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NOTE ON GROUPS WITH AND WITHOUT FULL BANACH MEAN VALUE

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In this note I shall prove two theorems. Theorem 1 is a simple consequence of the Main Theorem in [4, p. 245]. It shows that there exists an upper mean value in any group with a full Banach mean value (see [4, p. 243]), which is quite analogous to the usual upper mean value in abelian groups and has similar properties. As to Theorem 2, let me remark that a group G has a full Banach mean value except when there exists a function $H(x)$ which is $\geqq 1$ for all x and has the form

$$H(x) = h_1(x) - h_1(xa_1) + \ldots + h_n(x) - h_n(xa_n) , \tag{0}$$

where $h_1, \ldots, h_n$ are bounded functions on G and $a_1, \ldots, a_n$ elements from G; see [2, Theorem 4, p. 14]. Theorem 2 contains a surprisingly stronger result.

Professor B. Sz.-Nagy has kindly called my attention to some overlapping between my papers [2] and [4] and an earlier paper by J. Dixmier [1]; in particular Theorem 4 of [2, p. 14] (with L the space of all bounded functions) and part of the sufficiency statement in the Main Theorem of [4, p. 245] are contained in Dixmier's paper.

For every bounded function $f(x)$ on a group G with full Banach mean value we put

$$\overline{M}_0 f = \inf_{\mathcal{A}} \sup_x \Sigma\, \alpha_n f(xa_n) ,$$

where the infimum is to be taken over all $\mathcal{A} = \{\alpha_1, \ldots, \alpha_N;\ a_1, \ldots, a_N\}$, $\alpha_n > 0,\ \Sigma \alpha_n = 1,\ a_n \in G$.

THEOREM 1. *In a group G with full Banach mean value, the expression $\overline{M}_0 f$ has the properties*

$$\overline{M}_0 f \leqq \sup_x f(x) , \tag{1}$$

$$\overline{M}_0 \{\lambda f\} = \lambda \overline{M}_0 f, \qquad \lambda \geqq 0 , \tag{2}$$

$$\overline{M}_0 \{f+g\} \leqq \overline{M}_0 f + \overline{M}_0 g , \tag{3}$$

$$\overline{M}_0 \{f(x) - f(xa)\} = 0 . \tag{4}$$

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Every right-invariant Banach mean value Mf on the space of all bounded functions on G [2, p. 14] *satisfies*

$$(5) \qquad -\overline{M}_0(-f) \leqq Mf \leqq \overline{M}_0 f\,,$$

and for any fixed f the Banach mean value Mf can be chosen arbitrarily in the interval (5).

PROOF. If Mf is a right-invariant Banach mean value, we have

$$Mf = M\{\Sigma\alpha_n f(xa_n)\} \leqq \sup_x \Sigma\alpha_n f(xa_n)$$

and hence $Mf \leqq \overline{M}_0 f$. The other part of (5) follows by replacing f by $-f$ in the inequality just obtained. The existence of a right-invariant Banach mean value Mf on the space of all bounded functions on G, which for an arbitrary fixed f can be chosen arbitrarily in the interval (5), is a consequence of (1), (2), (3), (4), and the theorem of Banach stated in [3].

In order to prove (1), (2), (3), (4) we consider

$$\overline{M}f = \inf_H \sup_x \big(f(x)+H(x)\big)\,,$$

where the infimum is taken over all H of the form (0). It is finite and has all the properties stated for $\overline{M}_0 f$ in (1), (2), (3), (4); see [2, pp. 14–15]. We shall prove that $\overline{M}_0 f = \overline{M}f$. Since the function

$$\Sigma\alpha_n f(xa_n) = f(x) + \Sigma\alpha_n\big(f(xa_n)-f(x)\big)$$

has the form $f(x)+H(x)$, it follows from the definitions of $\overline{M}_0 f$ and $\overline{M}f$ that $\overline{M}f \leqq \overline{M}_0 f$.

In order to prove that $\overline{M}_0 f \leqq \overline{M}f$, let

$$H(x) = h_1(x)-h_1(xa_1)+\ldots+h_n(x)-h_n(xa_n)$$

be chosen such that $f(x)+H(x) \leqq \overline{M}f+\varepsilon$. By means of the Main Theorem in [4, p. 245], we can find, to every $\eta>0$, a finite set E of elements from G such that

$$N(E\cap Ea_i) > (1-\eta)N(E), \qquad i = 1,\ldots,n\,,$$

where $N(.)$ denotes the number of elements in the set between the brackets. Then

$$\begin{aligned}\overline{M}f+\varepsilon &\geqq N(E)^{-1}\sum_{a\in E}\big(f(xa)+H(xa)\big)\\ &= N(E)^{-1}\sum_{a\in E} f(xa) + \sum_{i=1}^{n} N(E)^{-1}\sum_{a\in E}\big(h_i(xa)-h_i(xaa_i)\big)\\ &\geqq N(E)^{-1}\sum_{a\in E} f(xa) - 2\eta\sum_{i=1}^{n}\sup_x |h_i(x)|\,.\end{aligned}$$

Choosing η sufficiently small we get

$$N(E)^{-1} \sum_{a \in E} f(xa) \leqq \overline{M}f + 2\varepsilon ,$$

and the inequality $\overline{M}_0 f \leqq \overline{M}f$ follows. This completes the proof of Theorem 1.

REMARK. Theorem 1 remains valid when the word "right-invariant" is replaced by "bi-invariant", $\overline{M}_0 f$ by

$$\overline{M}_2 f = \inf \sup_x \Sigma \alpha_n f(b_n x a_n) ,$$

and (4) by $\overline{M}_2 \{f(x) - f(bxa)\} = 0$.

This may be proved in a similar way as Theorem 1.

THEOREM 2. *In a group G without full Banach mean value every bounded function can be uniformly approximated by functions of the form* (0).

More generally: If L is a right-translation invariant linear space of bounded functions on G which is closed with respect to the formation of maximum and minimum between two functions, contains the constants, and has no right-invariant Banach mean value, then every function from L can be uniformly approximated by functions

$$H(x) = h_1(x) - h_1(xa_1) + \ldots + h_n(x) - h_n(xa_n) ,$$

where the h's belong to L.

PROOF. For a function f from L we put

$$(6) \qquad \overline{M}f = \inf{}^* \sup_x \big(f(x) + H(x)\big) ,$$

where the infimum is to be taken over all functions H of the form (0) with h's belonging to L and such that $f(x) + H(x) \geqq 0$. We remark that since L has no right-invariant Banach mean value, there exists an $H_0(x)$ of the form (0) with h's from L such that $\inf_x H_0(x) = 1$ (see [2, p. 14]), and consequently the function

$$H(x) = |\inf_x f(x)| \, H_0(x)$$

has the form (0) with h's in L and satisfies

$$\begin{aligned} \inf_x \big(f(x) + H(x)\big) &\geqq \inf_x f(x) + \inf_x H(x) \\ &= \inf_x f(x) + |\inf_x f(x)| \geqq 0 . \end{aligned}$$

Thus the set of H's over which the infimum in (6) is to be taken, is not empty. Further we obtain the inequality

$$0 \leqq \overline{M}f \leqq \sup_x f(x) + |\inf_x f(x)| \sup_x H_0(x)$$

so that

(7) $0 \leqq \overline{M}f \leqq C \sup_x |f(x)|$ with $C = 1 + \sup_x H_0(x) \quad (\geqq 2)$.

We first show that

$$\overline{M}(f+g) \leqq \overline{M}f + \overline{M}g \,. \tag{8}$$

Let $\varepsilon > 0$ be given. We choose $H_1(x)$ and $H_2(x)$ of the form (0) with h's in L such that

$$0 \leqq f(x) + H_1(x) \leqq \overline{M}f + \varepsilon \quad \text{and} \quad 0 \leqq g(x) + H_2(x) \leqq \overline{M}g + \varepsilon \,.$$

Then

$$0 \leqq f(x) + g(x) + H_1(x) + H_2(x) \leqq \overline{M}f + \overline{M}g + 2\varepsilon \,.$$

Since $H_1 + H_2$ is an H with h's in L, we get

$$\overline{M}(f+g) \leqq \overline{M}f + \overline{M}g + 2\varepsilon \,,$$

and (8) follows.

Next we show that

$$\overline{M}\{f(xa)\} = \overline{M}\{f(x)\} \,. \tag{9}$$

We have

$$\begin{aligned} \overline{M}\{f(x)\} &= \inf{}^* \sup_x \big(f(x) + H(x)\big) \\ &= \inf{}^* \sup_x \big(f(xa) + \{f(x) - f(xa) + H(x)\}\big) \\ &= \overline{M}\{f(xa)\}, \end{aligned}$$

where the infimum is to be taken over all H's of the form (0) with h's in L and $f(x) + H(x) \geqq 0$.

Next we show that

$$\overline{M}\{\lambda f(x)\} = \lambda \overline{M}\{f(x)\} \,, \qquad \lambda \geqq 0 \,. \tag{10}$$

In the case $\lambda = 0$ we get

$$\overline{M}\{0f\} = \inf{}^* \sup_x H(x) = 0 = 0\overline{M}f \,,$$

where the infimum is to be taken over all H's of the form (0) with h's in L and $H(x) \geqq 0$. We have used that $H(x) = 0$ is admitted. In the case $\lambda > 0$ we have

$$\overline{M}\{\lambda f\} = \inf{}^* \sup_x \big(\lambda f(x) + \lambda H(x)\big) \,,$$

where we consider H's with h's in L and $\lambda f(x) + \lambda H(x) \geqq 0$. Thus (10) is clear also in this case.

Next we show that

$$\overline{M}\{f(x) - f(xa)\} = 0 \,. \tag{11}$$

On the one hand, from (8) and (9) we get

$$\overline{M}\{f(x)-f(xa)\} \geqq \overline{M}f(x) - \overline{M}f(xa) = 0\,,$$

on the other hand, from (6) we get

$$\overline{M}\{f(x)-f(xa)\} \leqq \sup_x \{f(x)-f(xa)+f(xa)-f(x)\} = 0$$

since $0 \geqq 0$. Hence (11) follows.

We want to show that $\overline{M}f=0$ for all f in L. We assume, to the contrary, that there exists an f_0 in L with $\overline{M}f_0 \neq 0$ (incidentally $\overline{M}f_0 > 0$ on account of (7)). By (8), (10), and the theorem of Banach stated in [3] we can determine a linear functional Mf on L with $Mf \leqq \overline{M}f$ for all f in L and $Mf_0 = \overline{M}f_0 > 0$. It follows from (11) that Mf is right-invariant. In a well-known manner we proceed as follows in order to write Mf as the difference between two positive linear functionals M^+f and $-M^-f$.

For functions $f \geqq 0$ in L we put

$$M^+f = \sup_{\substack{0 \leqq g \leqq f,\\ g \in L}} Mg\,.$$

From $M0=0$ and (7) applied to g we obtain the estimate

$$0 \leqq M^+f \leqq C \sup_x f(x)\,.$$

Further $M^+\{\lambda f\} = \lambda M^+f$ when $\lambda \geqq 0$, and $M^+\{f(xa)\} = M^+\{f(x)\}$. For functions $f \geqq 0$ and $g \geqq 0$ in L it is clear that

$$M^+(f+g) \geqq M^+f + M^+g\,.$$

In order to prove the converse inequality, let h be in L and let $0 \leqq h \leqq f+g$. It suffices to show that $h = f_1 + g_1$ where f_1 and g_1 are in L and satisfy the inequalities $0 \leqq f_1 \leqq f$ and $0 \leqq g_1 \leqq g$. Clearly, $f_1 = \min(h, f)$ and $g_1 = h - f_1$ have the desired properties.

Every function f in L can be written $f = f_1 - f_2$ with non-negative f_1 and f_2 in L, in particular $f = f^+ - f^-$, where

$$f^+ = \max(f, 0), \qquad -f^- = \min(f, 0)\,.$$

The equation

$$M^+f = M^+f_1 - M^+f_2$$

defines M^+f in a unique manner, and for arbitrary f and g in L we get

$$M^+\{\lambda f\} = \lambda M^+ f$$

(where λ is arbitrary), $M^+(f+g) = M^+f + M^+g$, and $M^+\{f(xa)\} = M^+\{f(x)\}$. Further

$$M^+f = M^+f^+ - M^+f^- \leqq M^+f^+ \leqq C \sup_x f^+(x)\,.$$

This estimate, however, is not sufficient for our purpose. Using that the constants belong to L we get instead

$$M^{+}f \leqq M^{+}(\sup_x f(x)) = M^{+}(1)\sup_x f(x) .$$

For functions $f \geqq 0$ in L we put

$$M^{-}f = \inf_{\substack{0 \leqq g \leqq f, \\ g \in L}} Mg .$$

It follows from (7) that

$$Mg \geqq -\overline{M}(-g) \geqq -C\sup_x|-g(x)| \geqq -C\sup_x f(x)$$

so that

$$0 \geqq M^{-}f \geqq -C\sup_x f(x) .$$

Continuing as above we define $M^{-}f$ for all f in L and show that $-M^{-}f$ has the same properties as those listed for $M^{+}f$.

We shall prove that

$$Mf = M^{+}f + M^{-}f . \tag{12}$$

It suffices to do it for functions $f \geqq 0$ in L. Let $\varepsilon > 0$ be given. We choose g in L such that $0 \leqq g \leqq f$ and

$$Mg > M^{+}f - \varepsilon .$$

Then $0 \leqq f-g \leqq f$ so that $M(f-g) \geqq M^{-}f$. Hence

$$Mf > M^{+}f + M^{-}f - \varepsilon .$$

Analogously we get

$$Mf < M^{+}f + M^{-}f + \varepsilon .$$

Thus (12) is proved.

Since $M^{+}f_0 + M^{-}f_0 = Mf_0 > 0$, either $M^{+}f_0$ or $M^{-}f_0$ is $\neq 0$. Assume, for instance, that $M^{+}f_0 < 0$. Then $M^{+}1 \neq 0$ (and hence > 0) on account of the inequality

$$M^{+}(-f_0) \leqq M^{+}(1)\sup_x(-f_0(x)) ,$$

and $M^{+}f/M^{+}1$ is a right-invariant Banach mean value on L. Thus we have arrived at a contradiction.

We have shown that $\overline{M}f = 0$ for all f in L. Thus to every $\varepsilon > 0$ there exists an H with h's in L such that

$$\varepsilon \geqq f(x) + H(x) \geqq 0 .$$

In other words: $f(x)$ can be uniformly approximated by functions $H(x)$ with h's in L. This completes the proof of Theorem 2.

REMARK. Theorem 2 remains valid when "right-" is replaced by "bi-" and

$$h_1(x)-h_1(xa_1)+\ldots+h_n(x)-h_n(xa_n)$$

by

$$h_1(x)-h_1(b_1xa_1)+\ldots+h_n(x)-h_n(b_nxa_n)\ .$$

This may be proved in a similar way as Theorem 2.

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