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NOTE ON A q-IDENTITY

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Fjeldstad [4] proved the identity

$$(1)\qquad \sum_{s=0}^{2m} (-1)^s \binom{2m}{s} \binom{2n}{n-m+s} \binom{2p}{p-m+s} = (-1)^m \frac{(m+n+p)!\,(2m)!\,(2n)!\,(2p)!}{(m+n)!\,(m+p)!\,(n+p)!\,m!\,n!\,p!}$$

and later Carlitz [2] gave its q-analog

$$(2)\qquad \sum_{s=0}^{2m} (-1)^s \begin{bmatrix}2m\\ s\end{bmatrix} \begin{bmatrix}2n\\ n-m+s\end{bmatrix} \begin{bmatrix}2p\\ p-m+s\end{bmatrix} q^{\frac{1}{2}\{3(s-m)^2+(s-m)\}} = (-1)^m \frac{[m+n+p]!\,[2m]!\,[2n]!\,[2p]!}{[m+n]!\,[m+p]!\,[n+p]!\,[m]!\,[n]!\,[p]!}.$$

Carlitz obtained (2) by specializing Jackson's formula

$$\sum_{s=0}^{2m} (-1)^s \begin{bmatrix}2m\\ s\end{bmatrix} (a)_s (b)_s (a)_{2m-s} (b)_{2m-s} q^{\frac{1}{2}s(2m-s+1)} = (a)_m (b)_m (q^{m+1})_m (abq^m)_m ,$$

where

$$(a)_s = (1-a)(1-aq)\,\ldots\,(1-aq^{s-1}), \qquad (a)_0 = 1 ,$$

$$\begin{bmatrix}n\\ s\end{bmatrix} = \frac{(q)_n}{(q)_s (q)_{n-s}}, \qquad [n]! = (q)_n .$$

Hence it might be of interest to see what is implied by Jackson's more general formula [5]

$$(3)\qquad \sum_{s=0}^{2m} (-1)^s \begin{bmatrix}2m\\ s\end{bmatrix} (a)_s (b)_s (c)_s (d)_s (a)_{2m-s} (b)_{2m-s} (c)_{2m-s} (d)_{2m-s} q^{\frac{1}{2}s(6m-3s+1)}$$
$$= (-1)^m (a)_m (b)_m (c)_m (d)_m (q^{m+1})_m (abq^m)_m (bcq^m)_m (acq^m)_m d^m q^{\frac{1}{2}m(3m-1)} ,$$

where $abcd = q^{1-3n}$.

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Put in (3)

$$a=q^{-m-p_1},\qquad b=q^{-m-p_2},\qquad c=q^{-m-p_3},\qquad \text{and}\qquad d=q^{1+p},$$

where

$$p=p_1+p_2+p_3\,.$$

Hence (3) reduces to

$$(4)\quad \sum_{s=0}^{2m}(-1)^s\begin{bmatrix}2m\\ s\end{bmatrix}\begin{bmatrix}2p_1\\ p_1-m+s\end{bmatrix}\begin{bmatrix}2p_2\\ p_2-m+s\end{bmatrix}\begin{bmatrix}2p_3\\ p_3-m+s\end{bmatrix}\begin{bmatrix}p+s\\ s\end{bmatrix}\begin{bmatrix}p+2m\\ s\end{bmatrix}^{-1}q^{\frac{1}{2}s(3s-6m+1)}$$

$$=(-1)^m\frac{[2p_1]!\,[2p_2]!\,[2p_3]!\,[m+p]!(q^{m+1})_m(q^{p+1})^m}{[m+p_1]!\,[m+p_2]!\,[m+p_3]!\,[p+2m]!\,[p_1]!\,[p_2]!\,[p_3]!}\cdot$$

$$\cdot\frac{[m+p_1+p_2]!\,[m+p_1+p_3]!\,[m+p_2+p_3]!}{[p_1+p_2]!\,[p_1+p_3]!\,[p_2+p_3]!}q^{-\frac{1}{2}m(2p+3m+1)}.$$

In particular if $p_1=p_2=p_3=m$, (4) reduces to

$$(5)\quad \sum_{s=0}^{2m}(-1)^s\begin{bmatrix}2m\\ s\end{bmatrix}^4\begin{bmatrix}3m+s\\ s\end{bmatrix}\begin{bmatrix}5m\\ s\end{bmatrix}^{-1}q^{\frac{1}{2}s(3s-6m+1)}$$

$$=(-1)^m\begin{bmatrix}3m\\ m\end{bmatrix}^3\frac{[4m]!\,(q^{m+1})_m\,(q^{p+1})^m}{[5m]!}q^{-\frac{1}{2}m(9m+1)}.$$

Now if we take the limit as $q\to 1$, formulas (4) and (5) reduce respectively to

$$(6)\quad \sum_{s=0}^{2m}(-1)^s\binom{2m}{s}\binom{2p_1}{p_1-m+s}\binom{2p_2}{p_2-m+s}\binom{2p_3}{p_3-m+s}\binom{p+s}{s}\binom{p+2m}{s}^{-1}$$

$$=(-1)^m\frac{(2p_1)!\,(2p_2)!\,(2p_3)!\,(m+p)!\,(2m)!\,(m+p_1+p_2)!}{(m+p_1)!\,(m+p_2)!\,(m+p_3)!\,(p+2m)!\,p_1!\,p_2!\,p_3!}\cdot$$

$$\cdot\frac{(m+p_1+p_3)!\,(m+p_2+p_3)!}{(p_1+p_2)!\,(p_1+p_3)!\,(p_2+p_3)!},$$

$$(7)\quad \sum_{s=0}^{2m}(-1)^s\binom{2m}{s}^4\binom{3m+s}{s}\binom{5m}{s}^{-1}=(-1)^m\binom{3m}{m}^3\frac{(4m)!\,(2m)!}{(5m)!\,m!}.$$

Finally we remark that if we let $p_3\to\infty$, then formula (6) reduces to Fjeldstad's formula (1), and (4) reduces to (2).

It is not clear whether (6) follows from Dougall's theorem (see [3] and [1, § 4.3]).

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