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SOME IDENTITIES INVOLVING THE PARTITION FUNCTION

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1. Introduction. In a paper concerning certain arithmetical properties of $p(n)$, the number of unrestricted partitions of n, Ramanujan [13] stated without proof the identities

$$\sum_{n=0}^{\infty} p(5n+4)x^n = 5\,\frac{\{(1-x^5)(1-x^{10})(1-x^{15})\ldots\}^5}{\{(1-x)(1-x^2)(1-x^3)\ldots\}^6},$$

$$\sum_{n=0}^{\infty} p(7n+5)x^n = 7\,\frac{\{(1-x^7)(1-x^{14})\ldots\}^3}{\{(1-x)(1-x^2)\ldots\}^4} + 49x\,\frac{\{(1-x^7)(1-x^{14})\ldots\}^7}{\{(1-x)(1-x^2)\ldots\}^8}.$$

The first proofs were given by Darling [4] and Mordell [9] respectively. See also Watson [14], Rademacher and Zuckerman [12], Rademacher [11], Kruyswijk [7], Bailey [2] [3], Newman [10] and Fine [5].

Other identities involving the partition function have been found by Watson [14], Zuckerman [15], Rademacher [11], Lehner [8] and Fine [5]. Atkin and Swinnerton-Dyer [1] have deduced several remarkable congruences for the moduli 5, 7 and 11.

In the following we shall prove some identities which seem to be new. We also give new proofs of the Ramanujan identities and four of the congruences of Atkin and Swinnerton-Dyer.

2. Definitions and lemmas. We use the notation

$$\varphi(x) = \prod_{n=1}^{\infty} (1-x^n)\,.$$

Then we have the well-known identities (with $p(0)=1$)

$$\varphi(x)^{-1} = \sum_{n=0}^{\infty} p(n)x^n \qquad \text{(Euler)}\,, \tag{2.1}$$

$$\varphi(x) = \sum_{n=-\infty}^{\infty} (-1)^n x^{\frac{1}{2}n(3n+1)} \qquad \text{(Euler)}\,, \tag{2.2}$$

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$$(2.3)\qquad \varphi(x)^3 = \sum_{n=0}^{\infty} (-1)^n (2n+1) x^{\frac{1}{2}n(n+1)} \qquad \text{(Jacobi)}.$$

In the following q always denotes a prime. Now let q be given. We then define, for $s=0, 1, \ldots, q-1$:

$$(2.4)\qquad g_s = \sum_{\frac{1}{2}n(3n+1)\equiv s \pmod q} (-1)^n x^{\frac{1}{2}n(3n+1)},$$

$$(2.5)\qquad h_s = \sum_{\substack{\frac{1}{2}n(n+1)\equiv s \pmod q \\ n\geqq 0}} (-1)^n (2n+1) x^{\frac{1}{2}n(n+1)},$$

that is, we split up the power series into q parts, such that each part consists of those terms whose exponents belong to the same residue class $(\bmod q)$.

LEMMA 1. *If* $24s+1$ *is a quadratic non-residue* $(\bmod q)$, *then* $g_s=0$. *If* $24s+1\equiv 0 \pmod q$, *then*

$$g_s = (-1)^{[\frac{1}{6}(q+1)]} x^{\frac{1}{24}(q^2-1)} \varphi(x^{q^2}).$$

PROOF. We have

$$24\cdot\tfrac{1}{2}n(3n+1) + 1 = (6n+1)^2.$$

Hence, if $24s+1$ is a quadratic non-residue $(\bmod q)$, we have for all n

$$\tfrac{1}{2}n(3n+1) \not\equiv s \pmod q,$$

and therefore $g_s=0$.

Now let $24s+1\equiv 0 \pmod q$. Then g_s consists of those terms in (2.2) where

$$(2.6)\qquad 24\cdot\tfrac{1}{2}n(3n+1) + 1 \equiv 0 \pmod q,$$

that is

$$6n+1 \equiv 0 \pmod q.$$

Obviously, $q>3$. Let

$$n_0 = \tfrac{1}{6}(-1\pm q),$$

where the upper sign is used when $q\equiv 1 \pmod 6$, the lower sign when $q\equiv -1 \pmod 6$. Now the general solution of (2.6) can be written

$$n = qk+n_0, \qquad k = 0, \pm 1, \pm 2, \ldots.$$

Thus we have

$$\begin{aligned} g_s &= \sum_{k=-\infty}^{\infty} (-1)^{qk+n_0} x^{\frac{1}{2}(qk+n_0)(3qk+3n_0+1)} \\ &= (-1)^{n_0} x^{\frac{1}{2}n_0(3n_0+1)} \sum_{k=-\infty}^{\infty} (-1)^k x^{\frac{1}{2}k(3k\pm 1)q^2} \\ &= (-1)^{[\frac{1}{6}(q+1)]} x^{\frac{1}{24}(q^2-1)} \varphi(x^{q^2}). \end{aligned}$$

LEMMA 2. *If* $8s+1$ *is a quadratic non-residue* (mod q), *then* $h_s=0$. *If* $8s+1\equiv 0 \pmod q$, *then*

$$h_s = (-1)^{\frac{1}{2}(q-1)} q x^{\frac{1}{8}(q^2-1)} \varphi\left(x^{q^2}\right)^3 .$$

PROOF. The first part of the lemma follows immediately from the identity

$$8\cdot\tfrac{1}{2}n(n+1) + 1 = (2n+1)^2 .$$

Let $8s+1\equiv 0 \pmod q$. Then we have to solve the congruence

$$8\cdot\tfrac{1}{2}n(n+1) + 1 \equiv 0 \pmod q ,$$

where now $n \geqq 0$. (Obviously, $q>2$). The solution is

$$n = qk + \tfrac{1}{2}(q-1), \qquad k = 0, 1, 2, \ldots .$$

Hence

$$\begin{aligned} h_s &= \sum_{k=0}^{\infty} (-1)^{qk+\frac{1}{2}(q-1)} (2qk+q) x^{\frac{1}{2}\{qk+\frac{1}{2}(q-1)\}\{qk+\frac{1}{2}(q+1)\}} \\ &= (-1)^{\frac{1}{2}(q-1)} q x^{\frac{1}{8}(q^2-1)} \sum_{k=0}^{\infty} (-1)^k (2k+1) x^{\frac{1}{2}k(k+1)q^2} \\ &= (-1)^{\frac{1}{2}(q-1)} q x^{\frac{1}{8}(q^2-1)} \varphi\left(x^{q^2}\right)^3 . \end{aligned}$$

We define

$$D = \begin{vmatrix} g_0 & g_1 & \cdots & g_{q-1} \\ g_{q-1} & g_0 & \cdots & g_{q-2} \\ \cdots & \cdots & \cdots & \cdots \\ g_1 & g_2 & \cdots & g_0 \end{vmatrix} ,$$

$$D_s = \begin{vmatrix} g_{-s} & g_{-s+1} & \cdots & g_{-s+q-2} \\ g_{-s-1} & g_{-s} & \cdots & g_{-s+q-3} \\ \cdots & \cdots & \cdots & \cdots \\ g_{-s-q+2} & g_{-s-q+3} & \cdots & g_{-s} \end{vmatrix} , \quad s = 0, 1, \ldots, q-1 ,$$

where we have to put $g_r=g_s$ when $r\equiv s \pmod q$.

LEMMA 3. *We have*

$$D = \frac{\varphi(x^q)^{q+1}}{\varphi\left(x^{q^2}\right)} .$$

PROOF. Let $\omega=e^{2\pi i/q}$. Then, by a well-known theorem we have

$$D = \prod_{k=0}^{q-1} \left(g_0+\omega^k g_1+\ldots+\omega^{(q-1)k} g_{q-1}\right) . \tag{2.7}$$

But from the definition of $g_s = g_s(x)$ it follows that

$$\omega^{sk} g_s(x) = g_s(\omega^k x) ,$$

and hence

$$\begin{aligned} g_0 + \omega^k g_1 + \ldots + \omega^{(q-1)k} g_{q-1} \\ = g_0(\omega^k x) + g_1(\omega^k x) + \ldots + g_{q-1}(\omega^k x) = \varphi(\omega^k x) . \end{aligned}$$

Now, returning to (2.7) we get

$$\begin{aligned} D &= \prod_{k=0}^{q-1} \varphi(\omega^k x) \\ &= \prod_{n=1}^{\infty} \prod_{k=0}^{q-1} (1 - \omega^{kn} x^n) \\ &= \prod_{n \not\equiv 0 \ (\mathrm{mod}\, q)} (1 - x^{qn}) \prod_{n \equiv 0 \ (\mathrm{mod}\, q)} (1 - x^n)^q , \end{aligned}$$

and lemma 3 follows.

LEMMA 4. *We have, for* $s = 0, 1, \ldots, q-1$,

$$\sum_{n=0}^{\infty} p(qn+s) x^{qn+s} = (-1)^{(q-1)s} \frac{\varphi(x^{q^2})}{\varphi(x^q)^{q+1}} D_s .$$

PROOF. Let

$$P_s = \sum_{n=0}^{\infty} p(qn+s) x^{qn+s} .$$

Now we have

$$\begin{aligned} P_0 + P_1 + \ldots + P_{q-1} &= \varphi(x)^{-1} , \\ g_0 + g_1 + \ldots + g_{q-1} &= \varphi(x) . \end{aligned}$$

By multiplication we obtain

$$\begin{aligned} g_0 P_0 + g_{q-1} P_1 + \ldots + g_1 P_{q-1} &= 1 , \\ g_1 P_0 + g_0 P_1 + \ldots + g_2 P_{q-1} &= 0 , \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ g_{q-1} P_0 + g_{q-2} P_1 + \ldots + g_0 P_{q-1} &= 0 . \end{aligned}$$

From this we easily find

$$P_s = (-1)^{(q-s)s} D^{-1} D_s ,$$

and lemma 4 follows.

3. The case $q=3$. We shall prove the identities

$$(3.1)\qquad \sum_{n=0}^{\infty} p(3n)x^n = \frac{\varphi(x^3)\,\varphi(x^9)^2}{\varphi(x)^4} \prod_{m=1}^{\infty} \{(1-x^{9m-5})(1-x^{9m-4})\}^2 - x\,\frac{\varphi(x^9)^2}{\varphi(x)^3} \prod_{m=1}^{\infty} \{(1-x^{9m-5})(1-x^{9m-4})\}^{-1},$$

$$(3.2)\qquad \sum_{n=0}^{\infty} p(3n+1)x^n = \frac{\varphi(x^9)^2}{\varphi(x)^3} \prod_{m=1}^{\infty} \{(1-x^{9m-8})(1-x^{9m-1})\}^{-1} + x\,\frac{\varphi(x^3)\,\varphi(x^9)^2}{\varphi(x)^4} \prod_{m=1}^{\infty} \{(1-x^{9m-8})(1-x^{9m-1})\}^2,$$

$$(3.3)\qquad \sum_{n=0}^{\infty} p(3n+2)x^n = \frac{\varphi(x^9)^2}{\varphi(x)^3} \prod_{m=1}^{\infty} \{(1-x^{9m-7})(1-x^{9m-2})\}^{-1} + \frac{\varphi(x^3)\,\varphi(x^9)^2}{\varphi(x)^4} \prod_{m=1}^{\infty} \{(1-x^{9m-7})(1-x^{9m-2})\}^2,$$

$$(3.4)\qquad \sum_{n=0}^{\infty} p(3n)x^n \sum_{n=0}^{\infty} p(3n+1)x^n \sum_{n=0}^{\infty} p(3n+2)x^n = 2\,\frac{\varphi(x^3)\,\varphi(x^9)^3}{\varphi(x)^7} + 9x\,\frac{\varphi(x^3)\,\varphi(x^9)^6}{\varphi(x)^{10}}.$$

Proofs. Lemma 4 yields

$$(3.5)\qquad \sum_{n=0}^{\infty} p(3n+s)x^{3n+s} = \frac{\varphi(x^9)}{\varphi(x^3)^4} D_s,$$

where

$$(3.6)\qquad \begin{cases} D_0 = g_0^2 - g_1 g_2, \\ D_1 = g_2^2 - g_0 g_1, \\ D_2 = g_1^2 - g_2 g_0. \end{cases}$$

In this case the conditions of lemma 1 are not fulfilled, but we can find expressions for g_0, g_1 and g_2 by using the following well-known identity of Jacobi:

$$(3.7)\qquad \sum_{k=-\infty}^{\infty} y^k z^{k^2} = \varphi(z^2) \prod_{m=1}^{\infty} \{(1+yz^{2m-1})(1+y^{-1}z^{2m-1})\}.$$

Now, the congruence $\frac{1}{2}n(3n+1)\equiv 0 \pmod 3$ has the solution $n=3k$, $k=0, \pm 1, \pm 2, \ldots$. Hence

$$g_0 = \sum_{k=-\infty}^{\infty} (-1)^{3k} x^{\frac{1}{2}\cdot 3k(9k+1)} = \sum_{k=-\infty}^{\infty} (-1)^k x^{\frac{3}{2}k+\frac{27}{2}k^2} .$$

Putting $y=-x^{3/2}$, $z=x^{27/2}$, we get from (3.7)

$$g_0 = \varphi(x^{27}) \prod_{m=1}^{\infty} \{(1-x^{27m-15})(1-x^{27m-12})\} .$$

Similarly we find

$$g_1 = -x\varphi(x^{27}) \prod_{m=1}^{\infty} \{(1-x^{27m-21})(1-x^{27m-6})\} ,$$

$$g_2 = -x^2\varphi(x^{27}) \prod_{m=1}^{\infty} \{(1-x^{27m-24})(1-x^{27m-3})\} .$$

Further, by multiplication

$$g_0 g_1 g_2 = x^3 \frac{\varphi(x^3)\varphi(x^{27})^3}{\varphi(x^9)} . \tag{3.8}$$

Now we have

$$D_0 = g_0^2 - \frac{g_0 g_1 g_2}{g_0} = \varphi(x^{27})^2 \prod_{m=1}^{\infty} \{(1-x^{27m-15})(1-x^{27m-12})\}^2 - $$
$$- x^3 \frac{\varphi(x^3)\varphi(x^{27})^2}{\varphi(x^9)} \prod_{m=1}^{\infty} \{(1-x^{27m-15})(1-x^{27m-12})\}^{-1} ,$$

and (3.1) follows from (3.5). (3.2) and (3.3) are deduced similarly.

It remains to prove (3.4). From (3.6) we get

$$D_0 D_1 D_2 = g_0 g_1 g_2 (g_0^3+g_1^3+g_2^3) - (g_0^3 g_1^3 + g_1^3 g_2^3 + g_2^3 g_0^3) . \tag{3.9}$$

Now, by lemma 3

$$g_0^3+g_1^3+g_2^3-3g_0 g_1 g_2 = \frac{\varphi(x^3)^4}{\varphi(x^9)} . \tag{3.10}$$

From lemma 2 we find

$$h_1 = -3x\varphi(x^9)^3, \qquad h_2 = 0 .$$

We have

$$\begin{aligned} \varphi(x)^3 &= (g_0+g_1+g_2)^3 \\ &= g_0^3+g_1^3+g_2^3+6g_0 g_1 g_2 + \\ &\quad + 3(g_0^2 g_1 + g_1^2 g_2 + g_2^2 g_0) + \\ &\quad + 3(g_0 g_1^2 + g_1 g_2^2 + g_2 g_0^2) . \end{aligned}$$

Hence, remembering the definitions of g_s and h_s, we conclude

$$g_0^2 g_1 + g_1^2 g_2 + g_2^2 g_0 = -x\varphi(x^9)^3 , \tag{3.11}$$

$$g_0 g_1^2 + g_1 g_2^2 + g_2 g_0^2 = 0 . \tag{3.12}$$

Multiplication of (3.11) and (3.12) yields

$$0 = g_0 g_1 g_2 (g_0^3 + g_1^3 + g_2^3) + 3 g_0^2 g_1^2 g_2^2 + g_0^3 g_1^3 + g_1^3 g_2^3 + g_2^3 g_0^3 . \tag{3.13}$$

We then get, by adding (3.9) and (3.13)

$$\begin{aligned} D_0 D_1 D_2 &= 2 g_0 g_1 g_2 (g_0^3 + g_1^3 + g_2^3) + 3 g_0^2 g_1^2 g_2^2 \\ &= 2 g_0 g_1 g_2 (g_0^3 + g_1^3 + g_2^3 - 3 g_0 g_1 g_2) + 9 (g_0 g_1 g_2)^2 \\ &= 2x^3 \frac{\varphi(x^3)^5 \varphi(x^{27})^3}{\varphi(x^9)^2} + 9x^6 \frac{\varphi(x^3)^2 \varphi(x^{27})^6}{\varphi(x^9)^2} . \end{aligned}$$

Here we have made use of (3.8) and (3.10). Now (3.4) follows by (3.5).

4. The case $q=5$. We shall prove the identities

$$\sum_{n=0}^{\infty} p(5n+4) x^n = 5 \frac{\varphi(x^5)^5}{\varphi(x)^6} \qquad \text{(Ramanujan)}, \tag{4.1}$$

$$\sum_{n=0}^{\infty} p(5n+1) x^n \sum_{n=0}^{\infty} p(5n+2) x^n = 2 \frac{\varphi(x^5)^4}{\varphi(x)^6} + 25x \frac{\varphi(x^5)^{10}}{\varphi(x)^{12}} , \tag{4.2}$$

$$\sum_{n=0}^{\infty} p(5n) x^n \sum_{n=0}^{\infty} p(5n+3) x^n = 3 \frac{\varphi(x^5)^4}{\varphi(x)^6} + 25x \frac{\varphi(x^5)^{10}}{\varphi(x)^{12}} , \tag{4.3}$$

$$3 \sum_{n=0}^{\infty} p(5n+1) x^n \sum_{n=0}^{\infty} p(5n+2) x^n - 2 \sum_{n=0}^{\infty} p(5n) x^n \sum_{n=0}^{\infty} p(5n+3) x^n = x \left(\sum_{n=0}^{\infty} p(5n+4) x^n \right)^2 , \tag{4.4}$$

$$\begin{aligned} &\sum_{n=0}^{\infty} \{ p(5n+1) x^{5n} + p(5n+2) x^{5n+1} \} \\ &= \frac{1}{\varphi(x^5)^6} \{ \varphi(x)^3 \varphi(x^{25})^2 + 5x\varphi(x)^2 \varphi(x^{25})^3 + 10x^2 \varphi(x) \varphi(x^{25})^4 + 10x^3 \varphi(x^{25})^5 \} , \end{aligned} \tag{4.5}$$

$$\begin{aligned} &\sum_{n=0}^{\infty} \{ p(5n) x^{5n} + p(5n+3) x^{5n+3} \} \\ &= \frac{1}{\varphi(x^5)^6} \{ \varphi(x)^4 \varphi(x^{25}) + 4x\varphi(x)^3 \varphi(x^{25})^2 + 10x^2 \varphi(x)^2 \varphi(x^{25})^3 + 15x^3 \varphi(x) \varphi(x^{25})^4 + 10x^4 \varphi(x^{25})^5 \} . \end{aligned} \tag{4.6}$$

We shall also deduce four of the congruences of Atkin and Swinnerton-Dyer, viz.

$$(4.7)\quad \sum_{n=0}^{\infty} p(5n)x^n \equiv \varphi(x)\prod_{m=1}^{\infty}\{(1-x^{5m-4})(1-x^{5m-1})\}^{-3} \pmod 5,$$

$$(4.8)\quad \sum_{n=0}^{\infty} p(5n+1)x^n \equiv \varphi(x^5)\prod_{m=1}^{\infty}\{(1-x^{5m-4})(1-x^{5m-1})\}^{-1} \pmod 5,$$

$$(4.9)\quad \sum_{n=0}^{\infty} p(5n+2)x^n \equiv 2\varphi(x^5)\prod_{m=1}^{\infty}\{(1-x^{5m-3})(1-x^{5m-2})\}^{-1} \pmod 5,$$

$$(4.10)\quad \sum_{n=0}^{\infty} p(5n+3)x^n \equiv 3\varphi(x)\prod_{m=1}^{\infty}\{(1-x^{5m-3})(1-x^{5m-2})\}^{-3} \pmod 5.$$

PROOFS. By lemma 1

$$(4.11)\qquad g_1 = -x\varphi(x^{25}), \qquad g_3 = g_4 = 0.$$

From

$$\varphi(x)^3 = (g_0+g_1+g_2)^3$$

we find

$$h_2 = 3g_0^2g_2+3g_1^2g_0.$$

But $h_2=0$, by lemma 2. Hence

$$(4.12)\qquad g_0g_2+g_1^2 = 0.$$

It is convenient to use the following notation:

$$\alpha = g_1^{-1}g_0, \quad \beta = g_1^{-1}g_2, \quad A = g_1^{-5}D, \quad A_s = g_1^{-4}D_s.$$

From (4.12) we now get

$$(4.13)\qquad \alpha\beta = -1,$$

and by (4.11)

$$(4.14)\qquad \alpha+\beta = -\frac{\varphi(x)}{x\varphi(x^{25})} - 1.$$

Further, using (4.13) we find

$$A = \begin{vmatrix} \alpha & 1 & \beta & 0 & 0 \\ 0 & \alpha & 1 & \beta & 0 \\ 0 & 0 & \alpha & 1 & \beta \\ \beta & 0 & 0 & \alpha & 1 \\ 1 & \beta & 0 & 0 & \alpha \end{vmatrix} = \alpha^5+\beta^5+11.$$

Hence, by lemma 3

$$\alpha^5+\beta^5 = -\frac{\varphi(x^5)^6}{x^5\,\varphi(x^{25})^6} - 11\,. \tag{4.15}$$

From lemma 4 we obtain

$$\sum_{n=0}^{\infty} p(5n+s)x^{5n+s} = x^4\frac{\varphi(x^{25})^5}{\varphi(x^5)^6}A_s\,. \tag{4.16}$$

We then evaluate the determinants A_s and find

$$A_0 = \alpha^4-3\beta\,, \tag{4.17}$$

$$A_1 = -a^3+2\beta^2\,, \tag{4.18}$$

$$A_2 = 2\alpha^2-\beta^3\,, \tag{4.19}$$

$$A_3 = -3\alpha+\beta^4\,, \tag{4.20}$$

$$A_4 = 5\,. \tag{4.21}$$

We can now easily prove the identities (4.1)–(4.6):
(4.1) follows immediately from (4.16) and (4.21).
Multiplication of (4.18) and (4.19) yields

$$A_1A_2 = -2(\alpha^5+\beta^5)+3\,.$$

Hence, by (4.15)

$$A_1A_2 = 2\frac{\varphi(x^5)^6}{x^5\varphi(x^{25})^6}+25\,,$$

and (4.2) follows by (4.16). The proof of (4.3) is similar.
(4.4) follows directly from (4.1), (4.2) and (4.3).
Further

$$\begin{aligned} A_1+A_2 &= -(\alpha^3+\beta^3)+2(\alpha^2+\beta^2) \\ &= -(\alpha+\beta)^3+2(\alpha+\beta)^2-3(\alpha+\beta)+4\,. \end{aligned}$$

We insert for $\alpha+\beta$ from (4.14) and evaluate. Then (4.5) follows by (4.16).
Similarly

$$\begin{aligned} A_0+A_3 &= \alpha^4+\beta^4-3(\alpha+\beta) \\ &= (\alpha+\beta)^4+4(\alpha+\beta)^2-3(\alpha+\beta)+2\,, \end{aligned}$$

and (4.6) follows in the same way.
We shall then prove the congruences (4.7)–(4.10). From (4.5) we get

$$\sum_{n=0}^{\infty} p(5n+1)x^{5n} \equiv \frac{\varphi(x^{25})^2}{\varphi(x^5)^6}h_0 \pmod 5\,. \tag{4.22}$$

Now

$$h_0 = \sum_{k=0}^{\infty} (-1)^{5k}(10k+1)x^{\frac{1}{2}\cdot 5k(5k+1)} + \sum_{k=1}^{\infty} (-1)^{5k-1}(10k-1)x^{\frac{1}{2}(5k-1)5k}$$

$$\equiv \sum_{k=-\infty}^{\infty} (-1)^k x^{\frac{1}{2}\cdot 5k(5k+1)} \pmod 5 .$$

We now make use of (3.7). Putting $y = -x^{5/2}$, $z = x^{25/2}$ we get

$$h_0 \equiv \varphi(x^{25}) \prod_{m=1}^{\infty} \{(1-x^{25m-15})(1-x^{25m-10})\} \pmod 5 .$$

This can also be written

$$h_0 \equiv \varphi(x^5) \prod_{m=1}^{\infty} \{1-x^{25m-20})(1-x^{25m-5})\}^{-1} \pmod 5 .$$

Hence, by (4.22)

$$\sum_{n=0}^{\infty} p(5n+1)x^n \equiv \frac{\varphi(x^5)^2}{\varphi(x)^5} \prod_{m=1}^{\infty} \{(1-x^{5m-4})(1-x^{5m-1})\}^{-1} \pmod 5 .$$

But $\varphi(x)^5 \equiv \varphi(x^5) \pmod 5$, and (4.8) follows.

The remaining three congruences are now easily proved:

(4.9) follows directly from (4.2) and (4.8).

Further, by (4.17), (4.18) and (4.19)

$$A_0A_2 - 2A_1^2 = 15\alpha - 5\beta^4 .$$

Hence

$$A_0 \equiv 2A_1^2 A_2^{-1} \pmod 5 ,$$

and we get by (4.16)

$$\sum_{n=0}^{\infty} p(5n)x^n \equiv 2\left(\sum_{n=0}^{\infty} p(5n+1)x^n\right)^2 \left(\sum_{n=0}^{\infty} p(5n+2)x^n\right)^{-1} \pmod 5 .$$

Now (4.7) follows by (4.8) and (4.9).

(4.10) follows from (4.3) and (4.7).

Remark. Several relations involving the quantities A_s can be established, which by (4.16) would yield other identities. For instance

$$\begin{aligned} A_0A_2 + A_3A_4 &= 2A_1^2 , \\ A_1A_3 + A_0A_4 &= 2A_2^2 , \\ A_0^2A_2 + A_3^2A_1 &= 2(\alpha^5+\beta^5)^2 + 4(\alpha^5+\beta^5) + 52 , \\ A_1^2A_0 + A_2^2A_3 &= (\alpha^5+\beta^5)^2 - 13(\alpha^5+\beta^5) - 14 . \end{aligned}$$

We also mention that elimination of α and β from the equations (4.13), (4.14) and (4.15) would give us the Ramanujan-Watson modular equation of the 5th order (see Watson [14]).

5. The case $q=7$. We have the identities

$$(5.1)\qquad \sum_{n=0}^{\infty} p(7n+5)x^n = 7\,\frac{\varphi(x^7)^3}{\varphi(x)^4} + 49x\,\frac{\varphi(x^7)^7}{\varphi(x)^8} \qquad \text{(Ramanujan)},$$

$$(5.2)\qquad \sum_{n=0}^{\infty} p(7n+1)x^n \sum_{n=0}^{\infty} p(7n+3)x^n \sum_{n=0}^{\infty} p(7n+4)x^n$$
$$= 15\,\frac{\varphi(x^7)^5}{\varphi(x)^8} + 12\cdot 7^2 x\,\frac{\varphi(x^7)^9}{\varphi(x)^{12}} + 24\cdot 7^3 x^2\,\frac{\varphi(x^7)^{13}}{\varphi(x)^{16}} +$$
$$+ 3\cdot 7^5 x^3\,\frac{\varphi(x^7)^{17}}{\varphi(x)^{20}} + 7^6 x^4\,\frac{\varphi(x^7)^{21}}{\varphi(x)^{24}},$$

$$(5.3)\qquad \sum_{n=0}^{\infty} p(7n)x^n \sum_{n=0}^{\infty} p(7n+2)x^n \sum_{n=0}^{\infty} p(7n+6)x^n$$
$$= 22\,\frac{\varphi(x^7)^5}{\varphi(x)^8} + 2\cdot 7^3 x\,\frac{\varphi(x^7)^9}{\varphi(x)^{12}} + 25\cdot 7^3 x^2\,\frac{\varphi(x^7)^{13}}{\varphi(x)^{16}} +$$
$$+ 3\cdot 7^5 x^3\,\frac{\varphi(x^7)^{17}}{\varphi(x)^{20}} + 7^6 x^4\,\frac{\varphi(x^7)^{21}}{\varphi(x)^{24}}.$$

PROOFS. The lemmas 1 and 2 give

$$g_2 = -x^2\varphi(x^{49}), \qquad g_3 = g_4 = g_6 = 0,$$
$$h_6 = -7x^6\varphi(x^{49})^3, \qquad h_2 = h_4 = h_5 = 0.$$

From

$$\varphi(x)^3 = (g_0+g_1+g_2+g_5)^3$$

we find

$$(5.4)\qquad 3(g_0^2g_2+g_1^2g_0+g_2^2g_5) = h_2 = 0,$$
$$(5.5)\qquad 3(g_1^2g_2+g_2^2g_0+g_5^2g_1) = h_4 = 0,$$
$$(5.6)\qquad 3(g_0^2g_5+g_2^2g_1+g_5^2g_2) = h_5 = 0,$$
$$(5.7)\qquad g_2^3+6g_0g_1g_5 = h_6 = 7g_2^3.$$

We put

$$\alpha = g_2^{-1}g_0, \quad \beta = g_2^{-1}g_1, \quad \gamma = g_2^{-1}g_5, \quad A = g_2^{-7}D, \quad A_s = g_2^{-6}D_s.$$

Now, from the equations (5.4)–(5.7) we obtain

$$(5.8)\qquad \alpha\beta^2+\alpha^2+\gamma = 0, \quad \beta\gamma^2+\beta^2+\alpha = 0, \quad \gamma\alpha^2+\gamma^2+\beta = 0,$$

$$(5.9)\qquad \alpha\beta\gamma = 1.$$

By lemma 3

$$(5.10)\qquad A = -\frac{\varphi(x^7)^8}{x^{14}\varphi(x^{49})^8}.$$

On the other hand, using (5.9), we find by evaluation of the determinant

$$(5.11)\; A = \alpha^7+\beta^7+\gamma^7 - 7(\alpha\beta^5+\beta\gamma^5+\gamma\alpha^5) + 14(\alpha^2\beta^3+\beta^2\gamma^3+\gamma^2\alpha^3) + 8 .$$

It is now convenient to introduce quantities y_s defined by

$$(5.12) \qquad y_1 = \alpha^3\beta, \quad y_2 = \beta^3\gamma, \quad y_3 = \gamma^3\alpha .$$

Then, by (5.8) and (5.9) we easily find

$$(5.13) \qquad y_1y_2 = -y_1-1, \quad y_2y_3 = -y_2-1, \quad y_3y_1 = -y_3-1 ,$$

$$(5.14) \qquad y_1y_2y_3 = 1 ,$$

$$(5.15) \quad \alpha^2\beta^3 = -y_1-1, \qquad \beta^2\gamma^3 = -y_2-1, \qquad \gamma^2\alpha^3 = -y_3-1 ,$$

$$(5.16) \quad \alpha\beta^5 = y_1-y_2+1, \qquad \beta\gamma^5 = y_2-y_3+1, \qquad \gamma\alpha^5 = y_3-y_1+1 ,$$

$$(5.17) \quad \alpha^7 = -y_1^2+y_1-y_3-1, \; \beta^7 = -y_2^2+y_2-y_1-1, \; \gamma^7 = -y_3^2+y_3-y_2-1 .$$

Now, returning to (5.11) we get

$$\begin{aligned} A &= -(y_1^2+y_2^2+y_3^2) - 14(y_1+y_2+y_3) - 58 \\ &= -(y_1+y_2+y_3)^2 - 16(y_1+y_2+y_3) - 64 \\ &= -(y_1+y_2+y_3+8)^2 . \end{aligned}$$

Hence, by (5.10)

$$(5.18) \qquad y_1+y_2+y_3+8 = \pm \frac{\varphi(x^7)^4}{x^7\varphi(x^{49})^4} .$$

Considering the first term in the expansions of α, β and γ in powers of x, we find

$$y_1 = -x^{-7}+\ldots, \qquad y_2 = -1+\ldots, \qquad y_3 = x^7+\ldots .$$

We therefore have to take the sign $-$ in (5.18). With the abbreviation

$$(5.19) \qquad T = \frac{\varphi(x^7)^4}{x^7\varphi(x^{49})^4}$$

we thus get

$$(5.20) \qquad y_1+y_2+y_3 = -T-8 .$$

Then, by (5.13)

$$(5.21) \qquad y_1y_2+y_2y_3+y_3y_1 = T+5 .$$

Together with (5.14) this enables us to express in terms of T any symmetric polynomial in y_1, y_2 and y_3.

From lemma 4 we get

$$(5.22)\qquad \sum_{n=0}^{\infty} p(7n+s)x^{7n+s} = x^{12}\frac{\varphi(x^{49})^7}{\varphi(x^7)^8} A_s .$$

Evaluation of the determinants A_s gives

$$(5.23)\quad A_0 = \alpha^6+\beta^4\gamma^2-\beta^5-5\gamma\alpha^4+4\alpha\beta^3+6\gamma^2\alpha^2+\gamma^3+\alpha^2\beta ,$$

$$(5.24)\quad A_1 = -\alpha^5\beta+\beta^2\gamma^4-\gamma^3\alpha^3+2\gamma^4\alpha-3\beta^3\gamma^2+\beta^4-6\alpha\beta^2+2\alpha^2-3\gamma ,$$

$$(5.25)\quad A_2 = \gamma^6+\alpha^4\beta^2-\alpha^5-5\beta\gamma^4+4\gamma\alpha^3+6\beta^2\gamma^2+\beta^3+\gamma^2\alpha ,$$

$$(5.26)\quad A_3 = -\beta^5\gamma+\gamma^2\alpha^4-\alpha^3\beta^3+2\alpha^4\beta-3\gamma^3\alpha^2+\gamma^4-6\beta\gamma^2+2\beta^2-3\alpha ,$$

$$(5.27)\quad A_4 = -\gamma^5\alpha+\alpha^2\beta^4-\beta^3\gamma^3+2\beta^4\gamma-3\alpha^3\beta^2+\alpha^4-6\gamma\alpha^2+2\gamma^2-3\beta ,$$

$$(5.28)\quad A_5 = -(\alpha\beta^5+\beta\gamma^5+\gamma\alpha^5)+4(\alpha^2\beta^3+\beta^2\gamma^3+\gamma^2\alpha^3) - 3(\alpha^3\beta+\beta^3\gamma+\gamma^3\alpha)+8 ,$$

$$(5.29)\quad A_6 = \beta^6+\gamma^4\alpha^2-\gamma^5-5\alpha\beta^4+4\beta\gamma^3+6\alpha^2\beta^2+\alpha^3+\beta^2\gamma .$$

We can now prove the Ramanujan identity (5.1): From (5.28) we get by (5.12), (5.15) and (5.16)

$$A_5 = -7(y_1+y_2+y_3) - 7 .$$

Hence, by (5.19) and (5.20)

$$A_5 = 7\frac{\varphi(x^7)^4}{x^7\varphi(x^{49})^4} + 49 ,$$

and (5.1) follows from (5.22).

The proof of (5.2) is more complicated. From (5.24) we find

$$\beta^3 A_1 = y_1{}^2-5y_2{}^2-8y_1+y_2-12 ,$$

and then, by (5.13),

$$y_1{}^2\beta^3 A_1 = y_1{}^4-8y_1{}^3-18y_1{}^2-11y_1-5 .$$

Denoting this polynomial by Q we have

$$y_1{}^2\beta^3 A_1 = Q(y_1) .$$

In the same way we find

$$y_2{}^2\gamma^3 A_3 = Q(y_2) ,$$

$$y_3{}^2\alpha^3 A_4 = Q(y_3) .$$

Now, by multiplication we get

$$A_1A_3A_4 = Q(y_1)Q(y_2)Q(y_3) ,$$

that is, a polynomial of degree 12, symmetrical in y_1, y_2 and y_3. From

(5.20), (5.21) and (5.14) we then conclude that $A_1A_3A_4$ is expressible as a polynomial in T, of degree 12 at most:

$$(5.30) \qquad A_1A_3A_4 = \sum_{j=0}^{12} a_j T^j .$$

The coefficients a_j could of course be computed directly, but the following method is simpler. From (5.22) and (5.30) we get

$$(5.31) \quad \sum_{n=0}^{\infty} p(7n+1)x^n \sum_{n=0}^{\infty} p(7n+3)x^n \sum_{n=0}^{\infty} p(7n+4)x^n = \sum_{j=0}^{12} a_j x^{4-j} \frac{\varphi(x^7)^{21-4j}}{\varphi(x)^{24-4j}} .$$

We thus immediately conclude that

$$a_{12} = a_{11} = \ldots = a_5 = 0 .$$

The remaining five a_j's can now be found by evaluating the first five terms in the power series expansions of the two sides of (5.31), and comparing the coefficients. Thus (5.2) is established.

The proof of (5.3) is quite analogous.

6. The case $q=2$. For the sake of completeness we add the following identities with $q=2$, although they are rather trivial:

$$(6.1) \qquad \sum_{n=0}^{\infty} p(2n)x^n$$

$$= \frac{\varphi(x^2)\varphi(x^{24})}{\varphi(x)^3} \left(\prod_{m=1}^{\infty} \{(1+x^{24m-13})(1+x^{24m-11})\} - x \prod_{m=1}^{\infty} \{(1+x^{24m-19})(1+x^{24m-5})\} \right),$$

$$(6.2) \qquad \sum_{n=0}^{\infty} p(2n+1)x^n$$

$$= \frac{\varphi(x^2)\varphi(x^{24})}{\varphi(x)^3} \left(\prod_{m=1}^{\infty} \{(1+x^{24m-17})(1+x^{24m-7})\} - x^2 \prod_{m=1}^{\infty} \{(1+x^{24m-23})(1+x^{24m-1})\} \right).$$

PROOFS. By lemma 4

$$\sum_{n=0}^{\infty} p(2n+s)x^{2n+s} = (-1)^s \frac{\varphi(x^4)}{\varphi(x^2)^3} g_s .$$

Now

$$g_0 = \sum_{k=-\infty}^{\infty} \{x^{2k(12k+1)} - x^{(4k+1)(6k+2)}\} ,$$

and using (3.7) we easily find

$$g_0 = \varphi(x^{48}) \left(\prod_{m=1}^{\infty} \{(1+x^{48m-26})(1+x^{48m-22})\} - x^2 \prod_{m=1}^{\infty} \{(1+x^{48m-38})(1+x^{48m-10})\} \right).$$

Hence (6.1) follows. The proof of (6.2) is similar.

7. Additional remarks. Atkin and Swinnerton-Dyer [1] have shown that the quantities g_s, for $q>3$, can always be expressed by certain infinite products. For instance, if $q=5$, we have (this special result was stated by Ramanujan [13], and first proved by Darling [4]):

$$g_0 = \varphi(x^5)\prod_{m=1}^{\infty}\{(1-x^{25m-20})(1-x^{25m-5})\}^{-2},$$

$$g_2 = -x^2\varphi(x^5)\prod_{m=1}^{\infty}\{(1-x^{25m-15})(1-x^{25m-10})\}^{-2}.$$

By (4.17)–(4.20) and (4.16) we thus get four identities with $q=5$, similar to the identities (3.1)–(3.3). One of them is:

$$\sum_{n=0}^{\infty} p(5n)x^n$$

$$= \frac{\varphi(x^5)}{\varphi(x)^2}\prod_{m=1}^{\infty}\{(1-x^{5m-4})(1-x^{5m-1})\}^{-8} - 3x\frac{\varphi(x^5)^6}{\varphi(x)^7}\prod_{m=1}^{\infty}\{(1-x^{5m-4})(1-x^{5m-1})\}^{2}.$$

From lemma 4 we now conclude that identities of this kind always exist, but when $q>5$ they become much more complicated.

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