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THE DECOMPOSITION OF A CONTINUOUS LINEAR FUNCTIONAL INTO NON-NEGATIVE COMPONENTS

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We consider a real topological vector space X in which there is distinguished a set C such that $C+C \subseteq C$ and $\alpha C \subseteq C$ for every non-negative scalar α. Defining

$$u \leq v \qquad \text{to mean} \qquad v-u \in C\,,$$

we get a partial ordering of X, in a slightly generalized sense: the relation $\leq$ is reflexive and transitive, but may not be antisymmetric. An *interval* in X is a set that contains x whenever it contains u and v with $u \leq x \leq v$. A linear functional f on X is defined to be *non-negative* if its values on C are non-negative; that is, if $f(x) \geq 0$ whenever $x \geq 0$. A class F of linear functionals is defined to be *equicontinuous* if the set

$$U_F = \{x \mid f(x) < 1, \text{ all } f \in F\}$$

is a neighbourhood of the origin. By a *decomposition* of a class F of linear functionals, we mean a class F^+ of non-negative linear functionals such that every $f \in F$ is of the form $f_1 - f_2$, where $f_1, f_2 \in F^+$.

A result obtained for normed spaces by Grosberg and Krein [3], and for locally convex spaces in general by Bonsall [1], can be stated as follows: *in order that every equicontinuous class of linear functionals on X should have an equicontinuous decomposition, it is necessary and sufficient that X should have arbitrarily small neighbourhoods of the origin which are intervals.*

In this note we examine the possibility of decomposing a single linear functional into continuous non-negative components. We find a necessary and sufficient condition for this, and our method yields a simple proof of Bonsall's result.

If f is of the form $f_1 - f_2$, where f_1 and f_2 are continuous non-negative linear functionals, then there is a convex neighbourhood U of the origin, for example the set $\{u \mid f_1(u) < 1, f(u) < 1\}$, such that $f(x) < 1$ whenever $0 \leq x \leq u$ and $u \in U$. We show that, on the other hand, the existence of

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such a neighbourhood of the origin, corresponding to a continuous linear functional f, is sufficient to ensure that f has a continuous decomposition.

Let f be a continuous linear functional, and U an open convex neighbourhood of the origin such that $f(x) < 1$ whenever $0 \leq x \leq u$ and $u \in U$. If $f(u) \leq 0$ for all $u \in C$, let $f_1 = 0$. Otherwise, C has a point u_0 such that $f(u_0) = 1$; in this case, let S be the set of all points u for which there exists x such that $0 \leq x \leq u$ and $f(x) \geq 1$. Evidently, S is a convex set which contains u_0 and does not meet U. By the Hahn-Banach theorem, there is a linear functional f_1 such that $f_1(x) < 1$ when $x \in U$ and $f_1(x) \geq 1$ when $x \in S$ (cf. [2], p. 71]). The first of these inequalities shows that f_1 is continuous. The second shows that f_1 is non-negative: for if $u \in C$ then $u_0 + \alpha u \in S$ for all $\alpha \geq 0$ (since $0 \leq u_0 \leq u_0 + \alpha u$ and $f(u_0) = 1$), and therefore

$$f_1(u_0) + \alpha f_1(u) \geq 1 \qquad \text{for all} \qquad \alpha \geq 0 ,$$

so that $f_1(u) \geq 0$. Let $f_2 = f_1 - f$. If $f(u) > 0$ for some $u \in C$, and $\alpha = 1/f(u)$, then $0 \leq \alpha u$ and $f(\alpha u) = 1$, so that $\alpha u \in S$ and therefore $\alpha f_1(u) \geq 1$; thus $f_1(u) \geq f(u)$. This inequality holds for all $u \in C$, since f_1 is non-negative; hence f_2 is non-negative. We thus have the required decomposition.

Now suppose that X is locally convex. If there are arbitrarily small neighbourhoods of the origin which are intervals, and F is an equicontinuous class of linear functionals, we can assume that U, in the argument we have just used, is contained in an interval which is contained in U_F (thus ensuring that $f(x) < 1$ whenever $0 \leq x \leq u$, $u \in U$, and $f \in F$). The functionals f_1 then form an equicontinuous class, and so we get an equicontinuous decomposition of F. On the other hand, suppose that an equicontinuous class F has an equicontinuous decomposition F^+, and let V be a symmetric neighbourhood of the origin such that $V \subseteq U_{F^+}$; suppose that

$$u \leq x \leq v, \qquad \text{where} \qquad u, v \in \tfrac{1}{2}V ,$$

and that

$$f_1, f_2 \in F^+, \qquad \text{with} \qquad f_1 - f_2 \in F .$$

Then

$$f_1(x) \leq f_1(v) < \tfrac{1}{2} \qquad \text{and} \qquad -f_2(x) \leq f_2(-u) < \tfrac{1}{2} ,$$

so that $(f_1 - f_2)(x) < 1$, and therefore $x \in U_F$. This shows that U_F contains an interval which contains $\frac{1}{2}V$. By the Hahn-Banach theorem, however, every open convex neighbourhood of the origin is of the form U_F, where F is equicontinuous (in fact $F = \{f \mid f(x) < 1, \text{ all } x \in U_F\}$). Thus if every equicontinuous class of linear functionals has an equicontinuous decomposition then X has arbitrarily small neighbourhoods of the origin which are intervals.

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