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A GENERALIZATION OF COURANT'S NODAL DOMAIN THEOREM

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The object of this paper is to extend Pleijel's nodal domain theorem [4] to Riemannian manifolds. The problem of stating sufficient conditions under which our generalization is valid is not considered. However, as in the case which was treated by Pleijel, it is easy to give examples of eigenvalue problems to which the theory can be applied.

1. Let $\mathscr{M}$ be a 2-dimensional Riemannian manifold. The Beltrami-Laplace operator in $\mathscr{M}$ is

$$\Delta = -g^{-\frac{1}{2}} \frac{\partial}{\partial x^j} \left(g^{\frac{1}{2}} g^{jk} \frac{\partial}{\partial x^k} \right),$$

where g_{jk} and g^{jk} are the covariant and contravariant components of the metric tensor in a local coordinate system and $g = \det g_{jk}$.

Let Ω be a relatively compact connected domain in $\mathscr{M}$. Consider the eigenvalue problem

$$\text{(1)} \qquad \Delta u - \lambda u = 0 \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega \text{ (boundary of } \Omega).$$

We intend to estimate the number of nodal domains N of the n-th eigenfunction of (1).

2. In this section we suppose that $\mathscr{M}$ is homeomorphic to a disc in the Euclidean plane. We shall first prove a slight generalization of an isoperimetric inequality due to Huber [2] which will be used in the sequel. The boundaries of the domains under consideration are understood to be sufficiently smooth.

THEOREM 1. *Let Ω_0 be the least simply connected domain containing Ω. Suppose that*

$$\text{(2)} \qquad V_0 \sup_{\Omega_0} K^+ \leqq \pi,$$

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where K is the Gaussian curvature, $K^+ = \max(K, 0)$, and V_0 is the area of Ω_0. Then

$$S^2 \geqq 4\pi V \left\{ 1 - \frac{1}{2\pi} \int_\Omega K^+ \, dV \right\}, \tag{3}$$

where S is the length of $\partial\Omega$ and V the area of Ω. Equality holds if and only if $K = 0$ and Ω is a circle.

PROOF. If Ω is simply connected ($\Omega = \Omega_0$) then (3) is Huber's result. If Ω is multiply connected then, applying Huber's theorem to Ω_0, we obtain

$$S_0^2 \geqq 4\pi V_0 \left\{ 1 - \frac{1}{2\pi} \int_{\Omega_0} K^+ \, dV \right\}, \tag{4}$$

S_0 being the length of $\partial\Omega_0$. Let Σ be the interior of $\Omega_0 - \Omega$ and put $U = V_0 - V$. Then we have

$$\begin{aligned} V_0 \int_{\Omega_0} K^+ dV &= V \int_\Omega K^+ \, dV + V \int_\Sigma K^+ dV + U \int_{\Omega_0} K^+ dV \\ &\leqq V \int_\Omega K^+ dV + U V \sup_\Sigma K^+ + U V_0 \sup_{\Omega_0} K^+ \\ &\leqq V \int_\Omega K^+ dV + 2U V_0 \sup_{\Omega_0} K^+ . \end{aligned}$$

Hence, by (2) and since $U = V_0 - V$,

$$V \left\{ 2\pi - \int_\Omega K^+ dV \right\} \leqq V_0 \left\{ 2\pi - \int_{\Omega_0} K^+ \, dV \right\}. \tag{5}$$

Formula (3) now follows from (4), (5) and $S \geqq S_0$. If equality holds in (3), then Ω must be simply connected and the last assertion of the theorem follows from Huber's result.

Next we prove a generalization of a theorem due to Faber [1] and Krahn [3]. Our method of proof is essentially the original one of Faber and Krahn.

THEOREM 2. *Suppose that* (3) *is satisfied and let λ_1 be the first eigenvalue of* (1). *Then*

$$\lambda_1 V \geqq \pi j^2 \left\{ 1 - \frac{1}{2\pi} \int_\Omega K^+ \, dV \right\}, \tag{6}$$

where j is the first positive zero of the Bessel function J_0. Equality holds if and only if $K=0$ and Ω is a circle.

Proof. Let $u=u_1$ be the first eigenfunction. Put

$$\Omega(\varrho) = \{x \mid u(x) > \varrho\}, \qquad 0<\varrho<\max u$$

$$D(\varrho) = \int_{\Omega(\varrho)} |\operatorname{grad} u|^2 dV ,$$

$$V(\varrho) = \int_{\Omega(\varrho)} dV ,$$

$$S(\varrho) = \int_{\mathscr{B}\Omega(\varrho)} dS ,$$

$$H(\varrho) = \int_{\Omega(\varrho)} u^2 dV .$$

Then

$$|D'(\varrho)| = -D'(\varrho) = \int_{\mathscr{B}\Omega(\varrho)} |\operatorname{grad} u|\, dS$$

and

$$|V'(\varrho)| = -V'(\varrho) = \int_{\mathscr{B}\Omega(\varrho)} |\operatorname{grad} u|^{-1} dS .$$

Now by Schwarz's inequality

$$(S(\varrho))^2 \leqq |D'(\varrho)|\,|V'(\varrho)|$$

and by theorem 1

$$\left\{1 - \frac{1}{2\pi}\int_{\Omega} K^+ dV\right\} 4\pi V(\varrho)/|V'(\varrho)|$$

$$\leqq \left\{1 - \frac{1}{2\pi}\int_{\Omega(\varrho)} K^+ dV\right\} 4\pi V(\varrho)/|V'(\varrho)| \leqq |D'(\varrho)| .$$

We apply a process of symmetrization replacing the domains $\Omega(\varrho)$ by concentric circles $\tilde{\Omega}(\varrho)$ with the same areas in the Euclidean plane. We replace the function u by a function $\tilde{u}$ which equals ϱ on $\mathscr{B}\tilde{\Omega}(\varrho)$. Clearly $\tilde{V}(\varrho)=V(\varrho)$ and $\tilde{V}'(\varrho)=V'(\varrho)$. Hence

$$\left\{1 - \frac{1}{2\pi}\int_{\Omega} K^+ dV\right\} 4\pi \tilde{V}(\varrho)/|\tilde{V}'(\varrho)| \leqq |D'(\varrho)| .$$

But evidently

$$4\pi \tilde{V}(\varrho)/|\tilde{V}'(\varrho)| = |\tilde{D}'(\varrho)| ;$$

for $(\tilde{S}(\varrho))^2 = |\tilde{D}'(\varrho)|\,|\tilde{V}'(\varrho)|$ and $4\pi\tilde{V}(\varrho) = |\tilde{S}(\varrho)|^2$. Hence we obtain

$$\left\{1 - \frac{1}{2\pi}\int_\Omega K^+ dV\right\} |\tilde{D}'(\varrho)| \leqq |D'(\varrho)| .$$

Integration over the interval $0 < \varrho < \max u$ finally yields

$$\left\{1 - \frac{1}{2\pi}\int_\Omega K^+ dV\right\} \tilde{D} \leqq D .$$

Moreover $\tilde{H}(\varrho) = H(\varrho)$ and $\tilde{H} = H$. Now it follows from Rayleigh's inequality that

$$\lambda_1 = D/H, \quad \tilde{\lambda}_1 \leqq \tilde{D}/\tilde{H};$$

which proves (6) since $\tilde{\lambda}_1 = \pi j^2/V$. The conditions for equality follow immediately from theorem 1.

Remark. The above symmetrization process combined with the inequality (3) of theorem 1 yields a number of inequalities similar to (6) and related to variational problems of the type

$$\int |\mathrm{grad}\, u|^2 dV = \min, \quad \int F(u) dV = \text{constant}$$

with boundary condition $u = \text{constant}$. Inequalities of this type have been studied in various cases (Euclidean plane, surface of a sphere, etc.) by Pólya and Szegö [5].

We are now in a position to obtain the desired extension of Pleijel's theorem. Let λ_n be the n-th eigenvalue and u_n the n-th eigenfunction. Let $\Omega_1, \ldots, \Omega_N$ be the nodal domains of u_n (for a definition of nodal domain see [4]). For each Ω_l the value λ_n is the lowest eigenvalue. Applying theorem 2 to each Ω_l, we obtain

$$\lambda_n V_l \geqq \pi j^2 \left\{1 - \frac{1}{2\pi}\int_{\Omega_l} K^+ dV\right\} .$$

Summation of these inequalities gives

$$\lambda_n V \geqq \pi j^2 \left\{N - \frac{1}{2\pi}\int_\Omega K^+ dV\right\} .$$

From the well-known fact that $\lim n^{-1}\lambda_n V = 4\pi$ it follows that

$$\limsup N/n \leqq (2/j)^2 < 1 . \tag{7}$$

Thus we have proved

Theorem 3. *There is a number* $\alpha < 1$ *such that*

$$\limsup N/n \leqq \alpha . \tag{8}$$

Remark. It follows from Rayleigh's inequality that (7) and (8) remain true if (1) is replaced by the somewhat more general eigenvalue equation

$$\Delta u + c(x)u = \lambda u ,$$

where $c(x)$ denotes a sufficiently well-behaved bounded function. In fact, it is clear from the proof of theorem 2 that

$$\big(\lambda_1 - \inf c(x)\big) V \geqq \pi j^2 \left\{ 1 - \frac{1}{2\pi} \int_\Omega K^+ dV \right\}$$

from which formula (7) can be deduced.

3. In special cases it is possible to get better results, e.g. we can drop the restriction (2). Consider the case K = constant throughout $\mathcal{M}$. Then it is known that

$$S^2 \geqq 4\pi V \left(1 - \frac{1}{4\pi} K V\right) ,$$

where equality holds if and only if Ω is a (geodesic) circle. This inequality yields

$$\lambda_1 V \geqq \pi j^2 \left(1 - \frac{1}{4\pi} K^+ V\right)$$

and

$$\limsup N/n \leqq (2/j)^2 .$$

Moreover we may deduce:

Among all domains with the same area the (geodesic) circle minimizes λ_1.

It is also possible to extend the results of section 2 to k-dimensional Riemannian manifolds of constant curvature. The general case of a Riemannian manifold of arbitrary dimension remains open, however, for we do not know any k-dimensional analogue of Huber's theorem.

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