\infty} c_n e^{in\theta}, \qquad \text{where} \qquad \sum_{-\infty}^{\infty} |c_n| < \varepsilon ,$$

such that, for some $\delta > 0$,

$$h(\theta) = 1 - e^{i\theta} \quad \text{if} \quad |1 - e^{i\theta}| < \delta . \qquad (\theta \text{ is real.})$$

PROOF OF THE LEMMA. We may of course assume that $\varepsilon \leqq 5$, hence that $b = \frac{1}{50}\pi\varepsilon^2 \leqq \frac{1}{2}\pi$. Let us then choose

$$h(\theta) = \begin{cases} 1 - e^{i\theta} & \text{if } |\theta| \leqq b , \\ 1 - e^{i(2b-\theta)} & \text{if } b < \theta < 2b , \\ 1 - e^{i(-2b-\theta)} & \text{if } -2b < \theta < -b , \\ 0 & \text{if } 2b \leqq |\theta| \leqq \pi . \end{cases}$$

Then, using the Cauchy inequality, we obtain

$$\begin{aligned} \sum_{-\infty}^{\infty} |c_n| &\leqq \left[\sum_{-\infty}^{\infty} (1 + n^2)^{-1} \sum_{-\infty}^{\infty} (1 + n^2) |c_n|^2 \right]^{\frac{1}{2}} \\ &< \left[(5/2\pi) \int_{-\pi}^{\pi} \left(|h(\theta)|^2 + |h'(\theta)|^2 \right) d\theta \right]^{\frac{1}{2}} \\ &\leqq \left[(5/2\pi) \int_{-2b}^{2b} (2^2 + 1^2)\, d\theta \right]^{\frac{1}{2}} \\ &= [(5/2\pi) \cdot 4b \cdot 5]^{\frac{1}{2}} = \varepsilon . \end{aligned}$$

PROOF OF THEOREM 2. We denote by Λ_φ the set of real numbers λ with the following property:

For every open interval on $-\infty < t < \infty$, containing λ, there exists a function $f \in L^1$ with the Fourier transform

$$\hat{f}(t) = \int_{-\infty}^{\infty} e^{-itx} f(x)\, dx$$

vanishing outside the interval, and such that $\varphi * f \not\equiv 0$.

We shall prove that this set is not empty. If it were empty, then we could cover $-\infty < t < \infty$ with open intervals I, such that $\varphi * f \equiv 0$ for any $f \in L^1$ with $\hat{f}$ vanishing outside one of these intervals. And any compact interval C on $-\infty < t < \infty$ could hence be covered by a finite number of these intervals, say $\{I_\nu\}_1^n$. It is wellknown that any function $f \in L^1$ with $\hat{f}$ vanishing outside C can be decomposed into a sum

$$f = \sum_1^n f_\nu ,$$

where for every ν, $f_\nu \in L^1$ and $\hat{f}_\nu$ vanishes outside I_ν (cf. the proof of lemma 6_{15}, p. 91 in Wiener [3]). But since for every ν

$$\varphi * f_\nu \equiv 0 ,$$

we thus obtain $\varphi * f \equiv 0$.

This would then be true for every $f \in L^1$ with $\hat{f}$ vanishing outside a compact interval. This subclass of L^1 is, however, dense in L^1 (cf. lemma 6_9, p. 82 in Wiener [3]) and hence $\varphi * f \equiv 0$ for every $f \in L^1$, and this is impossible if $\varphi \not\equiv 0$. Therefore Λ_φ is not empty.

Let us now choose an arbitrary $\lambda \in \Lambda_\varphi$. We shall show that $e^{i\lambda x}$ is included in the narrow closure of the functions $\varphi * f$, where $f \in L^1$, and this will then prove the theorem.

For that purpose let us choose an arbitrary $\varepsilon > 0$ and an arbitrary compact interval $(-a, a)$. The subset E of $-\infty < t < \infty$ where

$$|1 - e^{i(t-\lambda)x}| < \delta$$

for all $x \in (-a, a)$, δ being a number which fulfills the condition in the lemma with respect to the given ε, is a set which contains λ as an interior point. Hence there exists a function $f \in L^1$ with $\varphi_0 = \varphi * f \not\equiv 0$ and with $\hat{f}$ vanishing outside E. There is no real restriction in assuming that

$$\varphi_0(0) = 1 \qquad \text{and} \qquad 1 \leqq \|\varphi_0\| \leqq 1 + \varepsilon ,$$

for otherwise we can make these conditions fulfilled by a suitable translation of f and a multiplication of f with a proper constant. Then we have if $x_0 \in (-a, a)$

$$\begin{aligned} |e^{i\lambda x_0} - \varphi_0(x_0)| &= |\varphi_0(0)\, e^{i\lambda x_0} - \varphi_0(x_0)| \\ &\leqq \|\varphi_0(x)\, e^{i\lambda x_0} - \varphi_0(x+x_0)\| \\ &= \|\varphi(x) * [f(x)\, e^{i\lambda x_0} - f(x+x_0)]\| \,. \end{aligned}$$

But the function

$$g(x) = f(x)\, e^{i\lambda x_0} - f(x+x_0)$$

has the Fourier transform

$$\hat{g}(t) = \hat{f}(t)\, e^{i\lambda x_0} - \hat{f}(t)\, e^{itx_0} = \hat{f}(t)\, e^{i\lambda x_0}[1 - e^{i(t-\lambda)x_0}] \,,$$

and since $\hat{f}(t)$ vanishes outside the set where

$$|1 - e^{i(t-\lambda)x_0}| < \delta \,,$$

the lemma gives us

$$\begin{aligned} \hat{g}(t) &= \hat{f}(t)\, e^{i\lambda x_0} \sum_{-\infty}^{\infty} c_n e^{in(t-\lambda)x_0} \\ &= \hat{f}(t)\, e^{i\lambda x_0} \sum_{-\infty}^{\infty} c_n e^{-in\lambda x_0} e^{intx_0} \,. \end{aligned}$$

Hence

$$g(x) = e^{i\lambda x_0} \sum_{-\infty}^{\infty} c_n e^{-in\lambda x_0} f(x+nx_0) \,,$$

which yields

$$\begin{aligned} |e^{i\lambda x_0} - \varphi_0(x_0)| &\leqq \left\| \varphi(x) * \left[e^{i\lambda x_0} \sum_{-\infty}^{\infty} c_n e^{-in\lambda x_0} f(x+nx_0) \right] \right\| \\ &= \left\| \sum_{-\infty}^{\infty} c_n e^{-in\lambda x_0} \varphi_0(x+nx_0) \right\| \\ &\leqq \sum_{-\infty}^{\infty} |c_n|\, \|\varphi_0(x)\| \\ &\leqq \varepsilon(1+\varepsilon) \,. \end{aligned}$$

This is true for all $x_0 \in (-a, a)$. Hence

$$\max_{-a \leqq x_0 \leqq a} |e^{i\lambda x_0} - \varphi_0(x_0)| + \big| \|e^{i\lambda x}\| - \|\varphi_0\| \big| \leqq \varepsilon(1+\varepsilon) + \varepsilon \,,$$

and since ε and $(-a, a)$ were arbitrary this implies that $e^{i\lambda x}$ is included in the narrow closure of the class of elements $\varphi * f$.

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