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TWO REMARKS ON SET THEORY

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1. A formulation of the axiom of infinity.

The axioms of Zermelo's set theory mostly express that a class of the objects constituting the considered domain D is a set. This means that there is an object m in D such that all the members of the class and only these are in the relation $\in$ to m. Only two axioms are not of this form, viz. the axiom of infinity and the axiom of choice. However, the former of these can be put in the same form, namely a class declared to be a set. In an address [3] given at the International Congress of Mathematicians at Cambridge, Mass., 1950, I hinted how it could be done, but did not carry out the proof in detail. In this paper I intend to expose the proof.

For brevity I define the notions of b-element and f-element. We say that n is an f-element of m when the conjunction $n \in m$ & $\{n\} \bar{\in} m$ is true. We say that n is a b-element of m, if $n \in m$ but $n \neq \{x\}$ for every element x of m. Then we shall consider sets m with the following 4 properties:

1) $0 \in m$,
2) m possesses a single f-element,
3) every subset n of m has at least one f-element,
4) if $n \subseteq m$ and $0 \in n$, while n has just the same single f-element as m, then $n = m$.

Let $I(m)$ denote the conjunction of 1), 2), 3), 4).

THEOREM 1. *If $I(m)$, then m has no other b-element than* 0.

PROOF. Let us assume that $b \in m$, $b \neq 0$ and $b \neq \{x\}$ for every $x \in m$. Putting $m - \{b\} = n$ we have $n \subset m$, whence according to 3) it follows that n has at least one f-element. If a is an arbitrary f-element of n, we have $a \in n$, $\{a\} \bar{\in} n$, so that also $a \in m$. Now we must have $\{a\} \bar{\in} m$ which implies that a is also f-element of m, because otherwise $\{a\} \in m$ would yield $\{a\} \in n$, since $\{a\} \neq b$. It follows from 2) that a is the single f-element

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of n. Further $0 \in n$ because of $0 \in m$ and $b \neq 0$. Hence according to 4) $n = m$, which is absurd.

THEOREM 2. *If* $I(m)$, *then* $I(m + \{\{a\}\})$, *a denoting the f-element of m.*

PROOF. For brevity let m_1 denote $m + \{\{a\}\}$. Since $0 \in m$, we also have $0 \in m_1$. Thus m_1 has the property 1).

The only f-element of m being a, we have $a \in m$, $\{a\} \bar{\in} m$. I assert that $\{a\}$ is f-element of m_1. Indeed, were $\{a\}$ not f-element of m_1, then $\{\{a\}\} \in m_1$. Since necessarily $a \neq \{a\}$, also $\{a\} \neq \{\{a\}\}$ such that $\{\{a\}\} \in m$. Then, however, $\{\{a\}\}$ would be a b-element of m different from 0, contrary to Theorem 1. Therefore $\{a\}$ is f-element of m_1. On the other hand, let c be an arbitrary f-element of m_1, that is, $c \in m_1$ and $\{c\} \bar{\in} m_1$. If c were $\neq \{a\}$, then $c \in m$ and $\{c\} \bar{\in} m$, which means that c is the single f-element a of m, whence $\{c\} = \{a\} \in m_1$. Hence m_1 has the single f-element $\{a\}$, so that m_1 has the property 2).

Let n be a subset of m_1. If $n \subseteq m$, then according to 3) n has at least one f-element. Else $\{a\} \in n$, and because of $\{\{a\}\} \bar{\in} m_1$ we have $\{\{a\}\} \bar{\in} n$, so that $\{a\}$ is f-element of n. Thus m_1 has the property 3).

Let $n_1 \subseteq m_1$ & $0 \in n_1$, while n_1 has a single f-element identical with the f-element $\{a\}$ of m_1. Then $n_1 = n + \{\{a\}\}$, where $n \subseteq m$. If now $x \in n$, then $x \in n_1$, whence $\{x\} \in n_1$ because $x = \{a\}$ is impossible, since $\{a\} \bar{\in} n$. Hence n cannot possess any other f-element than a, so that $n = m$ according to 4). It follows that $n_1 = m_1$. Thus m_1 has the property 4).

The two following theorems are elucidating although not necessary for my present purpose.

THEOREM 3. *If* $I(m)$ & $n \subseteq m$ & $0 \in n$, *n possessing just one f-element, then* $I(n)$.

PROOF. Let M be the subset of m consisting of all elements a of m such that $I(n)$ is true for all $n \subseteq m$ containing 0 as element and possessing a as single f-element. First it is seen that $0 \in M$. Indeed, when 0 is the only f-element of n, then n must be $\{0\}$ and $I(\{0\})$ is true. This is clear because otherwise $n - \{0\}$, being a subset of m, should contain an f-element, with the consequence that n would possess at least two f-elements against supposition. Now let a be an arbitrary element of M. I assert that if $\{a\}$ is still element of m, or, in other words, a is not f-element of m, then $\{a\} \in M$. Let $0 \in n$, $n \subseteq m$ and let $\{a\}$ be the only f-element of n. I set $n - \{\{a\}\} = n_1$ and assert that a is the single f-element of n_1. Obviously a is f-element of n_1 since $a \neq \{a\}$. According to supposition, n contains no other f-element than $\{a\}$.

Let us assume the existence of an element c such that

$$c \in n_1 \ \& \ c \neq a \ \& \ \{c\} \bar{\in} n_1 .$$

Then $\{c\} \neq \{a\}$, and therefore $\{c\} \bar{\in} n$. Thus we have $c \in n \ \& \ \{c\} \bar{\in} n$, whence $c = \{a\}$, but $\{a\} \bar{\in} n_1$. This shows that n_1 does not contain any f-element but a. Our supposition $a \in M$ yields $I(n_1)$, whence also $I(n)$, according to Theorem 2. So far it is proved that $0 \in M$ and that M has the same single f-element as m. Then according to 4) $M = m$. Thus Theorem 3 is correct.

THEOREM 4. $I(m_1) \ \& \ I(m_2) \rightarrow m_1 \subseteq m_2 \vee m_2 \subseteq m_1$.

PROOF. Let a_1 and a_2 be the f-element of m_1 and m_2, respectively. Then a preliminary remark is that either $a_1 \in m_2$ or $a_2 \in m_1$. Otherwise $m_1 \cap m_2$ would be a subset of m_1 without f-element. Now $0 \in m_1 \cap m_2$ so that due to 4) $m_1 \cap m_2 = m_1$ if $m_1 \cap m_2$ had no f-element $\neq a_1$, and $m_1 \cap m_2 = m_2$ if $m_1 \cap m_2$ had no f-element $\neq a_2$. Hence, if we should neither have $m_1 \subseteq m_2$ nor $m_2 \subseteq m_1$, then $m_1 \cap m_2$ must possess an f-element $c \neq a_1, a_2$. Then $\{c\} \bar{\in} m_1 \cap m_2$, that is, either $\{c\} \bar{\in} m_1$ or $\{c\} \bar{\in} m_2$. In the first case m_1 would possess an f-element $\neq a_1$ and in the second case m_2 an f-element $\neq a_2$. This would contradict the supposition $I(m_1) \ \& \ I(m_2)$.

Using Theorem 2 we observe that the sets in the sequence

$$\{0\}, \{0, \{0\}\}, \Big\{0, \{0\}, \{\{0\}\}\Big\}, \ldots$$

all have the I-property. Each set m' in this sequence is obtained from the preceding set m by addition of $\{a\}$, where a is the f-element of m. Using Theorem 1 we see that no other sets m for which $I(m)$ is true can be built by successive addition of elements. The above sequence therefore contains all finite sets m for which $I(m)$ takes place. Further I assert:

THEOREM 5. *If $I(m)$ is true, then m is finite.*

PROOF. Let n be the subset of m consisting of all $a \in m$ such that every subset p of m containing a as its single f-element is finite. Then of course $0 \in n$. Letting a be an arbitrary element of n, every subset of m having $\{a\}$ as its single f-element will have the form $p_1 = p + \big\{\{a\}\big\}$, where p has a as its only f-element. Since p is finite, p_1 is also finite. Thus, when $a \in n$, $\{a\} \in m$, we get $\{a\} \in n$, or, in other words, n has the same single f-element as m, and hence 4) yields $n = m$.

It is now evident that the union of all sets m such that $I(m)$ is valid is the Zermelo number series

$$0, \{0\}, \big\{\{0\}\big\}, \ldots .$$

Therefore the axiom of infinity can be expressed by saying that the class of all sets m for which $I(m)$ takes place is a set, or in other words: *There exists a set M such that the equivalence*

$$m \in M \longleftrightarrow I(m)$$

is generally valid.

As explained in my cited address, every set m in a theory which is based exclusively on axioms of the form "The class so and so is a set" can be defined by putting

$$x \in m \longleftrightarrow \Phi(x) .$$

Here $\Phi(x)$ is a propositional function with x as free variable and possibly other variables $x_1, x_2, \ldots$ which are all bound, Φ being built by the operations of the predicate calculus from atomic propositions of the form $y \in z$. This leads to the following remarkable result for such theories:

THEOREM 6. *The definable sets constitute a denumerable class.*

Indeed, it is possible to enumerate the propositional functions $\Phi(x)$. This is clear because each $\Phi(x)$ is a finite sequence of letters. Perhaps the most natural and simple enumeration may be obtained by restricting the consideration to the expressions $\Phi(x)$ in prenex normal form, the matrix being taken in conjunctive normal form, for example. Then there is a one-to-one correspondence between the $\Phi(x)$ and the different combinations of: 1° a sequence of pairs (n, e), $e = 0$ or 1 according as x_n occurs in the prefix as a universal or an existential quantifier, and 2° a sequence of sequences of triples (n, m, e), where $e = 0$ or 1 according as the corresponding atomic proposition is $x_n \in x_m$ or $x_n \bar{\in} x_m$. It is obvious that all these combinations can be enumerated. Thus we get an enumeration of the definable sets. This result is similar to the well-known Löwenheim theorem and shows likewise the illusory character of the classical absolutistic conceptions in set theory.

2. The ordered n-tuples as sets.

It will often be convenient to be able to consider n-ary relations as sets of ordered n-tuples. Then, however, one might ask what kind of objects the n-tuples are. If we want to develop a set theory where we take into account sets of n-tuples, either the ordered n-tuples must be introduced as an independent notion which leads to some complications, or one has to define, for each n, the ordered n-tuple as a set. Now there is one rather trivial way of defining the ordered n-tuple as a set, namely as a set of n pairs $\{a_r, i_r\}$, where $(a_1, a_2, \ldots, a_n)$ is the n-tuple, while the i_r are n special individuals which are kept constant. We may say that $i_1, \ldots, i_n$ represent the places where the a_r are placed. However, it will often be inconvenient that the use of this definition is restricted by the necessity of keeping the ordered individuals $a_1, \ldots, a_n$ apart from the ordering individuals $i_1, \ldots, i_n$. Therefore other definitions have been tried. The notion of ordered pair (a, b) has been defined as the set $\{\{a, b\}, \{a\}\}$ by Kuratowski [1] and Wiener [4]. But in literature I have found no answer to the general question how to define the ordered n-tuple as a set. In the present short discussion I shall show that there are different possibilities for doing this, and it is not easy to decide which is the most advantageous.

In the first instance it would be natural to think that we could define the ordered triple (a, b, c) as the ordered pair of the ordered pair (a, b) and the element c, that is

$$(a, b, c) = \big((a, b), c\big).$$

If, however, we develop a set theory with types, this will be inconvenient, because (a, b) will be of higher type than c when a, b, c are all of the same type. One could try to remedy this by putting

$$(a, b, c) = \big((a, b), \{\{c\}\}\big),$$

or more explicitly

$$(a, b, c) = \big(\{\{a, b\}, \{a\}\}, \{\{c\}\}\big).$$

However, if we compare (a, a, b) and (b, b, a) we observe that both of them equals $\big(\{\{a\}\}, \{\{b\}\}\big)$, which of course in unacceptable.

It would also be natural to try to define (a, b, c) even simpler, analogous to the way in which we have defined (a, b) above, namely thus:

$$(a, b, c) = \big\{\{a, b, c\}, \{a, b\}, \{a\}\big\}.$$

But also this definition must be rejected, because it leads, for example, to the result that $(a, a, b) = (a, b, b) = (a, b)$ as is easily verified.

On the other hand, it is possible to define the ordered triple (a, b, c) as an ordered pair of ordered pairs, namely for example as $\big((a, c), (b, c)\big)$. Indeed, the equation

$$(a, b, c) = (d, e, f)$$

then requires that

$$(a, c) = (d, f) \quad \text{and} \quad (b, c) = (e, f),$$

whence

$$a = d, \quad b = e, \quad c = f.$$

It may also be noticed that

$$(a, a, a) = ((a, a), (a, a)) = \{\{(a, a)\}\},$$

so that we do not run the risk of confusing (a, a, a) and (a, a) except in the case of a pathological set a having the property of being element of an element of itself.

Now it is possible to extend recursively this definition to ordered n-tuples. If we assume that the notion of ordered n-tuple has been defined, we may define the ordered $(n+1)$-tuple $a_1, a_2, \ldots, a_{n+1}$ as the n-tuple of ordered pairs

$$((a_1, a_{n+1}), (a_2, a_{n+1}), \ldots, (a_n, a_{n+1})).$$

If it is known already that the equation

$$(b_1, b_2, \ldots, b_n) = (b_1', b_2', \ldots, b_n')$$

furnishes the equations $b_1 = b_1', \ldots, b_n = b_n'$, then

$$(a_1, a_2, \ldots, a_{n+1}) = (a_1', a_2', \ldots, a_{n+1}')$$

yields

$$(a_1, a_{n+1}) = (a_1', a_{n+1}'), \quad \ldots, \quad (a_n, a_{n+1}) = (a_n', a_{n+1}'),$$

whence

$$a_1 = a_1', \quad a_2 = a_2', \quad \ldots, \quad a_{n+1} = a_{n+1}'.$$

Further we may notice that the different elements of the set which is declared to be $(a_1, a_2, \ldots, a_n)$ are all of the type $t+2n-3$, when $a_1, a_2, \ldots, a_n$ are all of the same type t. This difference of type makes it clear that a confusion of the m-tuple $(a, a, \ldots, a)$ and the n-tuple $(a, a, \ldots, a)$ can occur, only in the case of a set theory without types, and only when a is pathological.

Another possible definition of the ordered triple would be to put

$$(a, b, c) = \{(a, b), (a, c), (b, c)\}.$$

If we are working inside a set theory with types, we observe that all three elements of the set constituting the triple are of type $t+2$ when a, b, c are of type t. As a consequence of this we avoid an identification of (a, a, a) and (a, a); indeed, $(a, a, a) = (a, a)$ can only take place when a is element of itself. Further it is easy to see that the equation $(a, b, c) = (d, e, f)$ implies $a = d$, $b = e$, $c = f$. If $a \neq b \neq c \neq a$, this is most easily seen by first noticing that d, e, f must also be different; for otherwise the set (d, e, f) would not contain 3 different elements as (a, b, c), and then a must be $= d$ and not $= e$ or f because only d occurs first in two of the three elements of (d, e, f), etc. However, the equation $(a, b, c) = (d, e, f)$ also implies $a = d$, $b = e$, $c = f$ in the case when two of the elements a, b, c coincide. Since in this case (a, b, c) contains at least one element of the form (g, g), also (d, e, f) must contain such an element, which means that two of the elements d, e, f must coincide. Looking at the expressions

$$\begin{aligned} (a, a, b) &= \{(a, a),\ (a, b)\}, & (d, d, e) &= \{(d, d),\ (d, e)\}, \\ (a, b, a) &= \{(a, b),\ (a, a),\ (b, a)\}, & (d, e, d) &= \{(d, e),\ (d, d),\ (e, d)\}, \\ (b, a, a) &= \{(b, a),\ (a, a)\}, & (e, d, d) &= \{(e, d),\ (d, d)\}, \end{aligned}$$

one verifies that a triple to the left can only coincide with one to the right when $a = d$ and $b = e$. This definition of the ordered triple is therefore in order.

After this it seems natural to define the ordered quadruple (a, b, c, d) as the set

$$\{(a, b, c),\ (a, b, d),\ (a, c, d),\ (b, c, d)\}$$

and continue this procedure for the definition of the ordered quintuples, sextuples, and so on. However, one could also define (a, b, c, d) as the set

$$\{\{(a, b),\ (a, c),\ (a, d),\ (b, c),\ (b, d),\ (c, d)\}\},$$

where the two applications of the operation $\{\ \}$ instead of only one have the effect that it is possible to distinguish between (a, a, a, a) and (a, a, a) except in the case of a pathological a.

A definition of ordered pairs and tuples can be found in [2, p. 281]. However, this definition seems rather artificial.

I shall not pursue these considerations here, but only emphasize that it is still a problem how the ordered n-tuple can be defined in the most suitable way.

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