MATH. SCAND. 5 (1957), 93–113

ESTIMATES OF THE FRIEDRICHS-LEWY TYPE FOR MIXED PROBLEMS IN THE THEORY OF LINEAR HYPERBOLIC DIFFERENTIAL EQUATIONS IN TWO INDEPENDENT VARIABLES

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The aim of this paper is to investigate the possibility of estimating solutions to boundary problems of mixed type by means of the boundary data, on one hand for a hyperbolic system of differential equations of the first order in two independent variables, on the other for a hyperbolic differential equation of arbitrary order in two independent variables.

In the theory of partial differential equations, estimates for solutions to boundary problems by means of the boundary data play a fundamental role. For a boundary problem to be correctly set in Hadamard's sense, it is required, beside the unique solvability of the boundary problem, that the solution in some sense depends continuously on the boundary data.

The method in this paper is named after Friedrichs and Lewy [3], who used energy integrals of the type considered here in proving the uniqueness of Cauchy's problem for a hyperbolic differential equation of the second order. The method has recently been applied to Cauchy's problem by Leray [10] and Gårding [4] [5] for hyperbolic equations of arbitrary order and by Friedrichs [2] and Lax [9] for symmetric hyperbolic systems. To problems of mixed type the method has been applied by Krzyzański and Schauder [7] [8] and recently Hörmander [6] for hyperbolic equations of the second order, and by the author [12] for a hyperbolic equation of order three in two independent variables.

Campbell and Robinson [1] showed that the integration of a problem of mixed type for an arbitrary linear hyperbolic differential equation in two independent variables can be reduced to the integration of a problem of mixed type for a system of differential equations of the first order, and they proved the existence and uniqueness of solutions to problems

Received March 23, 1957.

of these two types. The case of the n^{th} order equation is reduced to the case of a first order system in this paper, too. This reduction, however, is not exact, but involves errors of lower order, which can be estimated by means of the principal parts. The estimates for the n^{th} order equation can be deduced also directly, without passing over a first order system (cf. [12]), but since estimates for systems of equations have an interest of their own, this direct method has not been chosen here.

The plan of the present paper is as follows:

In Section 1 notations are introduced and the problem is formulated for a linear hyperbolic system by means of certain linear normed spaces and an operator $\mathbf{L}$, which is the direct sum of the differential operator and the boundary operators. Then in Theorem I two inequalities of the form

$$\|u\| \leqq C\|\mathbf{L}u\|$$

are stated, where C is a positive constant independent of u.

In Section 2 this inequality is used to deduce the uniqueness of solutions to a boundary problem of mixed type for a hyperbolic system of differential equations of the first order and the continuous dependence of the solutions on the boundary data.

In Section 3 Theorem I is proved.

In Section 4 certain restrictions made in Theorem I on the boundary of the region considered are discussed.

Sections 5–7 are analogous to Sections 1–3 and treat a boundary problem of mixed type for a linear hyperbolic differential equation of order n.

Finally, in Section 8, some particular boundary operators for the n^{th} order equation are discussed.

I wish to express my gratitude to Professor Lars Hörmander, who read the manuscript and suggested many valuable improvements.

I. The hyperbolic system of first order.

1. Notations and results. Let $C_m^i(\overline{V})$ $(i=0, 1, \ldots; m=1, 2, \ldots)$ be the set of real-valued vectors

$$u = u(x, t) = \big(u_1(x, t),\ u_2(x, t),\ \ldots,\ u_m(x, t)\big)\,,$$

which are i times continuously differentiable in the closure $\overline{V}$ of a simply connected region V in the xt-plane. For $m=1$ we shall usually omit the lower index and simply write $C^i(\overline{V})$.

With $u \in C_n^1(\overline{V})$ the equations

$$L_i u = (D_t - \alpha_i D_x)u_i + \sum_{k=1}^{n} a_{ik}u_k, \qquad i = 1, \ldots, n\,, \tag{1.1}$$

($D_t = \partial/\partial t$, $D_x = \partial/\partial x$) where $\alpha_i \in C^1(\overline{V})$ and $a_{ik} \in C^0(\overline{V})$, define a linear hyperbolic differential operator Lu from $C_n{}^1(\overline{V})$ into $C_n{}^0(\overline{V})$,

$$Lu = (L_1 u, \ldots, L_n u) .$$

For the sake of simplicity we shall suppose $\alpha_1 \leqq \alpha_2 \leqq \ldots \leqq \alpha_n$. — We remark that the form (1.1) is a canonical form for operators

$$D_t u_i - \sum_{k=1}^{n} c_{ik} D_x u_k + \sum_{k=1}^{n} d_{ik} u_k, \qquad i = 1, \ldots, n ,$$

in which the matrix (c_{ik}) has linear elementary divisors over the field of real numbers (cf. [11, p. 61]).

On the boundary S of V, which we suppose to have a continuously turning tangent, except at a finite number of points, which will be discussed later (condition (b)), certain linear forms are given. Before introducing these, we make a division of S into parts S_i ($i = 0, \ldots, n$) in the following way:

The characteristic form associated with the operator L,

$$\prod_{i=1}^{n} (\tau - \alpha_i \xi) , \tag{1.2}$$

divides the $\xi\tau$-plane into parts Σ_i ($i = 0, \ldots, n$) where Σ_i is the set of points making exactly i factors of (1.2) negative (fig. 1). We also distinguish between the parts of Σ_i which correspond to positive and negative values of ξ and denote these parts by Σ_i^- and Σ_i^+, respectively. In general we shall only use this distinction for $i = 1, \ldots, n-1$.

The division of the $\xi\tau$-plane now gives rise to a division of S into parts S_i so that a point of S belongs to S_i when the exterior normal $\nu = (\nu_\xi, \nu_\tau)$ of S at this point belongs to Σ_i. The notations S_i^- and S_i^+

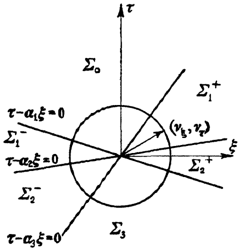

Fig. 1.

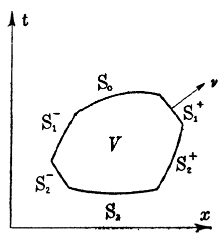

Fig. 2.

correspond in an obvious way to Σ_i^- and Σ_i^+. (See fig. 1 and 2.) In order to shorten the statements, we shall include the endpoints in the definitions of S_i^-, S_i^+ and S_i, so that these parts of the boundary are closed sets. We introduce the notations

$$S^- = S_1^- \cup S_2^- \cup \ldots \cup S_{n-1}^- \quad \text{and} \quad S^+ = S_1^+ \cup S_2^+ \cup \ldots \cup S_{n-1}^+ .$$

We now make the following assumptions concerning the boundary S:

(a) *S^- has a positive distance to S^+;*

(b) $\inf\limits_S |\lambda_i| > 0$, $i = 1, \ldots, n$, *where* $\lambda_i = \nu_\tau - \alpha_i \nu_\xi$.

Condition (b) means in particular that a passage from one part S_i of S to another is accompanied by a jump in the normal. In Section 4 we shall discuss the necessity of these assumptions.

Let s signify the arc length on S and let $C_m{}^i(S_k)$ ($i=0, \ldots ; k=0, \ldots, n;$ $m=1, \ldots$) be the set of real-valued vectors

$$u = u(s) = \big(u_1(s), \ldots, u_m(s)\big)$$

which are i times continuously differentiable on S_k. For $m=1$ we shall usually omit the lower index and simply write $C^i(S_k)$.

On S_i ($i=1, \ldots, n$) we give a linear operator $l_i u$ from $C_n{}^0(S_i)$ to $C_i{}^0(S_i)$,

$$l_i u = \big(l_{i1}(u), \ldots, l_{ii}(u)\big) ,$$

the components of which are linear forms in u with coefficients in $C^0(S_i)$, that is,

$$l_{ik}(u) = l_{ik}(u_1, \ldots, u_n) = \sum_{j=1}^{n} l_{ik}{}^j u_j, \qquad k = 1, \ldots, i , \tag{1.3}$$

where $l_{ik}{}^j \in C^0(S_i)$ ($k=1, \ldots, i; j=1, \ldots, n$) .

About the linear forms (1.3) we assume that

$$\begin{vmatrix} l_{i1}{}^1 & l_{i1}{}^2 & \ldots\ldots & l_{i1}{}^i \\ \ldots & \ldots & \ldots & \ldots \\ l_{ii}{}^1 & & \ldots\ldots & l_{ii}{}^i \end{vmatrix} \neq 0 \qquad \text{on } S_i^- \tag{1.4}$$

and

$$\begin{vmatrix} l_{i1}{}^{n-i+1} & l_{i1}{}^{n-i+2} & \ldots\ldots & l_{i1}{}^n \\ \ldots & \ldots & \ldots & \ldots \\ l_{ii}{}^{n-i+1} & & \ldots\ldots & l_{ii}{}^n \end{vmatrix} \neq 0 \qquad \text{on } S_i^+ . \tag{1.5}$$

On S_n conditions (1.4) and (1.5) are identical and simply mean that the linear forms (1.3) are linearly independent. On S_i, for $0 < i < n$, however, conditions (1.4) and (1.5) are stronger than linear independence. The condition (1.4) means that $u_1, \ldots, u_i$ are certain linear combinations of $u_{i+1}, \ldots, u_n$ and $l_{i1}(u), \ldots, l_{ii}(u)$ on S_i^-, and condition (1.5) means that $u_{n-i+1}, \ldots, u_n$ are certain linear combinations of $u_1, \ldots, u_{n-i}$ and $l_{i1}(u), \ldots, l_{ii}(u)$ on S_i^+.

With $R = V, S, S_0, S_1, \ldots$ or S_n we introduce norms defined by

$$\|\varphi\|_R^2 = \int_R \sum_{j=1}^m \varphi_j^2 dR, \qquad \varphi = (\varphi_1, \ldots, \varphi_m) \in C_m^0(R) .$$

Here dV is the euclidean measure in V, and dS and dS_i are the elements of arc on S and S_i. Further, we introduce the direct sum

$$C^0(\overline{V}, S) = \left(C_n^0(\overline{V}),\ C_1^0(S_1),\ C_2^0(S_2),\ \ldots,\ C_n^0(S_n)\right),$$

and for the elements $\varphi = (\varphi_{\overline{V}}, \varphi_{S_1}, \varphi_{S_2}, \ldots, \varphi_{S_n})$ of $C^0(\overline{V}, S)$ a norm defined by

$$\|\varphi\|_{V,S}^2 = \|\varphi_{\overline{V}}\|_V^2 + \sum_{i=1}^n \|\varphi_{S_i}\|_{S_i}^2 .$$

Now

$$\mathbf{L}u = (Lu, l_1u, \ldots, l_nu)$$

defines a linear operator from $C_n^1(\overline{V})$ to $C^0(\overline{V}, S)$. According to the earlier definitions we have

$$\|\mathbf{L}u\|_{V,S}^2 = \|Lu\|_V^2 + \sum_{i=1}^n \|l_iu\|_{S_i}^2$$

$$= \int_V \sum_{i=1}^n (L_iu)^2 dV + \sum_{i=1}^n \int_{S_i} \sum_{k=1}^i \left(l_{ik}(u)\right)^2 dS_i .$$

Then we can state

THEOREM I. *Let V be the region considered above with a boundary S satisfying conditions* (a) *and* (b), *and let $\mathbf{L}$ be the operator defined above, where l_iu $(i = 1, \ldots, n)$ satisfy conditions* (1.4) *and* (1.5). *Then there exist constants C such that for all $u \in C_n^1(\overline{V})$*

$$\|u\|_V \leq C\|\mathbf{L}u\|_{V,S} \tag{1.6}$$

and

$$\|u\|_S \leq C\|\mathbf{L}u\|_{V,S} . \tag{1.7}$$

Here and in what follows C means a positive constant independent of u, but it does not always mean the same constant even during the course of a proof. When necessary, we distinguish between different constants by using subscripts.

2. Two corollaries. Before proving Theorem I in Section 3, we shall in this section draw some conclusions from it.

Consider the system of differential equations

$$Lu = f, \tag{2.1}$$

where $f=(f_1, \ldots, f_n) \in C_n^0(\overline{V})$, with the boundary conditions

$$l_i u = g_i, \quad i = 1, \ldots, n, \tag{2.2}$$

where $g_i=(g_{i1}, \ldots, g_{ii}) \in C_i^0(S_i)$. This can be expressed concisely in the single equation

$$\mathbf{L}u = (f, g_1, \ldots, g_n).$$

From Theorem I we obtain the following corollaries:

COROLLARY Ia. *The system of differential equations* (2.1) *with the boundary conditions* (2.2) *has at most one solution* $u \in C_n^1(\overline{V})$.

To see this, assume that u^1 and u^2 are two solutions. This means that $\mathbf{L}(u^1-u^2)=(0, 0, \ldots, 0)$, and hence $\|\mathbf{L}(u^1-u^2)\|_{V,S}=0$. In view of the inequality (1.6) this implies $\|u^1-u^2\|_V=0$, that is, $u^1=u^2$.

COROLLARY Ib. *The solutions of* (2.1) *depend continuously on the boundary data* (2.2) *in the following sense: Let* u^1 *and* u^2 *be two solutions of* (2.1) *with the boundary data*

$$l_i u^k = g_i^k, \quad i = 1, \ldots, n;\ k = 1, 2.$$

Then

$$\|u^1-u^2\|_V \leqq C\left(\sum_{i=1}^{n} \|g_i^1-g_i^2\|_{S_i}^2\right)^{\frac{1}{2}} \tag{2.3}$$

For $\mathbf{L}(u^1-u^2)=(0, g_1^1-g_1^2, \ldots, g_n^1-g_n^2)$ so that (2.3) follows immediately from (1.6).

3. Proof of Theorem I. One easily shows the identity

$$\sum_{i=1}^{n} 2e^{-\gamma t} A_i u_i L_i u = \sum_{i=1}^{n} (D_t - D_x \alpha_i) e^{-\gamma t} A_i u_i^2 + \\ + e^{-\gamma t}\gamma \sum_{i=1}^{n} A_i u_i^2 + e^{-\gamma t} \sum_{i,k=1}^{n} b_{ik} u_i u_k, \tag{3.1}$$

where A_i $(i=1, \ldots, n)$ are functions in $C^1(\overline{V})$, which will be chosen later. Here we remark only that A_i will be >0 in $\overline{V}$. Further γ is a constant, which will be fixed later, and b_{ik}, which depends on A_i, α_i and a_{ik}, belong to $C^0(\overline{V})$ because of the assumptions.

Integrating the identity (3.1) over V and using Green's formula on the divergence terms, we get $\left(|\nu| = (\nu_\xi^2 + \nu_\tau^2)^{\frac{1}{2}} = 1\right)$

$$(3.2) \qquad \int_V e^{-\gamma t} \sum_{i=1}^n 2A_i u_i L_i u \, dV = \int_S e^{-\gamma t} \sum_{i=1}^n \lambda_i A_i u_i^2 dS + \int_V e^{-\gamma t} \left\{ \gamma \sum_{i=1}^n A_i u_i^2 + \sum_{i,k=1}^n b_{ik} u_i u_k \right\} dV .$$

We introduce the notations

$$S(u, u) = \sum_{i=1}^n \lambda_i A_i u_i^2 ,$$

$$V(u, u) = \gamma \sum_{i=1}^n A_i u_i^2 + \sum_{i,k=1}^n b_{ik} u_i u_k .$$

The object is to choose A_i and γ in such a way as to make $V(u, u)$ positive definite in $\bar{V}$ and $S(u, u)$ positive definite on S when the boundary conditions

$$(3.3) \qquad l_i u = 0 \quad \text{on } S_i ,$$

are satisfied, and then to deduce the desired estimates from the identity (3.2).

It is immediately seen, that if A_i are fixed functions which are >0 in $\bar{V}$, then the constant γ can be chosen so large that $V(u, u)$ is positive definite in $\bar{V}$.

We now show that A_i can be chosen positive and such that $S(u, u)$ becomes positive definite when the boundary conditions (3.3) are satisfied. We shall first choose A_i on S^- and S^+ and then extend them as positive, continuously differentiable functions in the whole of $\bar{V}$. Then $S(u, u)$ is non-negative also on S_n and S_0, for on S_n (3.3) implies that all u_i are $=0$ and thus $S(u, u) = 0$, and on S_0 all λ_i are >0.

Consider $S(u, u)$ on $S_{n-1}{}^+$. There we have $\lambda_k < 0$ for $k = 2, \ldots, n$ and $\lambda_1 > 0$. The boundary conditions are

$$l_{n-1,k}(u_1, \ldots, u_n) = 0, \qquad k = 1, \ldots, n-1 .$$

Beside $l_{n-1,k}(u)$ $(k = 1, \ldots, n-1)$ we then introduce one more linear form, viz.

$$l_{n-1,n}(u_1, \ldots, u_n) = u_1 ,$$

and make the linear transformation

$$u_{n-1,k} = l_{n-1,k}(u_1, \ldots, u_n), \qquad k = 1, \ldots, n .$$

Because of the condition (1.5) this transformation is non-singular. $S(u, u)$ can now be interpreted as a quadratic form in $u_{n-1,k}$ $(k=1, \ldots, n)$. When $u_{n-1,k}=0$ $(k=1, \ldots, n-1)$ we shall make $S(u, u)$ positive definite considered as a quadratic form in the remaining variable $u_{n-1,n}$. First, if with a fixed constant $A_1>0$ we put $A_k=0$ $(k=2, \ldots, n)$, we have

$$S(u, u) = \lambda_1 A_1 u_{n-1,n}^2 .$$

Now $\lambda_1 A_1$ has a positive lower bound on S_{n-1}^+ because of assumption (b). We thus have

$$S(u, u) \geqq C u_{n-1,n}^2$$

when $A_k=0$ $(k=2, \ldots, n)$, where C is a constant >0. We therefore see that there is an $\varepsilon_1>0$ such that if $0<A_k \leqq \varepsilon_1$ $(k=2, \ldots, n)$, we have

$$S(u, u) \geqq \tfrac{1}{2} C u_{n-1,n}^2 \qquad \text{on } S_{n-1}^+$$

if $u_{n-1,k}=0$ $(k=1, \ldots, n-1)$. The constant value of the function A_1 used on S_{n-1}^+ will be used on the whole of S^+.

We then consider S_{n-2}^+. Here $\lambda_k<0$ for $k=3, \ldots, n$ while $\lambda_1>0$ and $\lambda_2>0$. The boundary conditions are

$$l_{n-2,k}(u_1, \ldots, u_n) = 0, \qquad k = 1, \ldots, n-2 .$$

As on S_{n-1}^+ we extend the system of linear forms by two new forms, viz.

$$l_{n-2,n-1}(u_1, \ldots, u_n) = u_2 \qquad \text{and} \qquad l_{n-2,n}(u_1, \ldots, u_n) = u_1$$

and make the linear transformation

$$u_{n-2,k} = l_{n-2,k}(u_1, \ldots, u_n), \qquad k = 1, \ldots, n ,$$

which because of the condition (1.5) is non-singular. $S(u, u)$ can then be interpreted as a quadratic form in $u_{n-2,k}$ $(k=1, \ldots, n)$. When $u_{n-2,k}=0$ $(k=1, \ldots, n-2)$ we shall make $S(u, u)$ positive definite considered as a quadratic form in the remaining variables $u_{n-2,n-1}$ and $u_{n-2,n}$. First, if with A_1 fixed at the same positive value as on S_{n-1}^+ and A_2 fixed at a positive constant value $\leqq \varepsilon_1$, we put $A_k=0$ $(k=3, \ldots, n)$, we have

$$S(u, u) = \lambda_1 A_1 u_{n-2,n}^2 + \lambda_2 A_2 u_{n-2,n-1}^2 .$$

Now $\lambda_1 A_1$ and $\lambda_2 A_2$ have positive lower bounds on S_{n-2}^+ because of assumption (b). We thus have

$$S(u, u) \geqq C(u_{n-2,n-1}^2 + u_{n-2,n}^2)$$

when $A_k=0$ $(k=3, \ldots, n)$, where C is a constant >0. As on S_{n-1}^+ we realize that there exists an $\varepsilon_2 \leqq \varepsilon_1$ such that if $0<A_k \leqq \varepsilon_2$ $(k=3, \ldots, n)$, we have

$$S(u, u) \geqq \tfrac{1}{2} C (u_{n-2,n-1}{}^2 + u_{n-2,n}{}^2) \qquad \text{on } S_{n-2}{}^+$$

if $u_{n-2,k}=0$ $(k=1, \ldots, n-2)$. As we have seen, for any set of A_i satisfying these conditions, $S(u, u)$ is also positive definite on $S_{n-1}{}^+$. The constant value of the function A_2 used on $S_{n-2}{}^+$ will be used on the whole of S^+.

The process will now be continued so that we get n positive constants $A_1, \ldots, A_n$ with the property that $S(u, u)$ is positive definite on S^+ when the boundary conditions (3.3) are satisfied.

Then $A_1, \ldots, A_n$ are fixed on S^- after the same principle so that $S(u, u)$ becomes positive definite when the boundary conditions (3.3) are satisfied. This time, however, we fix A_n first, and then A_{n-1}, etc.

On $S_i = S_i^- \cup S_i^+$ $(i=1, \ldots, n-1)$ we then have

$$S(u, u) \geqq C\left\{(l_{i,i+1}(u))^2 + \ldots + (l_{in}(u))^2\right\}$$

when $l_{ik}(u)=0$ $(k=1, \ldots, i)$. Here $l_{ik}(u)$ $(k=i+1, \ldots, n)$ are auxiliary forms defined on S_i with the property that $l_{ik}(u)$ constitute a set of n linearly independent forms.

We now extend $A_1, \ldots, A_n$ to positive functions in $C^1(\overline{V})$ and choose the constant γ so large that $V(u, u)$ is positive definite in $\overline{V}$.

Then we return to the identity (3.2), which may be written

$$\int_V e^{-\gamma t} V(u, u)\, dV = \int_V e^{-\gamma t} \sum_{i=1}^{n} 2 A_i u_i L_i u\, dV - \int_S e^{-\gamma t} S(u, u)\, dS\,.$$

Since $V(u, u)$ is positive definite in $\overline{V}$ we obtain

$$\begin{aligned}
\|u\|_V{}^2 &\leqq C \int_V e^{-\gamma t} V(u, u)\, dV \\
&= C\left\{\int_V e^{-\gamma t} \sum_{i=1}^{n} 2 A_i u_i L_i u\, dV - \int_S e^{-\gamma t} S(u, u)\, dS\right\} \\
&\leqq C_1 \int_V \left\{\sum_{i=1}^{n} u_i{}^2\right\}^{\frac{1}{2}} \left\{\sum_{i=1}^{n} (L_i u)^2\right\}^{\frac{1}{2}} dV - C \int_S e^{-\gamma t} S(u, u)\, dS \\
&\leqq C_2 \|u\|_V \|Lu\|_V - C \int_S e^{-\gamma t} S(u, u)\, dS \\
&= \|u\|_V (C_2 \|Lu\|_V) - C \int_S e^{-\gamma t} S(u, u)\, dS \\
&\leqq \tfrac{1}{2} \|u\|_V{}^2 + \tfrac{1}{2} C_2{}^2 \|Lu\|_V{}^2 - C \int_S e^{-\gamma t} S(u, u)\, dS\,,
\end{aligned}$$

that is,

$$\|u\|_V^2 \leqq C_3\|Lu\|_V^2 - C_4\int_S e^{-\gamma t}S(u, u)dS . \tag{3.4}$$

In order to estimate the last term in (3.4) we observe that, since $S(u, u)$ is positive definite on S_i in the variables $l_{ik}(u)$ $(k=i+1, \ldots, n)$ when $l_{ik}(u)=0$ $(k=1, \ldots, i)$, we have on S_i $(i=1, \ldots, n)$

$$S(u, u) \geqq C\{(l_{i,i+1}(u))^2+\ldots+(l_{in}(u))^2\} + S_1(u, u) .$$

Here $S_1(u, u)$ is a quadratic form in $l_{ik}(u)$ $(k=1, \ldots, n)$ which is linear in $l_{ik}(u)$ $(k=i+1, \ldots, n)$. By completing squares we therefore see, that $S(u, u)$ becomes positive definite after addition of a sufficiently large multiple of $(l_{i1}(u))^2+\ldots+(l_{ii}(u))^2$, that is,

$$\begin{aligned} C_{1i}\{(l_{i1}(u))^2+\ldots+(l_{in}(u))^2\} \\ \leqq S(u, u) + C_{2i}\{(l_{i1}(u))^2+\ldots+(l_{ii}(u))^2\} . \end{aligned} \tag{3.5}$$

In order to prove the inequalities (1.6) and (1.7), we deduce from (3.5)

$$C_{3i}\|u\|_{S_i}^2 \leqq C_4\int_{S_i} e^{-\gamma t}S(u, u)dS_i + C_{4i}\|l_iu\|_{S_i}^2 , \tag{3.6}$$

where C_4 is the constant C_4 in (3.4). On S_0 it is obvious that

$$C_{30}\|u\|_{S_0}^2 \leqq C_4\int_{S_0} e^{-\gamma t}S(u, u)dS_0 . \tag{3.7}$$

Therefore, adding (3.4), (3.6) and (3.7) and using trivial estimates, we get

$$C_5\|u\|_S^2 + \|u\|_V^2 \leqq C_6\left\{\|Lu\|_V^2 + \sum_{i=1}^{n}\|l_iu\|_{S_i}^2\right\} ,$$

which contains inequalities (1.6) and (1.7).

4. The necessity of conditions (*a*) and (*b*). In this section we shall show by examples that the inequalities (1.6) and (1.7) with the norms defined in Section 1 are not always valid if one of the conditions (a) and (b) is deleted. We remark that in the proof of inequality (1.6), condition (b) is not used on S_0 and S_n and that in the proof of inequality (1.7), condition (b) is not used on S_n. In the more restrictive form given, condition (b) can, however, be used throughout the paper.

Let us consider the operator

$$Lu = (L_1u, L_2u) = ((D_t-D_x)u_1, (D_t+D_x)u_2)$$

in a region V and let us particularly consider this operator for functions of the type

$$u = (u_1, u_2) = \big(f_1(t+x),\ f_2(t-x)\big)\,, \tag{4.1}$$

where f_1 and f_2 are continuously differentiable functions. Thus u_1 is supposed to be constant on straight lines parallel to $t+x=0$ and u_2 on straight lines parallel to $t-x=0$. We then see that $Lu=(L_1u,\ L_2u)=(0,0)$, and thus

$$\|Lu\|_V = 0\,. \tag{4.2}$$

By making different choices of V and the boundary operators we shall now construct examples contradicting the inequalities (1.6) and (1.7) in cases where the conditions (a) or (b) are not fulfilled.

We first consider condition (a). Let S_2 be void (fig. 3), so that condition (a) is not fulfilled. Consider the boundary operator $l_1u=u_1-u_2$. For $u=(k,k)$, where k is a constant $\neq 0$, this implies a contradiction; for (4.2) combined with $\|l_1u\|_{S_1}=0$ gives $\|\mathbf{L}u\|_{V,S}=0$, while $\|u\|_V>0$, $\|u\|_S>0$, which contradicts inequalities (1.6) and (1.7).

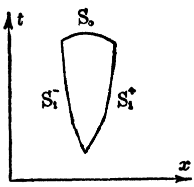

Fig. 3.

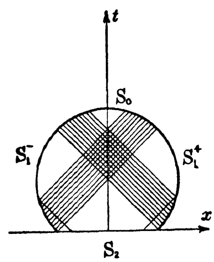

Fig. 4.

We then consider condition (b). Here let V be the segment of the circle $x^2+(t-1)^2=2$, which is situated above the x-axis (fig. 4). Since

$$\inf_S |\lambda_i| = 0, \qquad i = 1, 2\,,$$

$(\alpha_1=-1,\ \alpha_2=1)$ condition (b) is not fulfilled.

We shall consider the boundary operators $l_1u=u_1-u_2$ on S_1 and $l_2u=(u_1,\ u_2)$ on S_2. On S_2 let u be$=(\varphi_\omega(x),\ \varphi_\omega(x))$, where $\varphi_\omega(x)$ is a continuously differentiable function with the properties

$$\varphi_\omega(x) = \omega \quad \text{for} \quad \begin{cases} -1 \leqq x \leqq -1+1/(4\omega^2)\,, \\ 1-1/(4\omega^2) \leqq x \leqq 1\,, \end{cases}$$

and

$$\int_{-1}^{+1} \varphi_\omega^2(x)\,dx = 1\,.$$

When u is of the type (4.1) we realize that u_1 is defined in the region bounded by S_1^-, S_2 and the line $t+x=1$, thus particularly on S_1^-, and that u_2 is defined in the region bounded by S_1^+, S_2 and the line $t-x=1$, thus particularly on S_1^+. By putting $u_2=u_1$ on S_1^- and $u_1=u_2$ on S_1^+ we define u in the whole of V. Now we have

$$\|l_1 u\|_{S_1}^2 = \int_{S_1} (u_1-u_2)^2 dS_1 = 0\,,$$

$$\|l_2 u\|_{S_2}^2 = \int_{S_2} (u_1^2+u_2^2) dS_2 = 2\int_{-1}^{+1} \varphi_\omega^2(x)\,dx = 2\,.$$

Combined with (4.2) this gives

$$\|\mathbf{L}u\|_{V,S}^2 = 2\,. \tag{4.3}$$

However, $u_2=\omega$ in a band parallel to the line $t+x=0$, the breadth of which is $=1/(2\omega)$, and $u_1=\omega$ in a band parallel to the line $t-x=0$, the breadth of which is $=1/(2\omega)$. These bands have been shaded in fig. 4. Therefore we get

$$\|u\|_V^2 = \int_V (u_1^2+u_2^2)\,dV \geqq C\omega^{-1}\omega^2 = C\omega$$

and

$$\|u\|_S^2 \geqq \int_{S_0} (u_1^2+u_2^2)\,dS_0 \geqq C\omega^{-1}\omega^2 = C\omega\,.$$

Combined with (4.3) this contradicts inequalities (1.6) and (1.7).

II. The hyperbolic equation of order n.

5. Notations and results. Suppose that

$$Mu = \sum_{|p|\leqq n} a_p D^p u, \qquad u \in C^n(\overline{V}),$$

defines a linear hyperbolic differential operator in $\overline{V}$. Here $p=(p_t, p_x)$, $|p|=p_t+p_x$, $D^p=D_t^{p_t}D_x^{p_x}$, $a_p \in C^1(\overline{V})$ for $|p|=n$ and $a_p \in C^0(\overline{V})$ for $|p|<n$. The characteristic form associated with M is then, if $a_{(n,0)}=1$,

$$\sum_{|p|=n} a_p\, \tau^{p_t}\xi^{p_x} = \prod_{i=1}^{n} (\tau-\alpha_i\xi)\,, \tag{5.1}$$

where we assume $\alpha_1 < \alpha_2 < \ldots < \alpha_n$ and $\alpha_i \in C^1(\overline{V})$. — Observe that the assumption $\alpha_i \neq \alpha_k$ for $i \neq k$ is essential in this and the following sections; this was not the case in Sections 1–3.

The characteristic form (5.1) is the same as the characteristic form (1.2) in Section 1, and therefore we get the same division of S into parts as earlier. We assume as before that the boundary has a continuously turning tangent except at a finite number of points, and that the conditions (a) and (b) are valid.

Before we introduce our boundary operators we shall define n operators $M_i u$ $(i=1, \ldots, n)$ of order $n-1$,

$$M_i u = \sum_{|p|=n-1} b_{i,p} D^p u \tag{5.2}$$

with the characteristic forms

$$\sum_{|p|=n-1} b_{i,p} \tau^{p_t} \xi^{p_x} = \prod_{k \neq i} (\tau - \alpha_k \xi) . \tag{5.3}$$

$M_i u$ $(i=1, \ldots, n)$ constitute a set of n linearly independent forms in the derivatives $D^p u$ with $|p|=n-1$, and these derivatives can be expressed as linear combinations of $M_i u$ in the following way:

$$D^p u = \sum_{i=1}^{n} \alpha_i^{p_t} \beta_i M_i u, \quad |p| = n-1, \qquad \text{where} \qquad \beta_i = \prod_{k \neq i} (\alpha_i - \alpha_k)^{-1} .$$

This can be proved simply by means of Lagrange's interpolation formula.

On S_n we define the "Cauchy" operator $m_n u$ for $u \in C^{n-1}(\overline{V})$,

$$m_n u = (D^p u;\ |p| \leqq n-1) .$$

On S_i $(i=1, \ldots, n-1)$ we define a boundary operator $m_i u$ from $C^{n-1}(\overline{V})$ to $C_i^0(S_i)$,

$$m_i u = \big(m_{i1}(u), \ldots, m_{ii}(u)\big) ,$$

the components of which are linear forms in $D^p u$ with $|p| \leqq n-1$ and coefficients in $C^0(S_i)$,

$$m_{ik}(u) = \sum_{|p| \leqq n-1} m_{ik,p} D^p u, \qquad k = 1, \ldots, i . \tag{5.4}$$

Now any linear form in $D^p u$ with $|p|=n-1$ and coefficients in $C^0(S_i)$ can be represented as a linear combination of $M_i u$ $(i=1, \ldots, n)$. We can therefore find functions $m_{ik}{}^j \in C^0(S_i)$ such that

$$m_{ik}(u) = \sum_{j=1}^{n} m_{ik}{}^j M_j u - \sum_{|p| \leqq n-2} d_{ik,p} D^p u . \tag{5.5}$$

In order to make formula (5.5) valid also in the case $i=n$, we define

$$m_{nk}(u) = D_t^{k-1}D_x^{n-k}u = \sum_{j=1}^{n} \alpha_j^{k-1}\beta_j M_j u ,$$

and put $m_{nk}^j = \alpha_j^{k-1}\beta_j$.

About the linear operators $m_i u$ we assume that

$$\begin{vmatrix} m_{i1}^1 & m_{i1}^2 & \dots & m_{i1}^i \\ \dots & \dots & \dots & \dots \\ m_{ii}^1 & & \dots & m_{ii}^i \end{vmatrix} \neq 0 \quad \text{on } S_i^- \tag{5.6}$$

and

$$\begin{vmatrix} m_{i1}^{n-i+1} & m_{i1}^{n-i+2} & \dots & m_{i1}^n \\ \dots & \dots & \dots & \dots \\ m_{ii}^{n-i+1} & & \dots & m_{ii}^n \end{vmatrix} \neq 0 \quad \text{on } S_i^+ . \tag{5.7}$$

For $i=n$ (5.6) and (5.7) are consequences of the definition of $m_n u$ and of the fact that $\alpha_i \neq \alpha_k$ for $i \neq k$.

Consider the characteristic forms of (5.4) and $M_i u$ $(i=1, \dots, n)$, viz.

$$\hat{m}_{ik}(\xi, \tau) = \sum_{|p|=n-1} m_{ik,p}\, \tau^{p_t} \xi^{p_x}$$

and

$$\hat{M}_i(\xi, \tau) = \prod_{k \neq i} (\tau - \alpha_k \xi) ,$$

respectively. Conditions (5.6) and (5.7) can now be interpreted in the following way. Condition (5.6) means that on S_i^-, the augmented system $\hat{m}_{ik}(\xi, \tau)$ $(k=1, \dots, i)$, $\hat{M}_k(\xi, \tau)$ $(k=i+1, \dots, n)$ forms a set of linearly independent polynomials, that is, there do not exist functions $c_{ik} \in C^0(S_i)$ $(k=1, \dots, n)$, where not all of the $c_{ik}=0$, such that

$$\sum_{k=1}^{i} c_{ik}\, \hat{m}_{ik}(\xi, \tau) = \sum_{k=i+1}^{n} c_{ik}\, \hat{M}_k(\xi, \tau) .$$

$\hat{M}_k(\xi, \tau)$ $(k=i+1, \dots, n)$ being linearly independent, it is sufficient to require that not all of the c_{ik} $(k=1, \dots, i)$ vanish. Because $\hat{M}_k(\xi, \tau)$ $(k=i+1, \dots, n)$ all contain $\Pi_{j=1}^{i}(\tau - \alpha_j \xi)$ as a factor, and because every homogeneous polynomial in ξ and τ which contains $\Pi_{j=1}^{i}(\tau - \alpha_j \xi)$ as a factor can be written as a linear combination of $\hat{M}_k(\xi, \tau)$ $(k=i+1, \dots, n)$, we realize that condition (5.6) is equivalent to the following:

(5.8) *The linear forms $m_{ik}(u)$ $(k=1, \dots, i)$, given on S_i^- $(i=1, \dots, n)$, are such that no non-trivial linear combination of the polynomials $\hat{m}_{ik}(\xi, \tau)$ $(k=1, \dots, i)$ contains $\Pi_{j=1}^{i}(\tau - \alpha_j \xi)$ as a factor.*

Analogously, condition (5.7) is equivalent to the following:

(5.9) *The linear forms* $m_{ik}(u)$ $(k=1,\ldots,i)$ *given on* S_i^+ $(i=1,\ldots,n)$ *are such that no non-trivial linear combination of the polynomials* $\hat{m}_{ik}(\xi,\tau)$ $(k=1,\ldots,i)$ *contains* $\Pi_{j=n-i+1}^{n}(\tau-\alpha_j\xi)$ *as a factor.*

For functions $u \in C^k(\overline{V})$ we introduce norms defined by

$$\|u\|_{k,R}^2 = \int_R \sum_{|p|\leqq k} (D^p u)^2 dR, \qquad R = V, S, S_0, S_1, \ldots, S_n .$$

For $m_n u$ we take the norm

$$\|m_n u\|_{S_n} = \|u\|_{n-1,S_n},$$

and for $m_i u$ $(i=1,\ldots,n-1)$ we take a norm defined by

$$\|m_i u\|_{S_i}^2 = \sum_{k=1}^{i} \|m_{ik}(u)\|_{0,S_i}^2 = \sum_{k=1}^{i} \int_{S_i} (m_{ik}(u))^2 dS_i .$$

Like in Section 1 the linear operators in $\overline{V}$ and on S_i $(i=1,\ldots,n)$ can be expressed concisely in the single equation

$$\mathbf{M}u = (Mu, m_1 u, m_2 u, \ldots, m_n u),$$

and we take for the norm of $\mathbf{M}u$

$$\mathbf{M}\|u\|_{V,S}^2 = \|Mu\|_{0,V}^2 + \sum_{i=1}^{n} \|m_i u\|_{S_i}^2$$
$$= \int_V (Mu)^2 dV + \sum_{i=1}^{n-1} \int_{S_i} \sum_{k=1}^{i} (m_{ik}(u))^2 dS_i + \int_{S_n} \sum_{|p|\leqq n-1} (D^p u)^2 dS_n .$$

We can now state

THEOREM II. *Let* V *be the region considered above with a boundary* S *satisfying conditions* (a) *and* (b), *and let* $\mathbf{M}$ *be the operator defined above, where* Mu *is hyperbolic in* $\overline{V}$, *and where* $m_i u$ $(i=1,\ldots,n)$ *satisfy conditions* (5.8) *and* (5.9). *Then there exist constants* C *such that for all* $u \in C^n(\overline{V})$

(5.10) $$\|u\|_{n-1,V} \leqq C\|\mathbf{M}u\|_{V,S}$$

and

(5.11) $$\|u\|_{n-1,S} \leqq C\|\mathbf{M}u\|_{V,S} .$$

6. Two corollaries. Before proving Theorem II in Section 7, we shall in this section state two corollaries, analogous to Corollary Ia and Corollary Ib in Section 2.

Consider the hyperbolic differential equation

$$Mu = f, \tag{6.1}$$

where $f \in C^0(\overline{V})$ with the boundary conditions

$$m_i u = g_i \quad \text{on} \quad S_i, \qquad i = 1, \ldots, n, \tag{6.2}$$

where $g_i \in C_i^0(S_i)$ for $i=1, \ldots, n-1$ and g_n is a set of Cauchy data on S_n of a given function in $C^{n-1}(\overline{V})$. We then have

Corollary IIa. *The equation* (6.1) *with the boundary conditions* (6.2) *has at most one solution* $u \in C^n(\overline{V})$.

Corollary IIb. *The solutions of* (6.2) *depend continuously on the boundary data* (6.2) *in the following sense. Let* u^1 *and* u^2 *be two solutions of* (6.1) *with the boundary conditions*

$$m_i u^j = g_i^j, \qquad j = 1, 2;\ i = 1, \ldots, n.$$

Then

$$\|u^1 - u^2\|_{n-1,V} \leqq C \left\{ \sum_{i=1}^{n} \|g_i^1 - g_i^2\|_{S_i}^2 \right\}^{\frac{1}{2}}.$$

The proofs are analogous to those of Section 2.

7. Proof of Theorem II. The operators $M_i u$ $(i=1, \ldots, n)$ defined in (5.2) and (5.3) have the property that $(D_t - \alpha_i D_x) M_i u$ has the same principal part as Mu, that is

$$(D_t - \alpha_i D_x) M_i u = Mu + \sum_{|p| \leqq n-1} c_{i,p} D^p u, \tag{7.1}$$

where $c_{i,p} \in C^0(\overline{V})$. Introducing the operators

$$L_i(v) = (D_t - \alpha_i D_x) v_i, \qquad i = 1, \ldots, n,$$

where $v = (v_1, \ldots, v_n)$, (7.1) can be written

$$L_i(M_1 u, \ldots, M_n u) = Mu + \sum_{|p| \leqq n-1} c_{i,p} D^p u. \tag{7.2}$$

Here L_i is an operator of exactly the same kind as the operator L_i considered in Section 1. Introducing the operators

$$l_i v = \big(l_{i1}(v), \ldots, l_{ii}(v)\big), \qquad i = 1, \ldots, n,$$

with

$$l_{ik}(v) = \sum_{j=1}^{n} m_{ik}^j v_j, \qquad k = 1, \ldots, i;\ i = 1, \ldots, n,$$

we can write

$$(7.3)\quad l_{ik}(M_1 u, \ldots, M_n u) = m_{ik}(u) + \sum_{|p| \leqq n-2} d_{ik,p} D^p u, \qquad k = 1, \ldots, i .$$

(Campbell and Robinson [1] showed that the operators $M_i u$ can be modified with lower order terms so that the coefficients $c_{i,p}$ in (7.1) and $d_{ik,p}$ in (7.3) all vanish. To prove this, however, these authors need existence theorems for quasi-linear systems.)

The operators $L_i u$ and $l_i u$ fulfill the assumptions of Theorem I. In particular, conditions (1.4) and (1.5) are consequences of conditions (5.6) and (5.7), which are equivalent to conditions (5.8) and (5.9). We can therefore apply Theorem I and estimate $M_i u$ $(i = 1, \ldots, n)$. In order to be able to get rid of the terms of order $\leqq n-1$ in (7.2) and $\leqq n-2$ in (7.3) we shall not, however, use Theorem I on V, but on a part $V(\theta)$ of V to be defined presently.

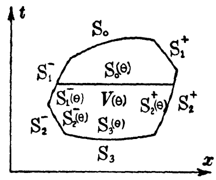

Fig. 5.

Let $V(\theta) = V \cap \{t \leqq \theta\}$ and let $S(\theta)$ be the boundary of $V(\theta)$. Then in the division of $S(\theta)$ into parts $S_i(\theta)$ $(i = 0, \ldots, n)$ after the direction of the exterior normal ν, $S_i(\theta)$ is a subset of S_i for $i = 1, \ldots, n$. With

$$\theta_0 = \inf_{(x,t) \in V} t, \qquad \theta_1 = \sup_{(x,t) \in V} t ,$$

$V(\theta)$ and $S(\theta)$ are void for $\theta < \theta_0$ and $V(\theta) = V$ and $S(\theta) = S$ for $\theta \geqq \theta_1$ (see fig. 5.).

We now use Theorem I for the operators $L_i u$ and $l_i u$ and for the region $V(\theta)$. The inequality (1.7) in Theorem I gives the following estimate

$$\sum_{i=1}^{n} \|M_i u\|_{0,S(\theta)}^2$$
$$\leqq C\left\{ \left\| Mu + \sum_{|p| \leqq n-1} c_{i,p} D^p u \right\|_{0,V(\theta)}^2 + \sum_{i=1}^{n} \sum_{k=1}^{i} \left\| m_{ik}(u) + \sum_{|p| \leqq n-2} d_{ik,p} D^p u \right\|_{0,S_i(\theta)}^2 \right\}$$
$$\leqq C\left\{ \|Mu\|_{0,V}^2 + \|u\|_{n-1,V(\theta)}^2 + \sum_{i=1}^{n} \sum_{k=1}^{i} \|m_{ik}(u)\|_{0,S_i}^2 + \|u\|_{n-2,S(\theta)}^2 \right\} .$$

It is easy to see, by examining the proof of Theorem I, that the constant C can be chosen independent of θ. Since every derivative of u of order $n-1$ is a linear combination of $M_i u$ $(i = 1, \ldots, n)$, we obtain

$$\|u\|_{n-1,S(\theta)}^2$$
$$(7.4)\qquad \leqq C\left\{ \|u\|_{n-2,S(\theta)}^2 + \sum_{i=1}^{n} \|M_i u\|_{0,S(\theta)}^2 \right\}$$
$$\leqq C\left\{ \|u\|_{n-1,V(\theta)}^2 + \|u\|_{n-2,S(\theta)}^2 + \|Mu\|_{0,V}^2 + \sum_{i=1}^{n} \sum_{k=1}^{i} \|m_{ik}(u)\|_{0,S_i}^2 \right\} .$$

The following well-known inequality is obtained by integrating the derivatives of order $n-1$ along curves from S_n to $S(\theta)$, followed by obvious estimates:

$$\|u\|_{n-2,S(\theta)}^2 \leqq C\{\|u\|_{n-1,V(\theta)}^2 + \|u\|_{n-2,S_n}^2\} .$$

Put into (7.4) this gives

$$\|u\|_{n-1,S(\theta)}^2 \leqq C\left\{\|u\|_{n-1,V(\theta)}^2 + \|Mu\|_{0,V}^2 + \sum_{i=1}^{n} \|m_i u\|_{S_i}^2\right\}. \tag{7.5}$$

We define

$$\psi(\theta) = \|u\|_{n-1,V(\theta)}^2 .$$

It is easy to see that $\psi(\theta)$ is a Lipschitz continuous, increasing function of θ, that is, $\psi(\theta)$ is an increasing function such that

$$\psi(\theta'')-\psi(\theta') \leqq C(\theta''-\theta') \quad \text{for all } \theta'' > \theta' ,$$

and that almost everywhere

$$\psi'(\theta) \leqq \|u\|_{n-1,S(\theta)}^2 .$$

Hence (7.5) implies

$$\psi'(\theta) \leqq C(\psi(\theta)+K) , \tag{7.6}$$

where C and K,

$$K = \|Mu\|_{0,V}^2 + \sum_{i=1}^{n} \|m_i u\|_{S_i}^2 ,$$

are independent of θ. Multiplying both sides of (7.6) by $e^{-C\theta}$, we can write

$$\frac{d}{d\theta}\left(e^{-C\theta}\psi(\theta)\right) \leqq CKe^{-C\theta} . \tag{7.7}$$

Integrating from θ_0 to θ_1, we obtain from (7.7)

$$e^{-C\theta_1}\psi(\theta_1)-e^{-C\theta_0}\psi(\theta_0) = e^{-C\theta_1}\psi(\theta_1) \leqq (e^{-C\theta_0}-e^{-C\theta_1})K .$$

With a new value of the constant C this gives

$$\psi(\theta_1) \leqq CK ,$$

that is,

$$\|u\|_{n-1,V}^2 \leqq C\left\{\|Mu\|_{0,V}^2 + \sum_{i=1}^{n} \|m_i u\|_{S_i}^2\right\}, \tag{7.8}$$

which is the inequality (5.10). From (7.5) with $\theta=\theta_1$ and (7.8) we get

$$\|u\|_{n-1,S}^2 \leqq C\left\{\|Mu\|_{0,V}^2 + \sum_{i=1}^{n} \|m_i u\|_{S_i}^2\right\},$$

which is the inequality (5.11).

8. Some particular boundary operators. In this section we shall first study the "standard" mixed problem, that is, the problem of solving equation (6.1) when u and its derivatives $D_\nu^{k-1}u$ $(k=1, \ldots, i)$ of orders $\leq i-1$ in the direction of the exterior normal ν are given on S_i. In particular on S_n we have Cauchy data. We assume that the boundary is $n-1$ times continuously differentiable except at a finite number of points, or, more exactly, that $\nu \in C_2^{n-2}(S_i)$ on S_i $(i=1, \ldots, n)$. On S_0 we still assume $\nu \in C_2^0(S_0)$. In order to reduce this problem to a problem of the kind considered in Theorem II, we shall differentiate $D_\nu^{k-1}u$ $n-k$ times with respect to the arc length s on S and consider the boundary forms

$$(8.1)\quad \begin{aligned} m_{ik}(u) &= D_s^{n-k}D_\nu^{k-1}u \\ &= (\nu_\xi D_t-\nu_\tau D_x)^{n-k}(\nu_\xi D_x+\nu_\tau D_t)^{k-1}u+\sum_{|p|=k}^{n-2} m_{ik,p}D^p u\,, \\ &\qquad k=1, \ldots, i \text{ on } S_i, \quad i=1, \ldots, n\,. \end{aligned}$$

The coefficients of these forms belong to $C^0(S_i)$ because $\nu \in C_2^{n-2}(S_i)$. Here $(\nu_\xi D_t-\nu_\tau D_x)^{n-k}(\nu_\xi D_x+\nu_\tau D_t)^{k-1}u$ shall be interpreted as a homogeneous differential expression of order $n-1$.

In order to see that the assumptions of Theorem II are fulfilled we proceed as follows. The polynomials

$$(8.2)\quad \begin{aligned} \hat{m}_{ik}(\xi, \tau) &= (\nu_\xi\tau-\nu_\tau\xi)^{n-k}(\nu_\xi\xi+\nu_\tau\tau)^{k-1}, \\ &\qquad k=1, \ldots, i \text{ on } S_i, \quad i=1, \ldots, n\,, \end{aligned}$$

associated with the linear forms (8.1), are clearly linearly independent. For in an arbitrary point of S_i, they correspond to i different derivatives of order $n-1$ with respect to the coordinates in the coordinate system defined by the normal and the tangent of S_i at this point. A non-trivial linear combination of the polynomials (8.2),

$$(8.3)\quad \sum_{k=1}^{i} c_{ik}\,\hat{m}_{ik}(\xi, \tau) = (\nu_\xi\tau-\nu_\tau\xi)^{n-i}\sum_{k=1}^{i} c_{ik}(\nu_\xi\tau-\nu_\tau\xi)^{i-k}(\nu_\xi\xi+\nu_\tau\tau)^{k-1}\,,$$

is therefore a non identically vanishing homogeneous polynomial of degree $n-1$. To see that on S_i^- this polynomial does not contain the polynomial $\Pi_{j=1}^{i}(\tau-\alpha_j\xi)$ of degree i as a factor, we have only to observe that, because

$$\sum_{k=1}^{i} c_{ik}(\nu_\xi\tau-\nu_\tau\xi)^{i-k}(\nu_\xi\xi+\nu_\tau\tau)^{k-1}$$

is a polynomial of degree $i-1$, $\tau-\alpha_j\xi$ would have to be a factor of $(\nu_\xi\tau-\nu_\tau\xi)^{n-i}$ for some value of j $(j=1, \ldots, i)$. This would imply

$$\lambda_j = \nu_\tau - \alpha_j \nu_\xi = 0 ,$$

which would contradict condition (b). We have therefore proved that the linear forms $m_{ik}(u)$ $(k=1, \ldots, i)$ satisfy condition (5.8). Similarly we prove that the forms $m_{ik}(u)$ $(k=1, \ldots, i)$ satisfy condition (5.9).

If instead of the "standard" mixed problem we consider the problem of solving equation (6.1) when i consecutive derivatives $D_\nu^{k+d_i-1}u$ $(k=1, \ldots, i)$ of u in the direction of the exterior normal ν of orders $\leqq n-1$ are given on S_i, we can find a simple necessary and sufficient condition on S for the assumptions of Theorem II to be satisfied. In this case, we need only assume that $\nu \in C^{n-d_i-2}(S_i)$ $(i=1, \ldots, n)$. The linear forms associated with the boundary problem are

$$(8.4) \quad \begin{aligned} m_{ik}(u) &= D_s^{n-d_i-k} D_\nu^{k+d_i-1} u \\ &= (\nu_\xi D_t - \nu_\tau D_x)^{n-d_i-k} (\nu_\xi D_x + \nu_\tau D_t)^{k+d_i-1} u + \sum_{|p|=k+d_i}^{n-2} m_{ik,p} D^p u , \\ & \qquad k = 1, \ldots, i; \; d_i \geqq 0 \text{ on } S_i, \quad i = 1, \ldots, n , \end{aligned}$$

and the polynomials $\hat{m}_{ik}(\xi, \tau)$ associated with the linear forms (8.4) are

$$\hat{m}_{ik}(\xi, \tau) = (\nu_\xi \tau - \nu_\tau \xi)^{n-d_i-k} (\nu_\xi \xi + \nu_\tau \tau)^{k+d_i-1} .$$

As before, these polynomials are linearly independent, and we form a non-trivial linear combination of them,

$$\begin{aligned} &\sum_{k=1}^{i} c_{ik} (\nu_\xi \tau - \nu_\tau \xi)^{n-d_i-k} (\nu_\xi \xi + \nu_\tau \tau)^{k+d_i-1} \\ &\quad = (\nu_\xi \tau - \nu_\tau \xi)^{n-d_i-i} (\nu_\xi \xi + \nu_\tau \tau)^{d_i} \sum_{k=1}^{i} c_{ik} (\nu_\xi \tau - \nu_\tau \xi)^{i-k} (\nu_\xi \xi + \nu_\tau \tau)^{k-1} . \end{aligned}$$

In the case $d_i > 0$ it is easily seen that, because $\lambda_j = \nu_\tau - \alpha_j \nu_\xi \neq 0$ $(j=1, \ldots, i)$, this polynomial never contains $\Pi_{j=1}^{i}(\tau - \alpha_j \xi)$ as a factor on S_i^-, that is, condition (5.8) is satisfied if and only if $\nu_\xi + \alpha_j \nu_\tau \neq 0$ $(j=1, \ldots, i)$ on S_i^-. Similarly, condition (5.9) is satisfied in the case $d_i > 0$ if and only if $\nu_\xi + \alpha_j \nu_\tau \neq 0$ $(j=n-i+1, \ldots, n)$ on S_i^+.

Remark. It is clear that the above results still hold for arbitrary $m_{ik}(u)$ with the same principal parts as (8.1), also if the boundary is only continuously differentiable except at a finite number of points. The assumption that $\nu \in C_2^{n-2}(S_i)$ $(i=1, \ldots, n)$ is only needed in order to deduce (8.1) from the original definition of the "standard" problem. The same remark applies to the boundary forms (8.4).

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