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AN INEQUALITY FOR FINITE SEQUENCES

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1. Let (a_ν) and (b_ν), $\nu=0, 1, 2, \ldots, m$, be two finite sequences of real and non-negative numbers. In this note we derive an inequality of Hilbert-Schwarz's type and, in the case of $a_\nu = b_\nu$, we get incidentally a refined form for the well-known Hilbert's inequality

$$\sum_{u=0}^{m} \sum_{v=0}^{m} \frac{a_u a_v}{u+v+1} \leqq \pi \sum_{u=0}^{m} a_u^2$$

for the finite sequence (a_ν). (Concerning this inequality and its generalisations cf. e. g. [2, pp. 117, 290] and [1, Chapt. IX].) Our result is as follows:

Theorem I. *Let (a_ν) and (b_ν) be the given sequences, then*

$$\sum_{u=0}^{m} \sum_{v=0}^{m} \frac{a_u b_v}{2u+2v+1} \leqq (m+1) \sin \frac{\pi}{2(m+1)} \left(\sum_{u=0}^{m} a_u^2\right)^{\frac{1}{2}} \left(\sum_{u=0}^{m} b_u^2\right)^{\frac{1}{2}}.$$

In the proof we require the following almost evident relation:

Lemma. *Let ϱ, u and v be positive integers. If $\varrho|u-v|<n$, then*

$$\sum_{r=0}^{n-1} e^{2i\varrho(u-v)r\pi/n} = \begin{cases} 0 & \text{if } u \neq v, \\ n & \text{if } u = v. \end{cases}$$

2. We construct a regular polygon C with an even number $n>2m$ of sides, inscribed in the unit circle $|z|=1$ and with the vertices $P_r = e^{2ir\pi/n}$, where $P_n = P_0$. Let L_r, $0 \leqq r \leqq n-1$, denote the chord joining the points P_r and P_{r+1}. On the chord L_r we have

$$z = \frac{\cos \pi/n \, e^{i\theta}}{\cos\big((2r+1)\pi/n - \theta\big)}, \qquad 2r\pi/n \leqq \theta \leqq (2r+2)\pi/n,$$

hence

$$|dz| = \frac{\cos \pi/n \, d\theta}{\cos^2\big((2r+1)\pi/n - \theta\big)}$$

if θ is measured in the positive sense of the angle.

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Let C_1 and C_2 be the parts of C above and below the real axis, respectively. We define two functions:

$$f(z) = \sum_{u=0}^{m} a_u z^{2u}, \qquad g(z) = \sum_{v=0}^{m} b_v z^{2v}.$$

Then we have

$$\sum_{u=0}^{m} \sum_{v=0}^{m} \frac{a_u b_v}{2u+2v+1} = \tfrac{1}{2}\int_{-1}^{1} f(x)g(x)\,dx = \tfrac{1}{2}\int_{C_1} f(z)g(z)\,dz$$

by Cauchy's theorem for contour integration. Let the last integral be denoted by I, then

$$\begin{aligned} I = \int_{C_1} f(z)g(z)\,dz &\leqq \tfrac{1}{2}\int_{C} |f(z)|\,|g(z)|\,|dz| \\ &\leqq \tfrac{1}{2}\left(\int_{C} |f(z)|^2\,|dz|\right)^{\frac{1}{2}} \left(\int_{C} |g(z)|^2\,|dz|\right)^{\frac{1}{2}} \\ &= \tfrac{1}{2} I_1^{\frac{1}{2}} I_2^{\frac{1}{2}}, \end{aligned}$$

say, by Schwarz's inequality. Now

$$\begin{aligned} I_1 &= \int_{C} f(z)f(\bar{z})\,|dz| \\ &= \sum_{r=0}^{n-1} \int_{L_r} f(z)f(\bar{z})\,|dz| \\ &= \sum_{r=0}^{n-1} \int_{L_r} \left(\sum_{u=0}^{m} a_u z^{2u}\right)\left(\sum_{v=0}^{m} a_v \bar{z}^{2v}\right)|dz| \end{aligned}$$

$$\sum_{r=0}^{n-1} \int_{2\pi r/n}^{(2r+2)\pi/n} \sum_{u=0}^{m} a_u \frac{\cos^{2u}\pi/n\; e^{2iu\theta}}{\cos^{2u}\big((2r+1)\pi/n-\theta\big)} \sum_{v=0}^{m} a_v \frac{\cos^{2v}\pi/n\; e^{-2iv\theta}}{\cos^{2v}\big((2r+1)\pi/n-\theta\big)} \cdot \frac{\cos\pi/n\; d\theta}{\cos^2\big((2r+1)\pi/n-\theta\big)}$$

$$\begin{aligned} &= \sum_{r=0}^{n-1} \int_{2\pi r/n}^{(2r+2)\pi/n} \sum_{u=0}^{m}\sum_{v=0}^{m} a_u a_v \frac{\cos^{2(u+v)}\pi/n\; e^{2i(u-v)\theta}}{\cos^{2(u+v)}\big((2r+1)\pi/n-\theta\big)} \cdot \frac{\cos\pi/n\; d\theta}{\cos^2\big((2r+1)\pi/n-\theta\big)} \\ &= \sum_{u=0}^{m}\sum_{v=0}^{m} a_u a_v \int_{-\pi/n}^{\pi/n} \frac{\cos^{2(u+v)+1}\pi/n\; e^{2i(u-v)(\pi/n-\varphi)}}{\cos^{2(u+v)+2}\varphi} \sum_{r=0}^{n-1} e^{4i(u-v)r\pi/n}\,d\varphi \\ &= n\sum_{u=0}^{m} a_u^2 \int_{-\pi/n}^{\pi/n} \frac{\cos^{4u+1}\pi/n}{\cos^{4u+2}\varphi}\,d\varphi \end{aligned}$$

by the lemma of section 1. Since

$$\int_{-\pi/n}^{\pi/n} \frac{d\varphi}{\cos^{4u+2}\varphi} \leqq \frac{1}{\cos^{4u}\pi/n} \int_{-\pi/n}^{\pi/n} \frac{d\varphi}{\cos^2\varphi} = 2\,\frac{\sin\pi/n}{\cos^{4u+1}\pi/n}$$

for $0 \leqq u \leqq m$, we obtain immediately that

$$I_1 \leqq 2n \sin\pi/n \sum_{u=0}^{m} a_u^2 .$$

A similar argument gives

$$I_2 \leqq 2n \sin\pi/n \sum_{u=0}^{m} b_u^2 .$$

From the above analysis it follows that

$$\sum_{u=0}^{m}\sum_{v=0}^{m} \frac{a_u b_v}{2u+2v+1} \leqq \tfrac{1}{2} n \sin\pi/n \left(\sum_{u=0}^{m} a_u^2\right)^{\frac{1}{2}} \left(\sum_{u=0}^{m} b_u^2\right)^{\frac{1}{2}}$$

for $n > 2m$, n even. Choosing $n = 2(m+1)$, we obtain theorem 1.

3. We can easily remove the restriction to non-negative a_ν and b_ν by observing that

$$\left|\sum_{u=0}^{m}\sum_{v=0}^{m} \frac{a_u b_v}{2u+2v+1}\right| \leqq \sum_{u=0}^{m}\sum_{v=0}^{m} \frac{|a_u|\,|b_v|}{2u+2v+1} .$$

Thus we obtain

THEOREM 2. *Let (a_ν) and (b_ν), $\nu = 0, 1, \ldots, m$, be two finite sequences of real numbers. Then*

$$\left|\sum_{u=0}^{m}\sum_{v=0}^{m} \frac{a_u b_v}{2u+2v+1}\right| \leqq (m+1) \sin\frac{\pi}{2(m+1)} \left(\sum_{u=0}^{m} a_u^2\right)^{\frac{1}{2}} \left(\sum_{u=0}^{m} b_u^2\right)^{\frac{1}{2}} .$$

REFERENCES

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