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ON CERTAIN DISTRIBUTIONS OF INTEGERS IN PAIRS WITH GIVEN DIFFERENCES

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1. A study of the structure of some triple systems of Steiner (cf. [2]) led me to consider the following problem: Is it possible to distribute the numbers $1, 2, \ldots, 2n$ in n pairs (a_r, b_r) such that we have $b_r - a_r = r$ for $r = 1, 2, \ldots, n$?

In the sequel, a set of pairs of this kind is called a $1, +1$ system because the differences $b_r - a_r$ begin with 1 and increase by 1 when r increases by 1. One finds very soon that such a system does not always exist. In the simplest case $n = 1$ we have only the two numbers 1, 2 which quite trivially form a pair that is a system of the kind considered. But already in the case $n = 2$ there is no $1, +1$ system. Indeed the only distributions of 1, 2, 3, 4 in two pairs are

$$(1, 2)\ (3, 4) \qquad (1, 3)\ (2, 4) \qquad (2, 3)\ (1, 4)$$

with the corresponding differences

$$1, 1 \qquad\qquad 2, 2 \qquad\qquad 1, 3$$

so that we never have just the differences 1, 2.

Also for $n = 3$ one easily finds that no $1, +1$ system exists. However, for $n = 4$ there is again such a system, namely

$$(6, 7)\ (1, 3)\ (2, 5)\ (4, 8)\ .$$

Thus the question arises: For which n does a $1, +1$ system of pairs exist? The complete answer is given by the two theorems:

Theorem 1. *If $n \equiv 2$ or 3 (mod 4), no $1, +1$ system exists.*

Theorem 2. *If $n \equiv 0$ or 1 (mod 4), then a $1, +1$ system always exists.*

Proof of Theorem 1. I give here a very short proof due to Professor Th. Bang, my own original proof being somewhat longer. If the pairs

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(a_r, b_r), $r=1, \ldots, n$, constitute a $1,+1$ system of the numbers $1, \ldots, 2n$, then we have the equations

$$b_r - a_r = r, \qquad r = 1, 2, \ldots, n ,$$

whence by summation

$$\Sigma b_r - \Sigma a_r = \tfrac{1}{2} n(n+1) .$$

On the other hand, since the collection of the numbers a_r and b_r is the set $1, 2, \ldots, 2n$, we also have

$$\Sigma b_r + \Sigma a_r = n(2n+1) .$$

Addition of the two equations yields

$$\Sigma b_r = \tfrac{1}{4} n(5n+3) ,$$

which is an integer only when $n \equiv 0$ or $1 \pmod 4$.

Proof of Theorem 2. Let first $n \equiv 0 \pmod 4$. It will then suffice to give a general description of a $1,+1$ system for an arbitrary $n=4m$. Such a description is the following: The system of pairs consists of

1) all pairs $(4m+r, 8m-r)$ for $r=0, 1, \ldots, 2m-1$,
2) the pairs $(2m+1, 6m)$ and $(2m, 4m-1)$,
3) the pairs $(r, 4m-1-r)$ for $r=1, 2, \ldots, m-1$,
4) the pair $(m, m+1)$,
5) the pairs $(m+2+r, 3m-1-r)$ for $r=0, 1, \ldots, m-3$.

The pairs 1) give all the even differences $2, 4, \ldots, 4m$. The two odd differences $2m-1$ and $4m-1$ are obtained from 2). The least difference 1 is got from 4), the differences $3, 5, \ldots, 2m-3$ from 5) and the remaining odd differences $2m+1, \ldots, 4m-3$ from 3).

Now let $n \equiv 1 \pmod 4$. As pointed out by Professor Bang it is possible also in this case to give a general description of a $1,+1$ system which is quite analogous to that given by me above for the case $n \equiv 0 \pmod 4$. Indeed, setting $n=4m+1$, the system consists of

1) all pairs $(4m+2+r, 8m+2-r)$ for $r=0, 1, \ldots, 2m-1$,
2) the pairs $(2m+1, 6m+2)$ and $(2m+2, 4m+1)$,
3) the pairs $(r, 4m+1-r)$ for $r=1, 2, \ldots, m$,
4) the pair $(m+1, m+2)$,
5) the pairs $(m+2+r, 3m+1-r)$ for $r=1, 2, \ldots, m-2$.

The pairs 1) give all even differences $2, 4, \ldots, 4m$. The two odd differences $2m-1$ and $4m+1$ are given by 2). The least difference 1 is obtained from 4), the differences $3, 5, \ldots, 2m-3$ from 5), and the odd differences $2m+1, \ldots, 4m-1$ from 3).

In the cases $n \equiv 0, 1 \pmod 4$ the number of $1, +1$ systems of pairs built from the integers $1, \ldots, 2n$ will probably increase indefinitely when n increases to infinity, but I will not here make any attempt to treat this question.

2. I shall now make some remarks concerning the extension of this problem to the whole number series. It is clear that in this case the existence of a $1, +1$ system is quite trivial. More generally it is obvious that $l, +m$ systems exist, that means systems of disjoint pairs such that the corresponding differences are the numbers $l, l+m, l+2m, \ldots$. The reason for my treatment of these systems is that some quite peculiar theorems may be proved in this connection.

The simplest procedure for constructing a $1, +1$ system of all the integers is as follows. The first pair may be (a_1, b_1), where a_1 is 1 and b_1 is 2. Then the nth pair (a_n, b_n) is built by recursion by letting a_n be the least integer which is different from all a_r and b_r for $r = 1, 2, \ldots, n-1$ and setting $b_n = a_n + n$. I list here the first 29 of these pairs:

(1, 2) (3, 5) (4, 7) (6, 10) (8, 13) (9, 15) (11, 18) (12, 20) (14, 23)
(16, 26) (17, 28) (19, 31) (21, 34) (22, 36) (24, 39) (25, 41) (27, 44)
(29, 47) (30, 49) (32, 52) (33, 54) (35, 57) (37, 60) (38, 62) (40, 65)
(42, 68) (43, 70) (45, 73) (46, 75).

I was a little surprised when I discovered that these pairs can be given by a simple formula. Indeed we have

$$a_n = [\tfrac{1}{2}(1+5^{\frac{1}{2}})n], \qquad b_n = [\tfrac{1}{2}(3+5^{\frac{1}{2}})n],$$

where $[\xi]$ as usual denotes the greatest integer $\leqq \xi$. This will be proved in Theorem **3a**.

I shall first prove some other statements. Let α be the positive root of the equation

$$\alpha^2 - \alpha - 1 = 0,$$

thus $\alpha = \frac{1}{2}(1+5^{\frac{1}{2}})$, $\alpha^2 = \frac{1}{2}(3+5^{\frac{1}{2}})$. Then the propositions are:

1. *If* $n = [\alpha m]$, *then* $[\alpha n] = [\alpha^2 m] - 1$.
2. *If* $n = [\alpha m] + 1$, *then* $[\alpha n] = [\alpha^2 m] + 1$.
3. *If* $n = [\alpha m]$, *then* $[\alpha(n+1)] = [\alpha n] + 2$.
4. *If* $n = [\alpha m] + 1$, *while* $[\alpha(m+1)] = [\alpha m] + 2$, *then* $[\alpha(n+1)] = [\alpha n] + 1$.

Proof of 1. From $n = [\alpha m]$ it follows that

$$n^2 - mn - m^2 < 0, \qquad (n+1)^2 - m(n+1) - m^2 > 0,$$

where the inequality to the right may be written

$$n^2 - mn - m^2 - m + 2n + 1 > 0 .$$

Hence

$$(n+m-1)^2 - n(n+m-1) - n^2 = -n^2 + mn + m^2 - 2m - n + 1 < 0 ,$$

because the left-hand side equals $-n^2 + mn + m^2 + m - 2n - 1 - 3m + n + 2$, and $-n^2 + mn + m^2 + m - 2n - 1$ and $-3m + n + 2$ are both negative. Further

$$(n+m)^2 - n(n+m) - n^2 = -n^2 + mn + m^2 > 0 .$$

Thus it is proved that

$$n + m - 1 = [\alpha^2 m] - 1 = [\alpha n] .$$

PROOF OF 2. From $n - 1 = [\alpha m]$ it follows that

$$(n-1)^2 - m(n-1) - m^2 < 0, \qquad n^2 - mn - m^2 > 0 .$$

The inequality to the left is

$$n^2 - mn - m^2 + m - 2n + 1 < 0 .$$

We have

$$(n+m)^2 - n(n+m) - n^2 = -n^2 + mn + m^2 < 0 .$$

Further

$$\begin{aligned}(n+m+1)^2 - n(n+m+1) - n^2 &= -n^2 + mn + m^2 + 2m + n + 1 \\ &= -n^2 + mn + m^2 - m + 2n - 1 + 3m - n + 2 ,\end{aligned}$$

which is positive. Thus we have proved that

$$n + m = [\alpha^2 m] + 1 = [\alpha n] .$$

PROOF OF 3. The statement may be written

$$[\alpha^2 m] + 1 = [\alpha(n+1)]$$

under the same hypothesis as in proposition 1. Now

$$(n+m+1)^2 - (n+1)(n+m+1) - (n+1)^2 = -n^2 + mn + m^2 + m - 2n - 1 ,$$

which (see the proof of 1.) is negative. On the other hand

$$(n+m+2)^2 - (n+1)(n+m+2) - (n+1)^2 = -n^2 + mn + m^2 + 3m - n + 1 ,$$

which is positive because $-n^2 + mn + m^2 > 0$ and $3m - n + 1 > 0$. Thus

$$n + m + 1 = [\alpha^2 m] + 1 = [\alpha(n+1)] .$$

PROOF OF 4. The statement may be written

$$[\alpha^2 m] + 2 = [\alpha(n+1)] .$$

We have the two inequalities expressing the hypothesis of proposition 2 and further

$$(n+1)^2-(m+1)(n+1)-(m+1)^2 < 0 ,$$
$$(n+2)^2-(m+1)(n+2)-(m+1)^2 > 0 ,$$

that is,

$$n^2-mn-m^2-3m+n-1 < 0, \qquad n^2-mn-m^2-4m+3n+1 > 0 .$$

Now we have

$$(n+m+1)^2-(n+1)(n+m+1)-(n+1)^2$$
$$= -n^2+mn+m^2+m-2n-1 < 0$$

because $-n^2+mn+m^2<0$ (see the proof of 2.) and $m-2n-1<0$. Further,

$$(n+m+2)^2-(n+1)(n+m+2)-(n+1)^2$$
$$= -n^2+mn+m^2+3m-n+1 > 0 .$$

Thus

$$n+m+1 = [\alpha^2 m]+2 = [\alpha(n+1)] .$$

It is now easy to prove the following theorem:

THEOREM 3a. *Every positive integer is of one and only one of the two forms*

$$[\alpha n], \qquad [\alpha^2 n] ,$$

where n denotes some positive integer. Further the pairs obtained by the procedure explained above are just the pairs $([\alpha n], [\alpha^2 n])$.

PROOF. My first proof of this theorem was based on the preceding four lemmas. However, a reproduction of this proof here is superfluous because it is easily verified that the second statement in Theorem 3a is a special case of Theorem 4, which is proved below. Then the first proposition in Theorem 3a is proved by the following simple argument: Since every integer is of one of the two forms $[\alpha n]$ or $[\alpha^2 n]$, the least integer which does not belong to any of the pairs $([\alpha r], [\alpha^2 r])$, $r=1, 2, \ldots, n-1$, must occur as the least integer in the pairs $([\alpha s], [\alpha^2 s])$ for $s=n, n+1, \ldots$. It is then evident that a_n is just this number, which means that the pairs obtained by the recursive procedure explained above are just the pairs $([\alpha n], [\alpha^2 n])$.

A more general theorem is:

THEOREM 3b. *Let m be an arbitrary natural number and l one of the numbers* 1, ... m. *Further let N_1 be the set of integers of the form*

$$f(n) = \left[\tfrac{1}{2}\left(2-m+(m^2+4)^{\frac{1}{2}}\right)\left(n-\frac{m-l}{m}\right)+\frac{2(m-l)}{m}\right]$$

and N_2 *the set of integers of the form*

$$g(n) = \left[\tfrac{1}{2}\left(2+m+(m^2+4)^{\frac{1}{2}}\right)\left(n-\frac{m-l}{m}\right)+\frac{2(m-l)}{m}\right].$$

Then N_1 *and* N_2 *are complementary subsets of the natural number series* N *and the pairs* $\left(f(n),\, g(n)\right)$ *constitute a* $l,+m$ *system.*

I omit the proof, which can be performed by considerations analogous to those in the proof of Theorem 3a.

3. The relation between two sets of the forms $[\alpha n]$ and $[\beta n]$ may be very different in different cases. I should like to give an example, where one of the two sets is contained in the other. Of course this phenomenon is trivial in the case that α/β or β/α is an integer. It is worth noticing, however, that it can also happen when α/β is irrational, which is shown by the following example: Every integer of the form $[(1+2^{\frac{1}{2}})n]$ is also of the form $[2^{\frac{1}{2}}n]$. Indeed, I shall prove the general validity of the formula

$$[(1+2^{\frac{1}{2}})m] = [2^{\frac{1}{2}}l],$$

where

$$l = [(2^{-\frac{1}{2}}+1)m+\tfrac{1}{2}],$$

so that l is the integer nearest to $(2^{-\frac{1}{2}}+1)m$.

Let $n=[2^{\frac{1}{2}}m]$, so that we have

$$2^{\frac{1}{2}}m = n+\varepsilon, \qquad 0<\varepsilon<1.$$

Then

$$l = [m+\tfrac{1}{2}(n+\varepsilon)+\tfrac{1}{2}].$$

I take first the case $n=2\nu$. Then

$$l = m+\nu$$

so that

$$2^{\frac{1}{2}}l = 2^{\frac{1}{2}}m+2^{\frac{1}{2}}\nu = 2^{\frac{1}{2}}m+2^{-\frac{1}{2}}n = m+(2^{\frac{1}{2}}-1)m+2^{-\frac{1}{2}}n.$$

Since

$$m > 2^{-\frac{1}{2}}n,$$

we obtain

$$2^{\frac{1}{2}}l > m+(1-2^{-\frac{1}{2}})n+2^{-\frac{1}{2}}n = m+n.$$

Since

$$2^{\frac{1}{2}}m < n+1 \qquad \text{and} \qquad 2^{-\frac{1}{2}}n < m,$$

we have

$$2^{\frac{1}{2}}l < n+1+2^{\frac{1}{2}}\nu = n+1+2^{-\frac{1}{2}}n < m+n+1$$

so that in this case

$$[2^{\frac{1}{2}}l] = m+n = [(1+2^{\frac{1}{2}})m].$$

Then let n be odd$=2\nu+1$. We obtain

$$l = m+\nu+1 \,,$$

whence

$$2^{\frac{1}{2}}l = 2^{\frac{1}{2}}m+2^{\frac{1}{2}}\tfrac{1}{2}(n+1) = m+(2^{\frac{1}{2}}-1)m+2^{-\frac{1}{2}}(n+1)$$
$$< m+(1-2^{-\frac{1}{2}})(n+1)+2^{-\frac{1}{2}}(n+1) = m+n+1 \,,$$

and on the other hand

$$2^{\frac{1}{2}}l > m+n$$

because

$$2^{\frac{1}{2}}m > n \qquad \text{and} \qquad 2^{-\frac{1}{2}}(n+1) > m \,.$$

Hence in this case as well we have

$$[2^{\frac{1}{2}}l] = m+n = [(1+2^{\frac{1}{2}})m] \,.$$

It is quite curious to observe that whereas every integer of the form $[(1+2^{\frac{1}{2}})n]$ is also of the form $[2^{\frac{1}{2}}n]$, no integer of the form $[(2+2^{\frac{1}{2}})n]$ is of the form $[2^{\frac{1}{2}}n]$. Indeed, the two latter sets of integers, those of form $[2^{\frac{1}{2}}n]$ and those of form $[(2+2^{\frac{1}{2}})n]$, are two complementary subsets of the natural number series. (See Theorem 3b for $l=m=2$ or Theorem 4.)

It is clear that these considerations can be extended in different directions. For example, one might ask if it is possible, also for $m>2$, to find m different irrational numbers $\alpha_1, \ldots, \alpha_m$, such that

$$[\alpha_1 n], \ldots, [\alpha_m n] \,,$$

for $n=1, 2, \ldots$ in infinitum, furnish m mutually disjoint sets with the whole number series as their union. I shall show below (Section 7) that the answer to this question is negative.

4. Instead of pairs with given differences one might consider triples (a_n, b_n, c_n) such that the second differences $a_n-2b_n+c_n$ for $n=1, 2, \ldots$ have given values. I mention an example.

Let (a_1, b_1, c_1) be (1, 2, 4) so that the second difference here is 1, and let a_n, b_n, c_n be determined recursively by letting a_n be the least integer different from all a_r, b_r, c_r, where $r<n$, b_n the least integer different from all a_r, b_r, c_r with $r<n$ and from a_n, while c_n is so chosen that

$$a_n-2b_n+c_n = n \,.$$

The first twelve of these triples are

(1, 2, 4) (3, 5, 9) (6, 7, 11) (8, 10, 16) (12, 13, 19) (14, 15, 22)
(17, 18, 26) (20, 21, 30) (23, 24, 34) (25, 27, 39) (28, 29, 41)
(31, 32, 45).

I have attempted to find a general formula for the nth of these triples by the aid of the operation [ ], buth without success.

With more success I have treated the triples obtained by the following recursion. Let $a_1=1$, $b_1=2$, $c_1=3$. Whenever a_r, b_r, c_r are already determined for $r \leqq n$, a_{n+1} is chosen as the least integer different from all those a_r, b_r, c_r, then b_{n+1} is chosen as the $(n+1)$th integer different from all the a_r, b_r, c_r and from a_{n+1}, while c_{n+1} is put $=b_{n+1}+n+1$. The first eight of these triples are

$$(1, 2, 3)\ (4, 6, 8)\ (5, 10, 13)\ (7, 14, 18)\ (9, 17, 22)$$
$$(11, 21, 27)\ (12, 25, 32)\ (15, 29, 37)$$

Here I have found the general formulas:

$$a_{2n} = [\tfrac{1}{2}(3+21^{\frac{1}{2}})n + \tfrac{1}{6}(-3+21^{\frac{1}{2}})], \qquad a_{2n+1} = [\tfrac{1}{2}(3+21^{\frac{1}{2}})n + \tfrac{1}{3}\cdot 21^{\frac{1}{2}}]$$
$$b_n = [\tfrac{1}{2}(3+21^{\frac{1}{2}})n] - 1, \qquad c_n = [\tfrac{1}{2}(3+21^{\frac{1}{2}})n] + n-1 .$$

5. My colleague I. Johansson pointed out to me that it could be seen almost immediately that N_1 and N_2 are disjoint, N_1 being the integers $[\alpha n]$ for integral n, N_2 the integers $[\beta n]$, if α and β are positive irrational numbers such that

$$\alpha^{-1}+\beta^{-1} = 1 .$$

Indeed, the proof is simply this: Let us assume that integers m and n exist such that $l=[\alpha m]=[\beta n]$. Then we have

$$l < \alpha m < l+1, \qquad l < \beta m < l+1 ,$$

or

$$\alpha^{-1}l < m < \alpha^{-1}(l+1), \qquad \beta^{-1}l < n < \beta^{-1}(l+1) ,$$

whence by addition, taking into account that $\alpha^{-1}+\beta^{-1}=1$, we obtain $l<m+n<l+1$, which is impossible.

By the way one observes at once that this can be generalized by putting

$$\alpha^{-1}+\beta^{-1}=c^{-1}$$

and here supposing only that c is a positive integer. Indeed this assumption leads by the same development to the inequality

$$l < c(m+n) < l+1$$

which is impossible in integers l, m, n, c.

One might now perhaps be tempted to believe that the last sufficient condition for the non-existence of elements common to N_1 and N_2 also is necessary. However, the situation is not so simple, as can be seen from the fact mentioned in Section 3 that all numbers $[(1+2^{\frac{1}{2}})n]$ are also numbers $[2^{\frac{1}{2}}n]$, whereas no number of the last form is of the form $[(2+2^{\frac{1}{2}})n]$: The sum

$$\frac{1}{1+2^{\frac{1}{2}}}+\frac{1}{2+2^{\frac{1}{2}}} = 2^{-\frac{1}{2}}$$

is not a number of the form 1 divided by an integer. Indeed, a still weaker sufficient condition, which is also necessary, is given in Theorem 8.

6. Some theorems can suitably be added. In these N, N_1 and N_2 retain their earlier meanings.

Theorem 4. *If $\alpha^{-1}+\beta^{-1}=1$, then N_1 and N_2 are complementary subsets of N.*

Proof. Since α and β shall be >0, they are both >1. We may also suppose that $1<\alpha<2$. Indeed if α and β were both >2, we would obtain $\alpha^{-1}+\beta^{-1}<1$. It is then evident that always

$$[\alpha(n+1)] = [\alpha n]+1 \quad \text{or} \quad [\alpha n]+2 \, .$$

In order to prove that every integer is either of the form $[\alpha n]$ or of the form $[\beta m]$ I have to show that when

$$[\alpha(n+1)] = [\alpha n]+2 \, ,$$

then

$$[\alpha n]+1 = [\beta m]$$

for a certain integer m. Let

$$\alpha = 1+\varkappa, \qquad 0 < \varkappa < 1 \, ,$$

and let k be a positive integer. Then for $n=[k/\varkappa]$

$$[\alpha n] = [n+n\varkappa] = n+[n\varkappa] = n+k-1$$

because

$$n\varkappa < k < n\varkappa+\varkappa < \varkappa n+1$$

so that obviously

$$[n\varkappa] = k-1 \, .$$

On the other hand

$$[\alpha(n+1)] = n+1+[(n+1)\varkappa] = n+1+k \, ,$$

since $k<(n+1)\varkappa<k+1$. It is clear that, for $k=1, 2, \ldots$ we have just the jumps by 2, which the value of $[\alpha n]$ makes when n increases by 1. Now, since

$$\beta = \frac{\alpha}{\alpha-1} = \frac{1+\varkappa}{\varkappa} = 1+\frac{1}{\varkappa} \, ,$$

we get

$$[k\beta] = k+[k/\varkappa] = n+k = [\alpha n]+1 \, .$$

Thus, when $[\alpha(n+1)] = [\alpha n] + 2$, then the intermediate number $[\alpha n] + 1$ has the form $[k\beta]$.

THEOREM 5. *If* $1, \alpha^{-1}, \beta^{-1}$ *are linearly independent (relative to the field of rationals), then* N_1 *and* N_2 *have an infinite number of common elements.*

PROOF. It is again clear that we can assume α and $\beta > 1$, because if $\alpha < 1$, every natural number is of the form $[\alpha n]$. It follows from well-known theorems in the theory of diophantine approximations that infinitely many triples of integers l, m, n exist such that

$$-\alpha^{-1} < \alpha^{-1}l - m < 0, \qquad -\beta^{-1} < \beta^{-1}l - n < 0\,,$$

whence

$$l < \alpha m < l+1, \qquad l < \beta n < l+1\,,$$

so that

$$[\alpha m] = [\beta n] = l\,.$$

THEOREM 6. *Let* α *and* β *be irrational numbers, but* $1, \alpha^{-1}, \beta^{-1}$ *linearly dependent in such a way that in the equation*

$$a\alpha^{-1} + b\beta^{-1} = c, \qquad c > 0, \qquad a, b, c \text{ integers}\,,$$

a and b have opposite signs. Then N_1 *and* N_2 *have an infinite number of common elements.*

PROOF. For a positive integer z let x_z and y_z denote the numbers $z\alpha^{-1}$, $z\beta^{-1}$ reduced modulo 1, so that $0 < x_z < 1$, $0 < y_z < 1$. Then the points (x_z, y_z) lie on a certain number of straight line segments crossing the unit square. All these segments have the same slope, namely $-a/b$, and one of them ends at the point $(1, 1)$. The points (x_z, y_z) lie everywhere dense on the segments. Therefore infinitely many (x_z, y_z) lie in the region

$$1 - \alpha^{-1} < x < 1, \qquad 1 - \beta^{-1} < y < 1\,,$$

which leads to the same conclusion as in the case of the preceding theorem.

THEOREM 7. *If the irrational numbers* α *and* β *are connected by an equation*

$$a\alpha^{-1} + b\beta^{-1} = c, \qquad a > 0,\ b > 0,\ c > 1, \qquad a, b, c \textit{ integers}\,,$$

where the greatest common divisor of a, b and c equals 1, then N_1 *and* N_2 *have infinitely many common elements.*

PROOF. Just as before, the points (x_z, y_z) lie on a number of line segments crossing the unit square, all of them possessing the same slope

$-a/b$. Since the points (x_z, y_z) lie everywhere dense on the segments, it follows that infinitely many of them are in the rectangle

$$1-\alpha^{-1} < x < 1, \qquad 1-\beta^{-1} < y < 1,$$

if it is shown that one of the lines enters into this region. This, however, is very easily seen, because either $a\alpha^{-1} > 1$ or $b\beta^{-1} > 1$, since $c \geqq 2$. Thus we have either $\alpha^{-1} > a^{-1}$ or $\beta^{-1} > b^{-1}$, while the line segment λ lying closest to the point (1, 1) connects the points $(1-a^{-1}, 1)$ and $(1, 1-b^{-1})$. Therefore λ must enter the said rectangle.

On the other hand it may be noticed that, if the irrational numbers α and β are connected by the equation

$$a\alpha^{-1}+b\beta^{-1} = 1,$$

where the integers a and b are > 0, then N_1 and N_2 are disjoint. This is seen by the fact that N_1 is contained in the set of all $[\alpha a^{-1} n]$ and N_2 contained in the set of all $[\beta b^{-1} n]$; for in Section 5 we have remarked that the two sets $[\alpha_1 n]$ and $[\beta_1 n]$ are disjoint when

$$\alpha_1^{-1}+\beta_1^{-1} = 1.$$

From all this follows

THEOREM 8. *A necessary and sufficient condition for the sets $[\alpha n]$ and $[\beta n]$ to be disjoint is that α and β are connected by an equation*

$$a\alpha^{-1}+b\beta^{-1} = 1,$$

where a and b are positive integers.

7. As an application we may prove the nonexistence of 3 irrationals α, β, γ such that the corresponding sets N_1, N_2, N_3 are mutually disjoint (N_1 being the set of all $[\alpha n]$, etc.). Indeed, if α, β, γ should possess this property, it follows from Theorem 8 that we should have 3 equations

$$a_1\alpha^{-1}+b_1\beta^{-1} = 1, \qquad a_2\alpha^{-1}+b_2\gamma^{-1} = 1, \qquad a_3\beta^{-1}+b_3\gamma^{-1} = 1$$

with positive coefficients a_i and b_i, $i = 1, 2, 3$. The elimination of γ between the two last equations yields

$$a_2 b_3 \alpha^{-1} - a_3 b_2 \beta^{-1} = b_3 - b_2,$$

where the coefficients have not all the same sign. Therefore the last equation is independent of $a_1\alpha^{-1}+b_1\beta^{-1} = 1$ so that α, β, γ must all be rational.

If the operation of taking the greatest integer $\leqq \xi$ is iterated, we may of course get expressions furnishing an arbitrary number of disjoint sets of integers. For example, the three expressions

$$[\alpha[\alpha n]], \qquad [\alpha[\alpha^2 n]], \qquad [\alpha^2 n],$$

where $\alpha = \frac{1}{2}(1+5^{\frac{1}{2}})$, yields three subsets of N which are mutually disjoint and have N as their union.

8. The theorems in the present paper concerning the sets $[\alpha n]$ are extended to the more general sets of the form $[\alpha n + \beta]$ in a paper to appear in Norske Vid. Selsk. Forh., Trondheim.

See also the following paper by Bang [1].

REFERENCES

1. Th. Bang, *On the sequence* $[n\alpha]$, $n = 1, 2, \ldots$. *Supplementary note to the preceding paper by Th. Skolem,* Math. Scand. 5 (1957), 69–76.
2. J. Steiner, *Combinatorische Aufgabe,* J. reine angew. Math. 45 (1853), 181–182 (= Gesammelte Werke II, Berlin, 1882, 437–438).

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