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BESICOVITCH ALMOST PERIODIC FUNCTIONS IN ARBITRARY GROUPS

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For every $p \geqq 1$ Besicovitch [1] has introduced a class of generalized almost periodic functions of a real variable, the class of B^p-a.p. functions, which has many properties in common with the class L^p of measurable p-integrable functions with period 2π. Thus, in particular, the Riesz-Fischer Theorem holds for the B^2-a.p. functions. A trigonometric series $\Sigma A_n e^{i\lambda_n x}$ is the Fourier series of a B^2-a.p. function if (and only if) $\Sigma|A_n|^2$ is convergent. It is easily seen that by simple modifications of Besicovitch's procedure one can obtain various other classes which have the same properties. Thus, when we consider Besicovitch's problem in an arbitrary group, we cannot expect a unique solution, and probably any general solution will be somewhat artificial.

The aim of the present paper is to prove the existence of a general solution of the problem, valid for any infinite group G; and for evident reasons we need not consider the problem in finite groups. A type of generalized almost periodic functions in an arbitrary group was introduced in [3]. These functions were called Weyl almost periodic (W-a.p.) functions on account of their analogy with the usual Weyl almost periodic functions. What we need now is a wider generalization of almost periodicity. To avoid misunderstanding we remark at once, that when G is the additive group of real numbers, the class of usual B^p-a.p. functions cannot be obtained by our method of definition for any $p \geqq 1$.

Our definition of the Besicovitch almost periodic functions depends on the choice of a sequence $E_1, E_2, \ldots$ of subsets of G subject to certain conditions, and this choice is not unique. The following lemma indicates the conditions and asserts that a choice of $E_1, E_2, \ldots$ according to the conditions is always possible. By a symmetric subset of G we mean a subset E for which $E^{-1} = E$.

LEMMA. *Let G be an arbitrary infinite group. Then there exist denumerably many disjoint symmetric subsets $E_1, E_2, \ldots$ of G with the property*

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that to arbitrary, finitely many elements $a_1, \ldots, a_N$ *from* G *and an arbitrary positive integer* h *there exists an element* x *such that* $a_n x a_m \in E_h$ *for* $n, m = 1, \ldots, N$.

Proof. Let ν_G be the smallest ordinal number belonging to a well-ordering of G and let $\aleph_G$ be the cardinal number of G. By S we denote an arbitrary finite subset of G. The set of all such S has also the cardinal number $\aleph_G$. We consider denumerably many copies of each S and distinguish them by an upper index $1, 2, \ldots$. We denote an arbitrary S provided with an arbitrary upper index by T. The set Z of all such T has also the cardinal number $\aleph_G$. We choose a well-ordering

$$T_0, T_1, \ldots, T_\nu, \ldots \qquad (\nu < \nu_G)$$

of Z with the ordinal number ν_G.

Let F be an arbitrary symmetric subset of G with smaller cardinal number than G, and let T be $S = \{a_1, \ldots, a_N\}$ with some upper index. By $T(F)$ we understand a set of the form

$$F \cup \{a_n x a_m \mid a_n, a_m \in S\} \cup \{a_m^{-1} x^{-1} a_n^{-1} \mid a_n, a_m \in S\},$$

where the element x is chosen so that none of the elements $a_n x a_m$ (and hence none of the elements $a_m^{-1} x^{-1} a_n^{-1}$) are in F. Thus x has to be chosen outside the set

$$\bigcup_{n,m=1}^{N} a_n^{-1} F a_m^{-1},$$

and since this set has a smaller cardinal number than G, this choice is always possible. Like F, the set $T(F)$ is symmetric, and the two sets are either finite or have the same cardinal number. Furthermore, $T(F) \supset F$.

On account of the definition of ν_G, every ordinal number $\nu < \nu_G$ corresponds to a cardinal number $< \aleph_G$. Hence, by transfinite induction, we can define for every $\nu < \nu_G$ the symmetric subset F_ν of G so that F_0 is the empty set, $T_\nu(F_\nu) = F_{\nu+1}$ for all $\nu < \nu_G$, and $F_\nu = \bigcup_{\mu<\nu} F_\mu$ for all limit numbers $\nu < \nu_G$, and so that F_ν for every ordinal number $\nu < \nu_G$ has the cardinal number corresponding to ν if ν is infinite, and is finite if ν is finite. Obviously $F_0 \subset F_1 \subset \ldots \subset F_\nu \subset \ldots$.

In order to define our denumerably many sets $E_1, E_2, \ldots$ we define $E_1 \cap F_\nu, E_2 \cap F_\nu, \ldots$ by transfinite induction for every $\nu < \nu_G$ and demand that $E_1, E_2, \ldots$ all be contained in $\bigcup_\nu F_\nu$. We let $E_1 \cap F_0, E_2 \cap F_0, \ldots$ be the empty set. For a given ordinal number $\nu < \nu_G$ we assume the sets $E_1 \cap F_\mu, E_2 \cap F_\mu, \ldots$ defined for $\mu < \nu$. If ν is a limit number we have $F_\nu = \bigcup_{\mu<\nu} F_\mu$ and we can put

$$E_h \cap F_\nu = \bigcup_{\mu < \nu} (E_h \cap F_\mu) .$$

If ν is not a limit number we have, for a certain element x_ν,

$$F_\nu = T_{\nu-1}(F_{\nu-1}) = F_{\nu-1} \cup \{a_n x_\nu a_m \mid a_n, a_m \in S\} \cup \{a_m^{-1} x_\nu^{-1} a_n^{-1} \mid a_n, a_m \in S\}$$

when $T_{\nu-1}$ is $S = \{a_1, \ldots, a_N\}$ with an upper index. Hence, if this index is k, we can put

$$\begin{aligned} E_h \cap F_\nu &= E_h \cap F_{\nu-1} && \text{for } h \neq k , \\ E_h \cap F_\nu &= (E_h \cap F_{\nu-1}) \cup \{a_n x_\nu a_m \mid a_n, a_m \in S\} \cup \\ &\qquad \cup \{a_m^{-1} x_\nu^{-1} a_n^{-1} \mid a_n, a_m \in S\} && \text{for } h = k . \end{aligned}$$

Obviously, the denumerably many sets $E_1, E_2, \ldots$ constructed in this way have the desired properties. This completes the proof of the lemma. For orientation we remark that when G is the additive group of real numbers, it suffices to choose the symmetric disjoint sets $E_1, E_2, \ldots$ so that each of the sets contains arbitrarily long intervals.

In our infinite group G we now make a fixed choice, once and for all, of the subsets $E_1, E_2, \ldots$ in accordance with the lemma.

For an arbitrary real-valued function f on G and an arbitrary positive integer h we put

$$\overline{M}_h f = \inf_{\mathcal{A}, B, C} \sup_{x, y} \sum_{n=1}^{N} \alpha_n f(x a_n y) ,$$

where

$$\begin{aligned} \mathcal{A} &= \{\alpha_1, \ldots, \alpha_N;\ a_1, \ldots, a_N\}, && \alpha_n > 0,\ \alpha_1 + \ldots + \alpha_N = 1,\ a_n \in G , \\ B &= (b_1, \ldots, b_R), && b_r \in G , \\ C &= (c_1, \ldots, c_R), && c_r \in G , \end{aligned}$$

and the sup is to be taken over the non-empty set of all those x, y for which all the NR elements $b_r x a_n y c_r$ are in E_h. It is clear that $\overline{M}_h f$ really depends only on the values of f in E_h. Next we put

$$\overline{M}_B f = \limsup_{h \to \infty} \overline{M}_h f .$$

Of course we must allow the values $+\infty$ and $-\infty$ for $\overline{M}_h$ and $\overline{M}_B$.

The following well-known rules hold for the (usual) upper mean value

$$\overline{M} f = \inf_{\mathcal{A}} \sup_{x, y} \sum_{n=1}^{N} \alpha_n f(x a_n y) ,$$

which plays an important role in [3]:

$$\overline{M}(f+g) \leqq \overline{M}f+\overline{M}g, \qquad \overline{M}\{\lambda f\} = \lambda \overline{M}f \quad (\lambda \text{ constant } \geqq 0)$$
$$\overline{M}1 = 1, \qquad \overline{M}f \leqq \overline{M}g \quad \text{for } f \leqq g$$
$$\overline{M}\{|fg|\} \leqq (\overline{M}\{|f|^p\})^{1/p}(\overline{M}\{|g|^q\})^{1/q} \qquad (p > 1,\ q > 1,\ p^{-1}+q^{-1} = 1)$$
$$(\overline{M}\{|f+g|^p\})^{1/p} \leqq (\overline{M}\{|f|^p\})^{1/p}+(\overline{M}\{|g|^p\})^{1/p} \qquad (p \geqq 1)$$
$$\overline{M}\{f(x)\} = \overline{M}\{f(x^{-1})\}, \qquad \overline{M}\{f(axb)\} = \overline{M}\{f(x)\}\,.$$

They hold also with $\overline{M}_h$ instead of $\overline{M}$, and consequently also with $\overline{M}_B$ instead of $\overline{M}$. The rules for $\overline{M}_h$ can be proved in a similar, and nearly as simple, way as the corresponding rules for $\overline{M}$.

Together with the upper B-mean value $\overline{M}_B f$ we consider the lower B-mean value $\underline{M}_B f = -\overline{M}_B(-f)$, and together with $\overline{M}_h f$ we consider $\underline{M}_h f = -\overline{M}_h(-f)$. In a similar way as usual we deduce the rules which contain $\underline{M}_h$ alone or both $\underline{M}_h$ and $\overline{M}_h$. From these rules follow the corresponding rules which contain $\underline{M}_B$ alone or both $\underline{M}_B$ and $\overline{M}_B$.

Obviously
$$\underline{M}f \leqq \underline{M}_B f \leqq \overline{M}_B f \leqq \overline{M}f$$

(even with h instead of B). If the sign of equality in the middle holds and $\overline{M}_B f$ is finite, we say that f has the B-mean value $M_B f = \underline{M}_B f = \overline{M}_B f$. If f has a (usual) mean value $Mf = \underline{M}f = \overline{M}f$, it has the same B-mean value. For a complex-valued function $f(x) = u(x)+iv(x)$ on G we put

$$M_B f = M_B u + i M_B v$$

when both $M_B u$ and $M_B v$ exist. Then for an arbitrary complex constant c we have $M_B\{cf\} = c\,M_B f$. Furthermore, if $M_B f$ and $M_B g$ exist, then so does $M_B(f+g)$, and $M_B(f+g) = M_B f + M_B g$. Finally we note that $|M_B f| \leqq \underline{M}_B\{|f|\}$. In fact, for a certain real θ we have

$$|M_B f| = e^{i\theta} M_B f = M_B\{e^{i\theta} f\} = M_B\{\mathrm{Re}\,(e^{i\theta} f)\} \leqq \underline{M}_B\{|f|\}\,.$$

For an arbitrary $p \geqq 1$ and an arbitrary complex-valued function f on G we define the B^p-norm

$$\|f\|_{B^p} = (\overline{M}_B\{|f|^p\})^{1/p}\,.$$

In particular, a polynomial of the form

$$t(x) = \sum_{\nu=1}^{N} s^{(\nu)} \sum_{\varrho,\sigma=1}^{s^{(\nu)}} a_{\varrho,\sigma} D^{(\nu)}_{\varrho,\sigma}(x)\,, \tag{1}$$

where the matrices $D^{(\nu)}(x) = \{D^{(\nu)}_{\varrho,\sigma}(x)\}$ of orders $s^{(\nu)}$ are inequivalent irreducible unitary representations of G, has its B^2-norm $\|t\|_{B^2}$ determined by

$$\|t\|_{B^2}{}^2 = M\{|t|^2\} = \sum_{\nu=1}^{N} s^{(\nu)} \sum_{\varrho,\sigma=1}^{s^{(\nu)}} |a_{\varrho,\sigma}|^2$$

(see [5, p. 122]).

We shall now prove that the space of all complex-valued functions on G is complete with respect to the B^p-norm. Let $f_1, f_2, \ldots$ be an arbitrary B^p-fundamental sequence. Then

$$\|f_n - f_m\|_{B^p}{}^p \leqq \varepsilon_n \qquad \text{for } m > n ,$$

where $\varepsilon_n \to 0$ for $n \to \infty$. We have to prove that there exists a function f such that $\|f - f_n\|_{B^p} \to 0$ for $n \to \infty$. For $m > n$ we can determine a non-negative integer $H(n, m)$ so that

$$\overline{M}_h\{|f_n - f_m|^p\} \leqq 2\varepsilon_n \qquad \text{for } h > H(n, m) .$$

When the sequence $f_1, f_2, \ldots$ consists of W-a.p. functions we have

$$\overline{M}_h\{|f_n - f_m|^p\} = M\{|f_n - f_m|^p\} \leqq \varepsilon_n \qquad \text{for } m > n \text{ and all } h ,$$

so that in this case we can choose all our $H(n, m) = 0$.

We now choose integers $H_0 = 0 < H_1 < H_2 < \ldots$ so that

$$H_1 \geqq H(1, 2), \quad H_2 \geqq \max\big(H(1, 3),\ H(2, 3)\big) ,$$
$$H_3 \geqq \max\big(H(1, 4),\ H(2, 4),\ H(3, 4)\big), \quad \ldots .$$

As f we choose a function which is equal to f_1 on $E_1, \ldots, E_{H_1}$, equal to f_2 on $E_{H_1+1}, \ldots, E_{H_2}$, etc., and arbitrary elsewhere.

For a given n we consider an $h > H_{n-1}$. Then we can determine $m \geqq n$ so that $H_{m-1} < h \leqq H_m$; hence in E_h we have $f = f_m$, and since

$$h > H_{m-1} \geqq \max\big(H(1, m),\ H(2, m),\ \ldots, H(m-1, m)\big) ,$$

we have $h > H(n, m)$ except when $m = n$. Thus

$$\overline{M}_h\{|f - f_n|^p\} = \overline{M}_h\{|f_m - f_n|^p\} \leqq 2\varepsilon_n .$$

Letting $h \to \infty$ we obtain

$$\overline{M}_B\{|f - f_n|^p\} \leqq 2\varepsilon_n .$$

Hence our B^p-fundamental sequence $f_1, f_2, \ldots$ B^p-converges to f, and our statement is proved.

When the sequence $f_1, f_2, \ldots$ consists of W-a.p. functions, we can choose $H_n = n$, and the construction of a B^p-limit function f becomes especially simple; we can put $f = f_1$ in E_1, $f = f_2$ in E_2, etc., and choose f arbitrarily elsewhere.

Being in possession of a B^p-norm for $p \geqq 1$, it is plain how we define a B^p-a.p. function on G. A function f on G is called B^p-a.p. if there exists a sequence of polynomials t_n of the form (1) which B^p-converges to f, that is, $\|f-t_n\|_{B^p} \to 0$. (If the group is considered with a topology, these polynomials are formed from *continuous* representations of the group.)

We can now proceed in the usual way by ascribing a Fourier series to every B^p-a.p. function and establish the related results (cf. [2, pp. 104–110] and [5, pp. 119–144]).

Only the auxiliary inequality

$$\|\sigma\|_{B^p} \leqq \|f\|_{B^p}$$

in [2, p. 107] for a Bochner-Fejér polynomial σ of a B^p-a.p. function f may need an explicit proof in the present case. We have

$$\sigma(x) = \underset{t}{M_B}\{f(xt^{-1})K(t)\},$$

where K is a non-negative polynomial of the form (1) with $M\{K\}=1$. From Hölder's inequality we get in the usual way

$$|\sigma(x)| \leqq \underset{t}{\underline{M}_B}\{|f(xt^{-1})|K(t)\} \leqq \big(\underset{t}{\underline{M}_B}\{|f(xt^{-1})|^p K(t)\}\big)^{1/p}\big(M\{K(t)\}\big)^{1-1/p}.$$

Hence

$$\|\sigma\|_{B^p}{}^p = M\{|\sigma|^p\} \leqq \underset{x}{\underline{M}}\,\underset{t}{\underline{M}_B}\{|f(xt^{-1})|^p K(t)\} = \underset{x}{\underline{M}}\,\underset{t}{\underline{M}_B}\{|f(t^{-1})|^p K(tx)\},$$

and when to a given $\varepsilon>0$ we choose $\alpha_n>0$ with $\alpha_1+\ldots+\alpha_N=1$, and $a_n \in G$ suitably, this is

$$\begin{aligned}
&\leqq \sum_{n=1}^{N} \alpha_n \underset{t}{\underline{M}_B}\{|f(t^{-1})|^p K(txa_n)\} + \varepsilon \\
&\leqq \underset{t}{\underline{M}_B}\left\{|f(t^{-1})|^p \sum_{n=1}^{N} \alpha_n K(txa_n)\right\} + \varepsilon \\
&\leqq (1+\varepsilon)\underset{t}{\underline{M}_B}\{|f(t^{-1})|^p\} + \varepsilon .
\end{aligned}$$

Thus

$$\|\sigma\|_{B^p}{}^p \leqq \underline{M}_B\{|f|^p\} \leqq \overline{M}_B\{|f|^p\} = \|f\|_{B^p}{}^p,$$

as we had to prove.

For $p=2$ we obtain the Riesz-Fischer Theorem. A series of the form

$$\sum_{\nu=1}^{\infty} s^{(\nu)} \sum_{\varrho,\sigma=1}^{s^{(\nu)}} a_{\varrho,\sigma} D_{\varrho,\sigma}^{(\nu)}(x),$$

where the matrices $D^{(\nu)}(x)=\{D^{(\nu)}_{\varrho,\sigma}(x)\}$ of orders $s^{(\nu)}$ are inequivalent irreducible (continuous) unitary representations of G, is the Fourier series of a B^2-a.p. function if (and only if)

$$\sum_{\nu=1}^{\infty} s^{(\nu)} \sum_{\varrho,\sigma=1}^{s^{(\nu)}} |a_{\varrho,\sigma}|^2$$

is convergent. A function f which in E_h is equal to the h^{th} partial sum of the first series for $h=1, 2, \ldots$, and arbitrary elsewhere, is a B^2-a.p. function with this series as its Fourier series.

The Correspondence Theorem in [4, pp. 19–20] can easily be transferred to the present general situation. Then it establishes a close correspondence between the B^p-a.p. points and the measurable p-integrable functions on the Bohr compactification of G by all (continuous) almost periodic functions on G. In particular, the "contraction" to G of the Fourier series of the measurable p-integrable functions on the Bohr compactification of G yields exactly the Fourier series of our B^p-a.p. functions.

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