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ON THE SEQUENCE $[n\alpha]$, $n=1,2,\ldots$. SUPPLEMENTARY NOTE TO THE PRECEDING PAPER BY TH. SKOLEM

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1. Let $\alpha \geqq 1$ be a real number, and let N_α denote the sequence $[\alpha], [2\alpha], [3\alpha], \ldots$ of positive integers, where, as usual, $[x]$ is the largest integer less than or equal to x. (The notation N_α will be more convenient for our purpose than the notation $N_1, N_2, \ldots$ used by Professor Skolem). Hence, in the sequel, N_1 (or simply N) denotes the sequence of the natural numbers, N_2 is the sequence of the positive even numbers, etc.

In Section 2 we shall give another proof of Skolem's beautiful Theorem 4 stating a sufficient condition that N_α and N_β be complementary subsets of N; the condition is also easily seen to be necessary. By Theorem 4 it is possible, for α irrational, to pass from a sequence N_α to its complement in N. Using this, we give in Section 3 necessary and sufficient conditions that $N_\alpha \cup N_\beta$ be N, and conditions for $N_\alpha \supseteqq N_\beta$. We describe also a simple geometrical connection between the numbers α and β occurring in the theorems. In Section 4 we show that for rational α and β we get exactly the same conditions for $N_\alpha \supseteqq N_\beta$ as in the irrational case. The sufficiency of the condition can easily be obtained from the irrational case by a passage to the limit, while the necessity of the condition needs an independent proof. Finally, in Section 5 we give a formula for the asymptotic density of $N_\alpha \cap N_\beta$, which could also be used to prove some of the previous results, a method applicable to more general sequences like $[n\alpha+\beta]$, $n=1, 2, \ldots$. The theorems are numbered in continuation of those in Skolem's paper.

We mention a few obvious properties of the sets N_α: If α is rational, $\alpha=p/q$, then N_α is periodical modulo p. If a is a positive integer, then $N_\alpha \supseteqq N_{a\alpha}$. If the ratio α/β is rational, then N_α and N_β have an infinity of common elements, namely $[M]$, where M runs through the common multiples of α and β. The number α is uniquely determined by the sequence N_α.

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2. We shall prove Theorem 4 (slightly strengthened):

The sequences N_α *and* N_β *are complementary in* N *if and only if* α *and* β *are irrational and*

$$\alpha^{-1}+\beta^{-1} = 1 . \tag{1}$$

In the proof we use the density function $\mu_\alpha(h)$ defined for all integers h as the number of elements of N_α which do not exceed h. The integer $\mu_\alpha(h)$ is easily found to be determined by

$$(h+1)\alpha^{-1}-1 \leqq \mu_\alpha(h) < (h+1)\alpha^{-1} , \tag{2}$$

and here the sign of equality only occurs if $(h+1)\alpha^{-1}$ is an integer. Obviously, $\mu_\alpha(h)/h$ tends to α^{-1} for $h \to \infty$.

That N_α and N_β are complementary in N is equivalent to

$$\mu_\alpha(h)+\mu_\beta(h) = h$$

for all h. If N_α and N_β are complementary we therefore have $\alpha^{-1}+\beta^{-1}=1$, which proves the necessity of (1). As remarked in Section 1 it is necessary that α/β is irrational, and this together with (1) shows that α and β are both irrational. This proves the necessity.

If α and β are irrational, then the sign of equality cannot occur in (2), and hence we have

$$(h+1)\alpha^{-1}-1+(h+1)\beta^{-1}-1 < \mu_\alpha(h)+\mu_\beta(h) < (h+1)\alpha^{-1}+(h+1)\beta^{-1}$$

and by (1)

$$h-1 < \mu_\alpha(h)+\mu_\beta(h) < h+1 ,$$

which proves that the integer $\mu_\alpha(h)+\mu_\beta(h)$ equals h. Hence the sets are complementary. This proves the sufficiency.

When α and β are rational and satisfy (1), we can write them as fractions p/q and p/r, where p, q and r are relatively prime. Just as above this implies $\mu_\alpha(h)+\mu_\beta(h)=h$, except for the case when p divides $h+1$. Thus N_α and N_β are "almost complementary" in the sense that they are complementary except for all multiples of p, which are contained in both of them, and the integers immediately preceding these, which belong to none of them.

3. Let α' denote the number related to α by the equation

$$\alpha^{-1}+\alpha'^{-1} = 1 .$$

Then $(\alpha')'=\alpha$. Theorem 4 states that the sets N_α and $N_{\alpha'}$ are complementary for irrational α.

Professor Skolem gives (Theorem 8) the conditions that N_β and N_γ be disjoint. The result can be stated in the following way:

The sequences N_β and N_γ are disjoint if and only if β and γ are irrational and there exist positive integers b and c such that

$$(\beta b^{-1})' = \gamma c^{-1}, \qquad \textit{that is,} \qquad b\beta^{-1} + c\gamma^{-1} = 1 .$$

Hence, the condition states that the sets can be enlarged to complementary sets $N_{\beta b^{-1}}$ and $N_{\gamma c^{-1}}$.

The result can be expressed in a simple geometrical way by means of the lattice of points (m, n), where m and n are positive integers (cf. fig. 1). The segment connecting α on the X-axis with α' on the Y-axis passes through the point $(1, 1)$. If β and γ are irrational, the sets N_β and N_γ are disjoint when, and only when, the segment connecting the points β and γ on the coordinate axes passes through a lattice point in the positive quadrant.

If we replace the sets by their complements, we get the following statement:

If α and δ are irrational and greater than 1, then the necessary and sufficient condition that $N_\alpha \cup N_\delta = N$ is that there exist positive integers a and d such that

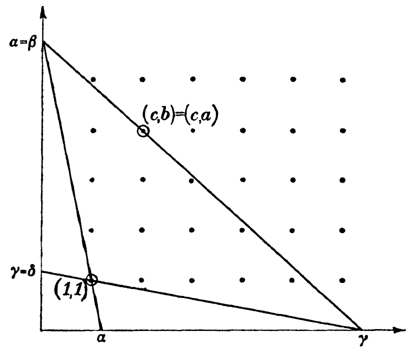

Fig. 1.

$$(\alpha' a^{-1})' = \delta' d^{-1}, \qquad \text{that is,} \qquad a(1-\alpha^{-1}) + d(1-\delta^{-1}) = 1 .$$

By replacing one of the sets by its complement we get

THEOREM 9. *If α or γ is irrational, then the necessary and sufficient condition that $N_\alpha \supseteqq N_\gamma$ is that there exist positive integers a and c such that*

$$(3) \qquad (\alpha' a^{-1})' = \gamma c^{-1}, \qquad \textit{that is,} \qquad a(1-\alpha^{-1}) + c\gamma^{-1} = 1 .$$

The condition is illustrated in fig. 1. To a given α there is a finite number of possible values of a and an infinite number of possible values of c. The condition obviously implies $\alpha \leqq \gamma$.

Suppose that (3) is fulfilled, that a_1 is a divisor of a, and that c_1 is a divisor of c. Then we have

$$(4) \qquad N_\alpha \supseteqq N_{(\alpha' a_1^{-1})'} \supseteqq N_{(\alpha' a^{-1})'} = N_{\gamma c^{-1}} \supseteqq N_{\gamma c_1^{-1}} \supseteqq N_\gamma .$$

We are going to prove that the sets in (4) are the only ones between N_α and N_γ.

THEOREM 10. *Suppose that* $N_\alpha \supseteqq N_\gamma$, *where* α *and* γ *are irrational, i.e., that* (3) *is satisfied. Then the numbers* ε *for which* $N_\alpha \supseteqq N_\varepsilon \supseteqq N_\gamma$ *are of two types, viz.* $\varepsilon = (\alpha' a_1^{-1})'$, *where* a_1 *divides* a, *and* $\varepsilon = \gamma c_1^{-1}$, *where* c_1 *divides* c.

The two cases correspond to $\varepsilon \leqq \gamma c^{-1}$ and $\varepsilon \geqq \gamma c^{-1}$, respectively.

The inclusion $N_\alpha \supseteqq N_\varepsilon$ means that there exists an equation

$$p(1-\alpha^{-1}) + q\varepsilon^{-1} = 1 ,$$

and $N_\varepsilon \supseteqq N_\gamma$ means that there exists an equation

$$r(1-\varepsilon^{-1}) + s\gamma^{-1} = 1 .$$

Elimination of ε yields

$$pr(1-\alpha^{-1}) + qs\cdot\gamma^{-1} = q+r-qr ,$$

and this can only be compatible with (3) when the right-hand side $q+r-qr$ is positive. Thus we get $q=1$ or $r=1$, and in both cases $q+r-qr=1$. In the case $\varepsilon \leqq \gamma c^{-1}$ we get $\varepsilon = (\alpha' p^{-1})'$ and $pr=a$, which shows that $p=a_1$ divides a; in the case $\varepsilon \geqq \gamma c^{-1}$ we get $\varepsilon = \gamma s^{-1}$ and $qs=c$, which shows that $s=c_1$ divides c. Thus the theorem is proved.

As a numerical example consider N_α with $\alpha = 2^{\frac{1}{2}}$ (cf. Skolem's paper). The complement of N_α is N_β, $\beta = 2+2^{\frac{1}{2}}$, and the sets N_β disjoint to N_α are the sets with $\beta = (2+2^{\frac{1}{2}})h$, where h is an integer. The sets N_γ contained in N_α are of three types, viz. $\gamma = 2^{\frac{1}{2}}h$, $\gamma = (1+2^{\frac{1}{2}})h$, and $\gamma = (4+3\cdot 2^{\frac{1}{2}})h$ (corresponding to the three possible values $a=1$, 2 or 3 in the formulas above). The number $\gamma = (4+3\cdot 2^{\frac{1}{2}})\cdot 6$ corresponds to $a=3$ and $c=6$ above and we get as intermediate sets

$$N_{2^{\frac{1}{2}}} \supseteqq N_{4+3\cdot 2^{\frac{1}{2}}} \supseteqq \left\{ \begin{array}{c} N_{(4+3\cdot 2^{\frac{1}{2}})\cdot 2} \\ N_{(4+3\cdot 2^{\frac{1}{2}})\cdot 3} \end{array} \right\} \supseteqq N_{(4+3\cdot 2^{\frac{1}{2}})\cdot 6} .$$

4. All the equations occurring in the preceding theorems associate with an irrational α irrational β, γ or δ. In the case of rational α, β, γ and δ the situation is more complicated, as already mentioned in Section 1. But it is worth-while to point out that Theorem 9 is valid also in the case of rational α, that is, we have the more general theorem:

THEOREM 11. *If* α *is greater than* 1, *then* $N_\alpha \supseteqq N_\gamma$ *if and only if there exist positive integers* a *and* c *such that*

(3) $\qquad (\alpha' a^{-1})' = \gamma c^{-1}, \qquad$ *that is,* $\qquad a(1-\alpha^{-1}) + c\gamma^{-1} = 1 .$

We need only prove the theorem for rational α and γ. The sufficiency of the condition can easily be proved by a passage to the limit from the irrational case. Indeed, if α tends *decreasingly* to α_0, then N_α converges to N_{α_0}, in the sense that every finite section of N_α is identical with the corresponding section of N_{α_0} when α is sufficiently close to α_0. Suppose now (3) is satisfied by rational α_0 and γ_0, then α_0 and γ_0 can be approximated simultaneously by decreasing sequences of irrational α and γ which satisfy (3) with the same a and c. Hence $N_\alpha \supseteqq N_\gamma$, and in the limit we get $N_{\alpha_0} \supseteqq N_{\gamma_0}$.

The necessity of the condition needs an independent proof. In the following we suppose that $N_\alpha \supseteqq N_\gamma$.

First, if α is an integer greater than 1, then it is easy to see directly that γ has to be an integer (moreover, a multiple of α; incidentally, this is in accordance with the equation (3)).

Let α and γ be written as fractions with a common numerator, $\alpha = p/q$ and $\gamma = p/r$, where the largest common divisor of p, q and r is 1. We shall prove that $p-q$ and r are relatively prime. Put $d = (p-q, r)$. Then p/d is a multiple of p/r and a multiple of $p/(p-q)$. Hence, using $N_\gamma \subseteqq N_\alpha$, that is, $N_{p/r} \subseteqq N_{p/q}$, we get

$$N_{p/d} \subseteqq N_{p/r} \cap N_{p/(p-q)} \subseteqq N_{p/q} \cap N_{p/(p-q)} ,$$

and, as remarked at the end of Section 2, this is the arithmetic progression N_h, where $h = p/(p, q)$ is an integer. From the preceding it follows that d divides p, and since d also divides $p-q$ and r, we have $d = 1$.

The inclusion $N_\gamma \subseteqq N_\alpha$ means that to each n there exists an integer m_n such that $[n\gamma] = [m_n\alpha]$. Hence

$$n\gamma - l \qquad \text{and} \qquad m_n\alpha - l$$

have the same sign for *all* integers l (if the sign of 0 is defined to be $+$). Inserting γ and α we get that

$$np - lr \qquad \text{and} \qquad m_n p - lq$$

have the same sign for all n and all l. Thus using the absolute values of these numbers as the positive integers a and c, we get

$$\begin{aligned} a(1-\alpha^{-1}) + c\gamma^{-1} &= |(np-lr)(1-q/p) + (m_n p - lq)\, r/p| \\ &= |n(p-q) + (m_n - l)r| \,. \end{aligned}$$

For n fixed, $m_n - l$ can assume all integral values. It is therefore possible to determine n and l such that the right-hand side equals the largest common divisor of $p-q$ and r, that is 1. This proves the theorem.

In the rational case the integers a and c in (3) are not uniquely determined, and therefore we cannot deduce a theorem analogous to Theorem 10. For instance, $\alpha = 12/11$ and $\gamma = 12$ satisfy (3) for all positive integers a and c such that $a + c = 12$, and hence $N_\alpha \supseteqq N_\gamma$. The numbers ε for which $N_\alpha \supseteqq N_\varepsilon \supseteqq N_\gamma$ are $\varepsilon = 12/p$, $1 \leqq p \leqq 11$; the way in which they are included in each other is represented in the following graph (fig. 2), where a line from p to q (directed to the right) indicates that $N_{12/p} \supseteqq N_{12/q}$.

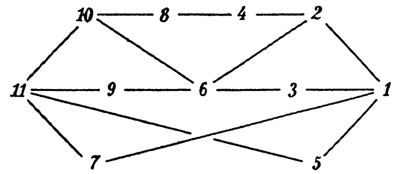

Fig. 2.

5. The asymptotic density δ of a set of positive integers M is defined as the limit of $\mu(h)/h$, where $\mu(h)$ denotes the number of elements from M not exceeding h. We have already remarked in Section 2 that $\delta(N_\alpha) = \alpha^{-1}$.

We shall now consider $\delta(N_\alpha \cap N_\beta)$. Following Skolem, we define, for a positive integer z, the number x_z as $z\alpha^{-1}$ reduced modulo 1, so that $0 < x_z \leqq 1$. In the same way y_z is defined as $z\beta^{-1}$ reduced modulo 1, so that $0 < y_z \leqq 1$. Then $z \in N_\alpha$ means that there exists an integer h such that $z \leqq h\alpha < z + 1$, that is

$$h - \alpha^{-1} < z\alpha^{-1} \leqq h, \qquad \text{or} \qquad 1 - \alpha^{-1} < x_z \leqq 1 .$$

Hence, $z \in N_\alpha \cap N_\beta$ means that the point (x_z, y_z) lies in the rectangle R in the XY-plane, defined by

$$1 - \alpha^{-1} < x \leqq 1, \qquad 1 - \beta^{-1} < y \leqq 1 .$$

On the other hand, each positive integer z yields a point (x_z, y_z) in the unit square S: $0 < x \leqq 1$, $0 < y \leqq 1$.

The points $(z\alpha^{-1} - h, z\beta^{-1} - k)$, where z, h and k are integers, form a vector modulus V, and the points (x_z, y_z) belong to $V \cap S$. Well-known theorems on vector modules state that the points (x_z, y_z) are equidistributed on the intersection between S and the closure $\overline{V}$ of V. Hence we have

$$\delta = \delta(N_\alpha \cap N_\beta) = \frac{m(\overline{V} \cap R)}{m(\overline{V} \cap S)} ,$$

where $m(A)$ is the measure of the point set A, this measure being defined in a proper way, depending on the nature of the closed vector modulus $\overline{V}$. There are three possibilities:

I. If α^{-1}, β^{-1} and 1 are rationally independent, then $\overline{V}$ is the whole plane. In this case $m(A)$ is the area of A, and we get

$$\delta(N_\alpha \cap N_\beta) = 1/\alpha\beta = \alpha^{-1}\beta^{-1},$$

that is, δ is the product of the densities of N_α and N_β.

II. If there exists one (and only one) relation

$$(5) \qquad a\alpha^{-1} + b\beta^{-1} = c,$$

where a, b and c are relatively prime integers, a and b not both equal to 0, then $\overline{V}$ is the collection of equidistant lines

$$ax + by = t; \qquad t = 0, \pm 1, \pm 2, \ldots .$$

In this case $m(A)$ is the total length of the line segments of $\overline{V} \cap A$.

In particular, if $b = 0$, that is, α rational, then $\overline{V}$ is a set of vertical lines, $x = t/a$ (t integral), and again we get $\delta = 1/\alpha\beta = \alpha^{-1}\beta^{-1}$, that is, the product of the densities of N_α and N_β. The same result is valid if β is rational.

In the general case of II, α and β are both irrational, and the lines have the slope $-a/b$; the figure shows the case where the slope is negative. Let the distance of two neighbouring lines be d. Then

$$\delta = \frac{d \cdot m(\overline{V} \cap R)}{d \cdot m(\overline{V} \cap S)},$$

and here the numerator is easily seen to be equal to the area spanned by the line segments inside R and the vertices $(1-\alpha^{-1}, 1-\beta^{-1})$ and $(1, 1)$ (the area hatched in fig. 3), while the denominator is seen in the same way to be the area of S, that is 1. A calculation yields

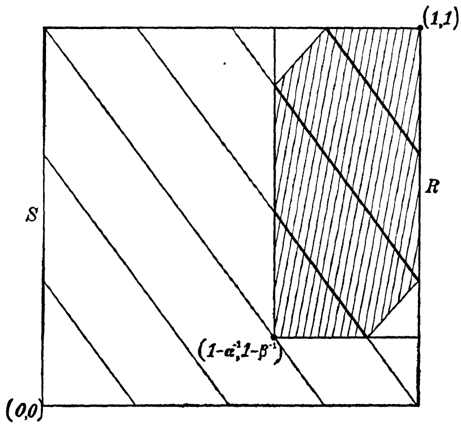

Fig. 3.

$$(6) \qquad \delta = \frac{1}{\alpha\beta} - \frac{1}{ab}\left(\frac{a}{\alpha} - \left[\frac{a}{\alpha}\right]\right)\left(\frac{b}{\beta} - \left[\frac{b}{\beta}\right]\right),$$

and this expression is also valid when the lines $\overline{V}$ have a positive slope. Since α and β are irrational, the second term of (6) does not vanish, and we have $\delta < 1/\alpha\beta$ or $\delta > 1/\alpha\beta$ according as a and b have the same sign or opposite signs.

Because of (5) and the irrationality of α and β, putting $\delta = 0$ in (6), we can again deduce the necessary condition, given in Theorem 8, that N_α and N_β be disjoint. By the same method, it follows that $N_\alpha \supseteqq N_\beta$ only if $\delta = \beta^{-1}$, and a calculation gives again condition (3).

III. If there is more than one relation of the form (5), then α and β are rational. The modulus V consists of isolated points, and $\overline{V} = V$; the measure $m(A)$ is obtained by counting the points in $V \cap A$.

In this case we cannot give a simple formula for δ. In some examples the expression (6) seems to be related to the resulting δ (the identity of the conditions in Theorems 9 and 11 points in the same direction), but the numbers a and b in (5) are not uniquely determined in this case, and it seems difficult to find a general rule.

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