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APPENDIX TO “REGULARITY OF VILLADSEN ALGEBRAS”: THE FAILURE OF THE CORONA FACTORIZATION PROPERTY FOR THE VILLADSEN ALGEBRA $\mathscr{V}_\infty$

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Abstract

In this appendix to M. S. Christensen, “Regularity of Villadsen algebras and characters on their central sequence algebras”, Math. Scand. 123 (2018), no. 1, 121–141, we prove that the Villadsen algebra $\mathscr{V}_\infty$ does not satisfy the Corona Factorization Property (CFP).

Appendix A. Failure of the Corona Factorization Property

In this appendix we prove that the Villadsen algebra $\mathscr{V}_\infty$ does not satisfy the Corona Factorization Property (CFP), thereby improving the result, from an earlier version of this paper, that $\mathscr{V}_\infty$ does not satisfy the ω-comparison property.

Both ω-comparison and the CFP may be regarded as comparison properties of the Cuntz semigroup invariant, and both properties are related to the question of when a given C^*-algebra is stable (see for instance [5, Proposition 4.8]). In particular, a simple, separable C^*-algebra A has the CFP if and only if, whenever $x, y_1, y_2, \dots$ are elements in $\mathrm{Cu}(A)$ and $m \geq 1$ is an integer satisfying $x \leq m y_j$ for all $j \geq 1$, we have $x \leq \sum_{i=1}^{\infty} y_i$ ([5, Theorem 5.13]). On the other hand, given a simple C^*-algebra A, $\mathrm{Cu}(A)$ has *ω-comparison* if and only if $\infty = x \in \mathrm{Cu}(A)$ whenever $f(x) = \infty$ for all functionals f on $\mathrm{Cu}(A)$ ([2, Proposition 5.5]). Recall that a *functional* f on the Cuntz semigroup $\mathrm{Cu}(A)$ of a C^*-algebra A is an ordered semigroup map $f\colon \mathrm{Cu}(A) \to [0, \infty]$ which preserves suprema of increasing sequences. In particular, the latter comparability condition is satisfied for all unital C^*-algebras A with finite radius of comparison by [1].

From the above characterization it follows that any separable C^*-algebra A whose Cuntz semigroup $\mathrm{Cu}(A)$ has the ω-comparison property also has the CFP (see [5, Proposition 2.17]). Whether the converse implication is true

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remains an open question. This question was considered by the first author of this appendix and Petzka in [2], where the failure of the converse implication was shown just in the algebraic framework of the category Cu. However, it was emphasized there that a more analytical approach will be necessary in order to verify (or disprove) the converse implication for any (simple) C^*-algebra A.

The Villadsen algebras have been used several times to certify bizarre behaviour in the theory of C^*-algebras; hence, after Ng and Kucerosvky showed in [4] that $\mathcal{V}_2$ satisfies the CFP, one wonders whether it satisfies the ω-comparison or not. From Corollary 4.4 (together with [1, Theorem 4.2.1]) one gets that, for all $1 \leq n < \infty$, the Cuntz semigroups $\mathrm{Cu}(\mathcal{V}_n)$ have the ω-comparison property and hence the CFP. But this is not the case for the C^*-algebra $\mathcal{V}_\infty$. As demonstrated below, it does not have the CFP (and hence $\mathrm{Cu}(\mathcal{V}_\infty)$ does not have ω-comparison). Notice that although $\mathcal{V}_\infty$ has a different structure than $\mathcal{V}_n$, it does not witness the potential non-equivalence of ω-comparison and the CFP for unital, simple and stably finite C^*-algebras.

THEOREM A.1. *Let $\mathcal{V}_\infty$ be given as above. Then $\mathcal{V}_\infty$ is a unital, simple, separable and nuclear C^*-algebras with a unique tracial state such that the Cuntz semigroup* $\mathrm{Cu}(\mathcal{V}_\infty)$ *does not have the Corona Factorization Property for semigroups.*

PROOF. We use the notation introduced above, with $k = \infty$ fixed and omitted. Additionally, for each $n \geq 1$, let $\lambda(n) := \kappa(\infty, n) = n\sigma(n) = n^2(n!)$, and $Y_n := \mathbb{C}P^{\lambda(1)} \times \cdots \times \mathbb{C}P^{\lambda(n)}$. Note that $X_n = \mathbb{D}^{n\sigma(n)^2} \times Y_n$, let $\overline{\pi}_n\colon X_n \to Y_n$ denote the coordinate projection and $\overline{\psi}_n\colon C(Y_n) \otimes \mathbb{K} \to C(X_n) \otimes \mathbb{K} \cong A_n \otimes \mathbb{K}$ denote the *-homomorphism induced by $\overline{\pi}$.

For each $n \geq 1$ and $1 \leq j \leq n$, let $\rho_{n,j}\colon Y_n \to \mathbb{C}P^{\lambda(j)}$ denote the projection map and let $\overline{\zeta}_{n,j}$ denote the vector bundle $\rho^*_{n,j}(\gamma_{\lambda(j)})$ over Y_n. To avoid overly cumbersome notation, we simply write $\overline{\zeta}_j$ for $\overline{\zeta}_{n,j}$ whenever $j \leq n$. Furthermore, for each $n \geq 1$, let $\overline{\xi}_n$ denote the vector bundle over Y_n given by $\theta_1 \oplus \sigma(1)\overline{\zeta}_1 \oplus \cdots \oplus \sigma(n)\overline{\zeta}_n$. Recall that, for each $j \geq 1$, ζ_j denotes the vector bundle $\pi_j^{2*}(\gamma_{\lambda(j)})$ over X_j. For brevity we also let ζ_j denote the vector bundle $\pi_n^{1*} \circ \cdots \circ \pi_{j+1}^{1*}(\zeta_j)$, whenever $n > j$. With this notation, the vector bundle ξ_n over X_n corresponding to the unit $p_n \in A_n$ may be written $\xi_n \cong \theta_1 \oplus \sigma(1)\zeta_1 \oplus \cdots \oplus \sigma(n)\zeta_n$. It is immediately verified that $\overline{\psi}_n(\overline{\zeta}_j) \cong \zeta_j$ for all $j \leq n$, and in particular $\psi_n^*(\overline{\xi}_n) \cong \xi_n$. Hence, if $q \in C(Y_n) \otimes \mathbb{K}$ is a projection corresponding to a vector bundle η satisfying $\overline{\xi}_n \precsim \eta$, then $p_n \precsim \psi_n(q)$.

Note that,

$$\lim_{n\to\infty} \frac{\dim(Y_n)}{2\lambda(n)} \leq \lim_{n\to\infty} \frac{n^2(n!) + n\left(\sum_{i=0}^{n-1} \sigma(i)\right)}{n^2(n!)} = 1 + \lim_{n\to\infty} \frac{1}{n} = 1. \quad \text{(A.1)}$$

Furthermore, it follows from (5), by induction, that for any $1 \le n < m$ and an arbitrary vector bundle η over X_n, we have

$$\varphi^*_{n,m}(\eta) \cong \mu^*_{m,n}(\eta) \oplus (n+1)\operatorname{rank}(\eta)\zeta_{n+1} \oplus \cdots \oplus \frac{\sigma(m)}{(n+1)!}\operatorname{rank}(\eta)\zeta_m, \quad \text{(A.2)}$$

where $\mu_{m,n} := \pi^1_{n+1} \circ \cdots \circ \pi^1_m \colon X_m \to X_n$. Moreover, we find that

$$\lim_{m\to\infty} \frac{\sigma(m)}{(n+1)!\lambda(m)} = \lim_{m\to\infty} \frac{1}{(n+1)!m} = 0. \quad \text{(A.3)}$$

Choose $\ell(1) \ge 1$ large enough that $\lambda(k)$ is divisible by 4 and $\frac{5}{4}\lambda(k) \ge \dim(Y_k)/2 > \operatorname{rc}(C(Y_k) \otimes \mathbb{K})$ for all $k \ge \ell(1)$, which is possible by (A.1). Set $k(1) := \frac{1}{2}\lambda(\ell(1))$. Define sequences $(\ell(n))_{n\ge1}$ and $(k(n))_{n\ge1}$ as follows: given $n \ge 2$ and $\ell(1), \dots, \ell(n-1)$ choose $\ell(n) > \ell(n-1)$ such that

$$\frac{\left(\sum_{j=1}^{\ell(n-1)} \lambda(j)\right)\sigma(\ell(n))}{(\ell(n-1)+1)!\lambda(\ell(n))} \le \frac{1}{2}, \quad \text{i.e.,} \quad \frac{\left(\sum_{j=1}^{\ell(n-1)} \lambda(j)\right)\sigma(\ell(n))}{(\ell(n-1)+1)!} \le \frac{\lambda(\ell(n))}{2}, \quad \text{(A.4)}$$

which is possible by (A.3), and set $k(n) := \frac{1}{2}\lambda(\ell(n))$. Finally, for each $n \ge 1$, let $\overline{q}_n \in A_{\ell(n)} \otimes \mathbb{K}$ be the projection corresponding to the vector bundle $\zeta_{\ell(n)}$ over $X_{\ell(n)}$ and $x_n := k(n)\langle\varphi_{\ell(n),\infty}(q_n)\rangle \in \operatorname{Cu}(\mathscr{V}_\infty)$. We aim to show that the sequence $(x_n)_{n\ge1}$ in $\operatorname{Cu}(\mathscr{V}_\infty)$ witnesses the failure of the Corona Factorization Property in $\operatorname{Cu}(\mathscr{V}_\infty)$.

First, we show that $5x_n \ge \langle \mathbf{1}_{\mathscr{V}}\rangle =: e$ for all $n \ge 1$. As noted above, it suffices to show that $5k(n)\overline{\zeta}_{\lambda(\ell(n))} \ge \overline{\xi}_{\lambda(\ell(n))}$. But, by choice of $k(n)$ and $\ell(n)$ we have that

$$\operatorname{rank}(5k(n)\overline{\zeta}_{\lambda(\ell(n))}) = \frac{5}{2}\lambda(\ell(n)) \ge \frac{\dim(Y_{\ell(n)})}{2} + \operatorname{rank}(\overline{\xi}_{\ell(n)}),$$

since $\operatorname{rank}(\overline{\xi}_{\ell(n)}) = (\ell(n)+1)! \le \dim(Y_{\ell(n)})/2$. The desired result therefore follows from [3, Theorem 2.5].

Next, we show that $e \not\le \sum_{i=1}^\infty x_i$. Proceeding as in the proof of Proposition 4.3(ii), it suffices to prove that

$$\langle p_j\rangle \not\le \left\langle \bigoplus_{i=1}^n \varphi_{\ell(i),j}(q_i)\right\rangle$$

for all $j \ge \ell(n)$ (recall that $p_j \in A_j$ denotes the unit, i.e., the projection corresponding to ξ_j). Since ξ_j dominates a trivial line bundle for each j, it suffices to prove that the vector bundle corresponding to the right-hand side above does not. We do this by proving that

$$\bigoplus_{i=1}^{n} \varphi^*_{\ell(i),j}(k(i)\zeta_{\ell(i)}) \precsim \bigoplus_{s=1}^{j} \lambda(s)\zeta_s.$$

Since the right-hand side does not dominate any trivial bundle, by the proof of Proposition 4.3(ii), this will complete the proof. Note that it also follows from the proof of Proposition 4.3(ii) that $\varphi^*_{j,m}\big(\bigoplus_{s=1}^{j} \lambda(s)\zeta_s\big) \precsim \bigoplus_{s=1}^{m} \lambda(s)\zeta_s$ for all $m \geq j$. Thus, it suffices to prove that

$$\bigoplus_{i=1}^{n-1} \varphi^*_{\ell(i),\ell(n)}(k(i)\zeta_{\ell(i)}) \oplus k(n)\zeta_{\ell(n)} \precsim \bigoplus_{s=1}^{\ell(n)} \lambda(s)\zeta_s$$

for all $n \geq 1$. We proceed by induction. Clearly the statement is true for $n = 1$, so suppose it is true for $n - 1$ with $n \geq 2$. Then

$$\bigoplus_{i=1}^{n-1} \varphi^*_{\ell(i),\ell(n)}(k(i)\zeta_{\ell(i)}) \oplus k(n)\zeta_{\ell(n)} \precsim \varphi^*_{\ell(n-1),\ell(n)}\left(\bigoplus_{s=1}^{\ell(n-1)} \lambda(s)\zeta_s\right) \oplus k(n)\zeta_{\ell(n)}.$$

by induction hypothesis.

Now, letting $N := \sum_{s=1}^{\ell(n-1)} \lambda(s) = \operatorname{rank}\big(\bigoplus_{s=1}^{\ell(n-1)} \lambda(s)\zeta_s\big)$, it follows by the choice of $\ell(n)$ and $k(n)$ (see (A.4)) that $k(n) + \frac{N\sigma(\ell(n))}{(\ell(n-1)+1)!} \leq \lambda(\ell(n))$. Hence, combining the above induction step with (A.2), one has:

$$\begin{aligned}
&\bigoplus_{i=1}^{n-1} \varphi^*_{\ell(i),\ell(n)}(k(i)\zeta_{\ell(i)}) \oplus k(n)\zeta_{\ell(n)} \\
&\quad \precsim \varphi^*_{\ell(n-1),\ell(n)}\Big(\bigoplus_{s=1}^{\ell(n-1)} \lambda(s)\zeta_s\Big) \oplus k(n)\zeta_{\ell(n)} \\
&\quad \overset{(A.2)}{\precsim} \Big(\bigoplus_{s=1}^{\ell(n-1)} \lambda(s)\zeta_s\Big) \oplus (\ell(n-1)+1)N\zeta_{\ell(n-1)+1} \oplus \\
&\qquad\qquad \cdots \oplus \frac{N\sigma(\ell(n))}{(\ell(n-1)+1)!}\zeta_{\ell(n)} \oplus k(n)\zeta_{\ell(n)} \\
&\quad \precsim \bigoplus_{s=1}^{\ell(n)} \lambda(s)\zeta_s.
\end{aligned}$$

Thus, the desired result follows.

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