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ON SOME FUNDAMENTAL PROBLEMS CONCERNING ISOMORPHISM OF BOOLEAN ALGEBRAS

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The main notions which will be discussed in this note are those of direct product and isomorphism.[1] The direct product of two algebraic systems $\mathfrak{A}$ and $\mathfrak{B}$ will be denoted as usual by $\mathfrak{A} \times \mathfrak{B}$; instead of $\mathfrak{A} \times \ldots \times \mathfrak{A}$ (with n factors) we shall write $\mathfrak{A}^n$. The fact that two algebraic systems $\mathfrak{A}$ and $\mathfrak{B}$ are, or are not, isomorphic will be expressed by the formula $\mathfrak{A} \cong \mathfrak{B}$, or $\mathfrak{A}\, \text{non-}\!\cong \mathfrak{B}$, respectively.

Until recently, various fundamental problems concerning isomorphism of direct products of algebraic systems remained unsolved. Thus, for instance, for various familiar classes $\boldsymbol{K}$ of algebraic systems, such as Boolean algebras and groups, the following problems were open:

Problem I. Does $\mathfrak{A} \cong \mathfrak{A} \times \mathfrak{B} \times \mathfrak{C}$ imply $\mathfrak{A} \cong \mathfrak{A} \times \mathfrak{B}$ for arbitrary systems $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ in $\boldsymbol{K}$? Or, in an equivalent formulation: If each of two systems of $\boldsymbol{K}$ is isomorphic to a direct factor of the other, does it follow that the two systems are isomorphic?

Problem II. Does $\mathfrak{A}^2 \cong \mathfrak{B}^2$ imply $\mathfrak{A} \cong \mathfrak{B}$ for any $\mathfrak{A}$ and $\mathfrak{B}$ in $\boldsymbol{K}$?[2]

A. Tarski suggested the following related problem:

Problem III. Does $\mathfrak{A} \cong \mathfrak{A} \times \mathfrak{B} \times \mathfrak{B}$ imply $\mathfrak{A} \cong \mathfrak{A} \times \mathfrak{B}$ for any $\mathfrak{A}$ and $\mathfrak{B}$ in $\boldsymbol{K}$?

Obviously, if Problem III is solved negatively, then so is Problem I. Tarski noticed that such a solution of Problem III also leads to a nega-

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[1] The results in this note were first stated in abstracts [1], [2], and [14]. Grateful acknowledgement is made to Professor Anne C. Davis and Professor Alfred Tarski for encouragement and assistance in the preparation of this paper. A large part of the work on this paper was done when the author was engaged in a research project (in the University of California, Berkeley) on the foundations of mathematics directed by Professor Tarski and sponsored by the National Science Foundation.

[2] For Boolean algebras, Problem I was stated by Sikorski [8, p. 242]; Problems I and II were stated independently and about the same time by Tarski [10, p. 100]. In the appendix to [11], pp. 311 f., these two problems are discussed for arbitrary algebraic systems and it is emphasized there that they are open for both Boolean algebras and groups. In Kaplanski [5], Problems I and II are listed among the three test problems on the structure of Abelian groups.

tive solution of Problem II. He also pointed out that it would be especially interesting to solve Problem III taking for $\mathfrak{B}$ either a finite system, if possible a two element system (Problem III′), or else the system $\mathfrak{A}$ itself (Problem III″).

Problems I and II (and hence also III) have been solved affirmatively only for rather special classes of algebraic systems. Thus for complete and atomistic Boolean algebras, Problems I and II reduce to some general set-theoretical problems whose solution has been known for a long time; the solution of Problem I is provided by the well-known Cantor-Bernstein theorem. More generally, these two problems have been solved affirmatively for countably complete Boolean algebras.[1]

For arbitrary Boolean algebras and, in fact, for denumerable Boolean algebras, a negative solution of Problem I was obtained a few years ago by S. Kinoshita (see [6]). Tarski suggested that it might be possible to modify Kinoshita's construction to obtain also a negative solution to Problem III. This indeed turned out to be the case and the result is stated in Theorem 1 below; the resulting negative solution of Problem II is contained in Theorem 2. R. L. Vaught has shown that the solution of Problem III′ for $\mathfrak{A}$ a denumerable Boolean algebra is affirmative; we reproduce this result in Theorem 3. Problem III″ for denumerable Boolean algebras still remains open. On the other hand, for the class of all Boolean algebras, the solution of both Problems III′ and III″ is negative and will be given here in Theorems 4 and 5. The Boolean algebra $\mathfrak{A}$ involved in these counterexamples is of the power of the continuum. The constructions used in Theorems 1, 4, and 5 can be applied to the solution of several other related problems. Some of these applications are mentioned in remarks following Theorem 5. One of them, due to C. C. Chang, is stated explicitely in Theorem 6; by this theorem, there are two partially ordered systems $\mathfrak{A}$ and $\mathfrak{B}$ and a finite partially ordered system $\mathfrak{C}$ such that $\mathfrak{A} \times \mathfrak{C} \cong \mathfrak{B} \times \mathfrak{C}$ but $\mathfrak{A}$ non-$\cong \mathfrak{B}$.[2] The articles Tarski [13] and Jónsson [4] immediately following this note are closely related to it in content. In [13] it is indicated how Theorems 1, 2, 4, and 5 can be used to obtain analogous results for commutative semi-groups. In [4] these theorems are applied to solve Problems I–III

[1] A solution of Problem I for countably complete Boolean algebras can be found in [8], [10], and in [11]; [10] and [11] contain also a solution of Problem II for the same class of algebras. (In [11] see pp. 215f, in particular Theorem 15.27.) Concerning the affirmative solution of these problems for other classes of algebras see remarks in [11, pp. 311f.].

[2] The results of R. L. Vaught and C. C. Chang mentioned above are published here for the first time (naturally, with the permission of the authors).

negatively for (non-Abelian) groups; [4] also contains some general formulations which comprehend the results for Boolean algebras, commutative semi-groups, and groups as particular cases. Finally we should like to mention that the article Jónsson [3] contains the negative solution of Problem II for Abelian groups.

The proofs of Theorems 1 and 3 which follow make essential use of the notion of a Boolean algebra with ordered basis.[1] We distinguish between a Boolean algebra $\mathfrak{B}$ and the set B of its elements although we often speak of an element of $\mathfrak{B}$ when we mean an element of B. A subset A of B will be said to form an ordered basis of the Boolean algebra $\mathfrak{B}$ if (i) A contains the zero but not the unit element of $\mathfrak{B}$, (ii) A generates $\mathfrak{B}$, that is, every element of B is a finite union of finite intersections of elements of A and their complements, and (iii) A is simply ordered by the inclusion relation of $\mathfrak{B}$. Let Γ be the set of all order types α of the form $1+\beta$, that is, α is the type of a simply ordering relation which has a first element. It is well known that every denumerable Boolean algebra has an ordered basis; this easily follows, for instance, from the topological representation of Boolean algebras to which we shall refer below. Furthermore, given an order type $\alpha \in \Gamma$, there exists a Boolean algebra $\mathfrak{B}$ having an ordered basis of type α (and $\mathfrak{B}$ will be denumerable just in case α is). In fact, we can construct $\mathfrak{B}$ as follows: Let A be a set which is of order type α under a relation $\leq$. Let B be the set of all subsets of A which are finite unions of intervals $[a, b)$ of A (by $[a, b)$ we mean the set of all $x \in A$ such that $a \leq x < b$). Clearly B forms a Boolean algebra under the usual set-theoretical operations. Furthermore, since $\alpha \in \Gamma$, A has a first element a_0 and the set of all intervals $[a_0, b)$ forms an ordered basis of type α for $\mathfrak{B}$.

It is easily seen that, although any two Boolean algebras with ordered bases of the same type are isomorphic, a Boolean algebra may have ordered bases of unequal order type. We shall write $\alpha \approx \beta$ if some Boolean algebra has both an ordered basis of type α and one of type β. If Boolean algebras $\mathfrak{A}$ and $\mathfrak{B}$ have ordered bases of type α and β respectively, then the direct product $\mathfrak{A} \times \mathfrak{B}$ has an ordered basis of type $\alpha+\beta$. From this we see that, for each α, $\beta \in \Gamma$, $\alpha+\beta \approx \beta+\alpha$. We will also make use of the formula

[1] A discussion of Boolean algebras with ordered bases is given in Mostowski-Tarski [7] although the notation and terminology of that paper are not followed in detail here. Most of the observations concerning Boolean algebras with ordered bases which are made in this and the next paragraph originate with Mostowski and Tarski even if they are not included in [7].

$$\sum_{n\in\omega} \alpha_n + \delta \approx \sum_{n\in\omega} \beta_n + \delta$$

which holds whenever α_n, $\beta_n \in \Gamma$ and $\alpha_n \approx \beta_n$, for each $n \in \omega$, and δ is an arbitrary order type. This statement can be proved by exhibiting the natural isomorphism which exists between the two Boolean algebras involved.

If $\mathfrak{A}$ is a Boolean algebra and a is one of its elements, we will denote by $\mathfrak{A}[a]$ the Boolean algebra formed by the principal ideal generated by a, that is, the Boolean algebra formed by the set of all elements of $\mathfrak{A}$ which are included in a and having the same operations as $\mathfrak{A}$ except that complementation is taken relative to a. Throughout the following discussion, ω will denote the set of natural numbers as well as the order type of the natural numbers under $\leq$. The order type of the rational numbers under $\leq$ will be denoted by η.

THEOREM 1. *There exist denumerable Boolean algebras $\mathfrak{A}$ and $\mathfrak{B}$ such that $\mathfrak{A} \cong \mathfrak{A} \times \mathfrak{B}^2$ but $\mathfrak{A}$ non-$\cong \mathfrak{A} \times \mathfrak{B}$. More generally, given a positive integer n, there exist denumerable Boolean algebras $\mathfrak{A}$ and $\mathfrak{B}$ such that, for each positive integer m, $\mathfrak{A}^m \cong \mathfrak{A}^m \times \mathfrak{B}^k$ just in case k is a multiple of n.*

PROOF.[1] Suppose n is given. We set

$$\varrho_i = \omega^{i+1} + \eta, \qquad \sigma_i = \sum_{j\in\omega} \varrho_{i+j}, \qquad \tau_i = \sum_{j\in\omega} (\varrho_{i+j}\cdot n)\ ,$$

for each $i \in \omega$. Let $\mathfrak{A}$ be a Boolean algebra with ordered basis of type α where

$$\alpha = \tau_0\cdot\omega + \sum_{i\in\omega} (\sigma_{i+1}\cdot n + \tau_{i+1})\ ,$$

and let $\mathfrak{B}$ be a Boolean algebra with ordered basis of type σ_0.

To show $\mathfrak{A}^m \cong \mathfrak{A}^m \times \mathfrak{B}^k$ whenever k is a multiple of n, it will clearly suffice to show that $\mathfrak{A} \cong \mathfrak{A} \times \mathfrak{B}^n$. Since $\omega^{i+1} = 1 + \omega^{i+1}$, all the order types defined above are elements of Γ. Furthermore $\tau_i = \varrho_i\cdot n + \tau_{i+1}$ and $\sigma_i = \varrho_i + \sigma_{i+1}$. Thus we have

$$\begin{aligned}\tau_i + \sigma_{i+1}\cdot n &= \varrho_i\cdot n + \tau_{i+1} + \sigma_{i+1}\cdot n\\ &\approx (\varrho_i + \sigma_{i+1})\cdot n + \tau_{i+1}\\ &= \sigma_i\cdot n + \tau_{i+1}\end{aligned}$$

and hence

[1] The proof of Theorem 1 was originally formulated in topological terms as was the proof of the analogous result of Kinoshita. It was later transposed (at the suggestion of Tarski) into an algebraic form using essentially the notion of an ordered basis. This has resulted in a considerable simplification of the fundamental construction but has involved some complications in other parts of the proof.

$$\sum_{i\in\omega}(\tau_i+\sigma_{i+1}\cdot n)\approx\sum_{i\in\omega}(\sigma_i\cdot n+\tau_{i+1})\,.$$

Making use of the general associative law, it follows from this that

$$\begin{aligned}\alpha &= \tau_0\cdot\omega+\sum_{i\in\omega}(\sigma_{i+1}\cdot n+\tau_{i+1})\\ &= (\tau_0+\tau_0\cdot\omega)+\sum_{i\in\omega}(\sigma_{i+1}\cdot n+\tau_{i+1})\\ &\approx \tau_0\cdot\omega+\left[\tau_0+\sum_{i\in\omega}(\sigma_{i+1}\cdot n+\tau_{i+1})\right]\\ &= \tau_0\cdot\omega+\sum_{i\in\omega}(\tau_i+\sigma_{i+1}\cdot n)\\ &\approx \tau_0\cdot\omega+\sum_{i\in\omega}(\sigma_i\cdot n+\tau_{i+1})\\ &= \tau_0\cdot\omega+\left[\sigma_0\cdot n+\sum_{i\in\omega}(\tau_{i+1}+\sigma_{i+1}\cdot n)\right]\\ &\approx \tau_0\cdot\omega+\sum_{i\in\omega}(\sigma_{i+1}\cdot n+\tau_{i+1})+\sigma_0\cdot n\\ &= \alpha+\sigma_0\cdot n\,.\end{aligned}$$

Thus $\mathfrak{A}$ and $\mathfrak{A}\times\mathfrak{B}^n$ have ordered bases of equivalent types and so are isomorphic.

It remains now to show that if $\mathfrak{A}^m\cong\mathfrak{A}^m\times\mathfrak{B}^k$, then k is a multiple of n. Suppose $\mathfrak{A}^m\cong\mathfrak{A}^m\times\mathfrak{B}^k$. Let $\mathfrak{C}$ be a Boolean algebra constructed from an ordered set of type $\alpha\cdot m+\sigma_0\cdot k$ by the method described in the remarks preceeding this theorem; $\mathfrak{C}$ is thus isomorphic to $\mathfrak{A}^m\times\mathfrak{B}^k$. By the type of an interval of $\mathfrak{C}$ we mean its type under the relation $\leq$ of the ordered set. By a maximal interval of a given type we mean an interval which is not properly included in any interval of the same type; $\mathfrak{C}$ clearly has maximal intervals of type ϱ_i, σ_i, τ_i, α, etc., as well as intervals of type ω, η, ϱ_i, etc., which are not maximal. Let a be the maximal interval of type $\alpha\cdot m$. $\mathfrak{A}^m$ is then isomorphic to $\mathfrak{C}[a]$. Let F be a function mapping $\mathfrak{C}[a]$ isomorphically onto $\mathfrak{C}$.

First we notice that for any element x of the Boolean algebra $\mathfrak{C}$, either (i) for some $i\in\omega$, an interval of type ϱ_i is included in x or (ii) for some elements y and z of $\mathfrak{C}$, $x=y\cup z$ and the algebra $\mathfrak{C}[y]$ is atomistic while $\mathfrak{C}[z]$ is atomless. This follows from the fact that every interval of $\mathfrak{C}$ satisfies (i) or (ii) and every element of $\mathfrak{C}$ is a finite union of intervals. Thus we see that given an interval x of type ϱ_j in $\mathfrak{C}[a]$, the element $F(x)$ must contain an interval of type ϱ_i for some $i\in\omega$; for if it did not, then $F(x)$ and hence x would satisfy (ii) which is impossible since if $x=y\cup z$ for some y and z of $\mathfrak{C}$, then either y or z contains in turn an interval of type ϱ_j. We wish now to show that i must equal j, that is, that the image under F of an interval of type ϱ_j must contain an interval of the same type.

By the derivative of a Boolean algebra we mean its quotient algebra modulo the ideal of all finite sums of atoms, that is, the algebra formed by identifying elements which differ by a finite sum of atoms. The derivative of a Boolean algebra with ordered basis of type $\omega^{i+1}+\eta$ has an ordered basis of type $\omega^i+\eta$. Thus if we take $j+1$ successive derivatives of $\mathfrak{C}[x]$ (where x is an interval of type ϱ_j), we obtain a Boolean algebra with ordered basis of type $1+\eta$, that is, an atomless Boolean algebra. Therefore the $(j+1)$st derivative of $\mathfrak{C}[F(x)]$ must also be atomless and so $\mathfrak{C}[F(x)]$ cannot contain any interval of type ϱ_i for any $i>j$. Thus $i\leq j$. By considering the function F^{-1} mapping $\mathfrak{C}$ isomorphically onto $\mathfrak{C}[a]$, we obtain $j\leq i$. Hence we obtain the desired conclusion that $j=i$.

Let b be the element of $\mathfrak{C}$ which is the union of the m maximal intervals of type $\tau_0\cdot\omega$. Let p_i be the number of maximal intervals x of type ϱ_i included in b and such that $F(x)$ contains an interval of type ϱ_i which is disjoint from b. Similarly, let q_i be the number of maximal intervals x of type ϱ_i which are disjoint from b but are such that $F(x)$ contains an interval of type ϱ_i included in b. Now there are $2ni$ maximal intervals of type ϱ_i in the algebra $\mathfrak{C}[a]$ which are disjoint from b. Thus there are $2ni-q_i$ maximal intervals x of type ϱ_i which are disjoint from b and are such that $F(x)$ contains an interval of type ϱ_i which is disjoint from b. For each maximal interval y of type ϱ_i in $\mathfrak{C}$, there can be only one maximal interval x of type ϱ_i in $\mathfrak{C}[a]$ such that $F(x)$ contains an interval of type ϱ_i included in y. Therefore the number, $2ni-q_i+p_i$, of maximal intervals x of type ϱ_i for which $F(x)$ contains an interval of type ϱ_i disjoint from b must equal $2ni+k$, the total number of maximal intervals of type ϱ_i which are disjoint from b. Thus $p_i=q_i+k$ for each $i\in\omega$.

Let u be the intersection of b with the complement of the inverse image of b under F; that is,

$$u = b\cap\overline{F^{-1}(b)}\,.$$

It is easy to see that p_i is the number of maximal intervals of type ϱ_i which contain an interval of type ϱ_i included in u. Now u is the finite union of intervals v each of which is included in b and hence has the property that there are at most two natural numbers i (corresponding to the two endpoints) such that the number of maximal intervals of type ϱ_i which contain an interval of type ϱ_i included in v is not a multiple of n. From this we conclude that there are only finitely many $i \in \omega$ such that p_i is not a multiple of n. Applying a similar argument to the element $b \cap \overline{F(b)}$ we obtain that there are only finitely many $i \in \omega$ such that q_i is not a multiple of n. Therefore, for some $i \in \omega$, both p_i and q_i are multiples of n. Hence we obtain the desired conclusion that $k = p_i - q_i$ is a multiple of n, which completes the proof of Theorem 1.

THEOREM 2 (A. Tarski). *There exist denumerable Boolean algebras* $\mathfrak{A}$ *and* $\mathfrak{C}$ *such that* $\mathfrak{A}^2 \cong \mathfrak{C}^2$ *but* $\mathfrak{A}$ non-$\cong \mathfrak{C}$. *More generally, given a positive integer* n, *there exist denumerable Boolean algebras* $\mathfrak{A}$ *and* $\mathfrak{C}$ *such that for every positive integer* k, $\mathfrak{A}^k \cong \mathfrak{C}^k$ *in case* k *is a multiple of* n, *and* $\mathfrak{A}^k$ non-$\cong \mathfrak{C}^k$ *otherwise.*

PROOF. This is an immediate consequence of Theorem 1 obtained by letting $\mathfrak{C} \cong \mathfrak{A} \times \mathfrak{B}$ (and by taking $m = k$ in the second part of that theorem).

THEOREM 3 (R. L. Vaught). *Let* $\mathfrak{A}$ *be a denumerable Boolean algebra or, more generally, a Boolean algebra with ordered basis.*

(i) *If* $\mathfrak{A}$ *has infinitely many atoms and* $\mathfrak{B}$ *is any finite Boolean algebra, then* $\mathfrak{A} \cong \mathfrak{A} \times \mathfrak{B}$.

(ii) *If* $\mathfrak{B}$ *and* $\mathfrak{C}$ *are finite Boolean algebras and* $\mathfrak{A} \cong \mathfrak{A} \times \mathfrak{B} \times \mathfrak{C}$, *then* $\mathfrak{A} \cong \mathfrak{A} \times \mathfrak{B} \cong \mathfrak{A} \times \mathfrak{C}$.

PROOF. (i) If B is an ordered basis for $\mathfrak{A}$, then corresponding to each atom of $\mathfrak{A}$ there is an element b of B such that b has an immediate successor b' in B. Therefore, there is either an infinite monotonically increasing or an infinite monotonically decreasing sequence of elements of B each of which has an immediate successor. We may assume without loss of generality that the sequence is increasing. Thus the set B is of order type

$$\beta = \sum_{i \in \omega} (\alpha_i + 1) + \delta$$

for some order types α_i and δ where α_i is an element of Γ (that is, α_i is the type of an ordered set which has a first element), for each $i \in \omega$. Thus we have

$$\begin{aligned} \beta &= \sum_{i \in \omega} (\alpha_i + 1) + \delta \\ &\approx \sum_{i \in \omega} (1 + \alpha_i) + \delta \\ &= 1 + \sum_{i \in \omega} (\alpha_i + 1) + \delta \\ &= 1 + \beta . \end{aligned}$$

Hence $\mathfrak{A}$ is isomorphic to its direct product with a two element Boolean algebra (which has an ordered basis of type 1) and the conclusion (1) follows by finite induction.

(ii) If now $\mathfrak{A} \cong \mathfrak{A} \times \mathfrak{B} \times \mathfrak{C}$ where $\mathfrak{B}$ and $\mathfrak{C}$ are finite (and hence not atomless) Boolean algebras, then $\mathfrak{A}$ obviously has infinitely many atoms and the conclusion (ii) follows immediately by part (i) of our theorem.

In the subsequent discussion, we will use the following notation: For any natural numbers n and i, let $a_{n,i}$ be the set consisting of the natural numbers $in, in+1, \ldots, in+(n-1)$. For each positive integer n let M_n be the family of all those subsets x of ω such that, for some $y \subseteq \omega$ and some finite $z \subseteq \omega$,

$$x = \bigcup_{i \in y} a_{n,i} \cup z .$$

It is easily seen that the set M_n, under the operations of union, $\cup$, intersection, $\cap$, and complementation with respect to ω, $\bar{\ }$, forms a Boolean algebra. We denote this algebra by $\mathfrak{M}_n$. $\mathfrak{M}_n$ is easily seen to be infinite (the power of the continuum) and atomistic, the atoms of the algebra being $a_{1,i}$ (the set consisting of just the single integer i) for each $i \in \omega$.

We now state in a series of lemmas the properties of $\mathfrak{M}_n$ which we shall need. In the following, $\mathfrak{T}$ will denote a two element Boolean algebra.

LEMMA 1. $\mathfrak{M}_n \cong \mathfrak{M}_n \times \mathfrak{T}^n$.

PROOF. Clearly it will suffice to show that an algebra obtained from $\mathfrak{M}_n$ by deleting n atoms is isomorphic to $\mathfrak{M}_n$. We will show therefore that $\mathfrak{M}_n[\overline{a_{n,0}}]$ is isomorphic to $\mathfrak{M}_n$. Let F be a function such that for each element x of the algebra $\mathfrak{M}_n[\overline{a_{n,0}}]$, $F(x)$ is the set of all those natural numbers i such that $i+n \in x$. It is easy to verify that F maps $\mathfrak{M}_n[\overline{a_{n,0}}]$ isomorphically onto $\mathfrak{M}_n$.

LEMMA 2. $\mathfrak{M}_n$ non-$\cong \mathfrak{M}_n \times \mathfrak{T}^k$ *for* $0 < k < n$.

PROOF. It will suffice to show that $\mathfrak{M}_n[\overline{a_{k,0}}]$ is not isomorphic to $\mathfrak{M}_n$. To do this, we will show that $\mathfrak{M}_n$ has a certain algebraic property which $\mathfrak{M}_n[\overline{a_{k,0}}]$ doesn't have. If B is the set of elements of a Boolean algebra $\mathfrak{B}$, then a subset A of B will be said to be a *complete partition* of $\mathfrak{B}$ if (i) Any two elements of A are disjoint, (ii) The least upper bound of A is the unit element of $\mathfrak{B}$, and (iii) If C is a subset of A, then the least upper bound of C is an element of $\mathfrak{B}$. It is clear that $\mathfrak{M}_n$ has a complete partition A each element of which is a sum of exactly n atoms, for we can take A to be the set of all $a_{n,i}$ for $i \in \omega$. We wish now to show that

$\mathfrak{M}_n[\overline{a_{k,0}}]$ has no complete partition each element of which is the sum of exactly n atoms.

Suppose, on the contrary, that A is such a partition of $\mathfrak{M}_n[\overline{a_{k,0}}]$. Let C be the set of all $x \in A$ such that $x \neq a_{n,i}$ for each $i \in \omega$. Suppose C is finite. Let c be the sum of elements of C. The number of elements of c is then finite and a multiple of n since it is the union of a finite number of disjoint sets each of which contains n elements. But c is also the union of a finite number of sets $a_{n,j}$ together with the set $a_{n,0}-a_{k,0}$ (which has $n-k$ elements) and so the number of elements of c cannot be a multiple of n. Therefore C must be infinite.

We will now show that there exists an infinite sequence

$$c_0, c_1, \ldots, c_j, \ldots$$

of distinct elements of C such that, for each $j \in \omega$,

$$(1) \qquad a_{n,i} \nsubseteq c_0 \cup c_1 \cup \ldots \cup c_j \qquad \text{for every} \quad i \in \omega \, .$$

Let c_0 be any element of C. Clearly (1) holds for $j=0$, since c_0 and $a_{n,i}$ each have n elements but they cannot be equal since $c_0 \in C$. Suppose now we have a sequence $c_0, c_1, \ldots, c_m$ such that (1) holds for all $j \leq m$. There are only finitely many $i \in \omega$ such that $a_{n,i}$ intersects $c_0 \cup c_1 \cup \ldots \cup c_m$ and so we can find an element c_{m+1} of C which does not intersect any of these $a_{n,i}$. With this choice of c_{m+1}, (1) is extended to hold for $j \leq m+1$. Proceeding in this way we obtain the desired sequence.

Let d be the union of $c_0, c_1, \ldots, c_j, \ldots$; d is then an element of $\mathfrak{M}_n[\overline{a_{k,0}}]$ since A is a complete partition of $\mathfrak{M}_n[\overline{a_{k,0}}]$ and each c_j is in A. Hence, for some $y \subseteq \omega$ and some finite $z \subseteq \omega$,

$$d = \bigcup_{i \in y} a_{n,i} \cup z \, .$$

But y must be empty since if $a_{n,i}$ were included in d, then it would be included in a finite union of the c_j contrary to (1). Therefore $d=z$. This is impossible since d is infinite and z is finite. Therefore, $\mathfrak{M}_n[\overline{a_{k,0}}]$ can have no complete partition each element of which is a sum of n atoms and so it cannot be isomorphic to $\mathfrak{M}_n$.

In connection with the foregoing proof, B. Jónsson noticed that the structures of the two algebras $\mathfrak{M}_n$ and $\mathfrak{M}_n[\overline{a_{k,0}}]$ can also be distinguished by means of the following rather simple and natural property: The algebra $\mathfrak{M}_n$ has an automorphism F such that F^n is the identity automorphism and no atom is a fixpoint of the automorphism F^m for $0<m<n$. $\mathfrak{M}[\overline{a_{k,0}}]$ has no such automorphism.

LEMMA 3. $\mathfrak{M}_n \cong \mathfrak{M}_n \times \mathfrak{M}_n$.

PROOF. For each element x of $\mathfrak{M}_n$, let $F_1(x)$ be the set of all natural numbers of the form $in+k$ where $0 \leq k < n$ and $2in+k \in x$. Let $F_2(x)$ be defined analogously but with the condition $2in+k \in x$ replaced by $(2i+1)n+k \in x$. Let $F(x)$ be the ordered pair $\langle F_1(x), F_2(x)\rangle$. It is easy to verify that F maps $\mathfrak{M}_n$ isomorphically onto $\mathfrak{M}_n \times \mathfrak{M}_n$.

THEOREM 4. *There exists a Boolean algebra* $\mathfrak{A}$ *such that* $\mathfrak{A} \cong \mathfrak{A} \times \mathfrak{T}^n$ *but* $\mathfrak{A}$ non-$\cong \mathfrak{A} \times \mathfrak{T}^k$ *for* $k = 1, 2, \ldots, n-1$. (*As before,* $\mathfrak{T}$ *is a two element Boolean algebra.*)

PROOF. Taking $\mathfrak{A} = \mathfrak{M}_n$, the result follows directly from Lemmas 1 and 2.

THEOREM 5. *There exists a Boolean algebra* $\mathfrak{A}$ *such that* $\mathfrak{A} \cong \mathfrak{A}^n$ *but* $\mathfrak{A}$ non-$\cong \mathfrak{A}^k$ *for* $k = 2, 3, \ldots, n-1$.

PROOF. Take $\mathfrak{A} = \mathfrak{M}_{n-1} \times \mathfrak{T}$. By induction, we obtain from Lemma 3 that

$$(\mathfrak{M}_{n-1})^k \cong \mathfrak{M}_{n-1} .$$

Hence

$$\mathfrak{A}^k = (\mathfrak{M}_{n-1} \times \mathfrak{T})^k \cong (\mathfrak{M}_{n-1})^k \times \mathfrak{T}^k \cong \mathfrak{M}_{n-1} \times \mathfrak{T}^k . \tag{1}$$

For $k = n$ we have therefore from the foregoing and Lemma 1 that

$$\mathfrak{A}^n \cong \mathfrak{M}_{n-1} \times \mathfrak{T}^n \cong (\mathfrak{M}_{n-1} \times \mathfrak{T}^{n-1}) \times \mathfrak{T} \cong \mathfrak{M}_{n-1} \times \mathfrak{T} .$$

Thus $\mathfrak{A} \cong \mathfrak{A}^n$. Suppose now that $\mathfrak{A} \cong \mathfrak{A}^k$ for $1 < k < n$. We then have from (1) and the definition of $\mathfrak{A}$ that

$$\mathfrak{M}_{n-1} \times \mathfrak{T} \cong \mathfrak{M}_{n-1} \times \mathfrak{T}^k ,$$

$$\mathfrak{M}_{n-1} \times \mathfrak{T} \times \mathfrak{T}^{n-2} \cong \mathfrak{M}_{n-1} \times \mathfrak{T}^k \times \mathfrak{T}^{n-2} ,$$

$$\mathfrak{M}_{n-1} \times \mathfrak{T}^{n-1} \cong (\mathfrak{M}_{n-1} \times \mathfrak{T}^{n-1}) \times \mathfrak{T}^{k-1} .$$

Hence by Lemma 1, we obtain from this

$$\mathfrak{M}_{n-1} \cong \mathfrak{M}_{n-1} \times \mathfrak{T}^{k-1} .$$

This is contrary to Lemma 2. Therefore $\mathfrak{A}$ non-$\cong \mathfrak{A}^k$.

From the constructions given in Theorems 1 and 4, we obtain the following further results:

There exist denumerable Boolean algebras $\mathfrak{A}_1$, $\mathfrak{A}_2$, *and* $\mathfrak{B}$ *such that* $\mathfrak{A}_1 \times \mathfrak{A}_2 \cong \mathfrak{A}_1 \times \mathfrak{A}_2 \times \mathfrak{B}$ *but* $\mathfrak{B}$ *is not isomorphic to a direct product* $\mathfrak{B}_1 \times \mathfrak{B}_2$ *where* $\mathfrak{A}_1 \cong \mathfrak{A}_1 \times \mathfrak{B}_1$ *and* $\mathfrak{A}_2 \cong \mathfrak{A}_2 \times \mathfrak{B}_2$.

There exist (non-denumerable) Boolean algebras $\mathfrak{A}_1$ *and* $\mathfrak{A}_2$ *such that* $\mathfrak{A}_1 \times \mathfrak{A}_2 \cong \mathfrak{A}_1 \times \mathfrak{A}_2 \times \mathfrak{T}$ *but neither* $\mathfrak{A}_1 \cong \mathfrak{A}_1 \times \mathfrak{T}$ *nor* $\mathfrak{A}_2 \cong \mathfrak{A}_2 \times \mathfrak{T}$.

To obtain the first of these results, we take for $\mathfrak{A}_1$ the algebra $\mathfrak{A}$ of Theorem 1 for the case $n = 2$, for $\mathfrak{A}_2$ we take the algebra $\mathfrak{A}$ of Theorem 1 for the case $n = 3$, and for $\mathfrak{B}$ we take the algebra $\mathfrak{B}$ of Theorem 1 (which is independent of n). The proof is a modification of the proof of Theorem 1. To obtain the second of the above results, we take for $\mathfrak{A}_1$ the algebra $\mathfrak{M}_2$ and for $\mathfrak{A}_2$ the algebra $\mathfrak{M}_3$; the result then follows directly from Lemmas 1 and 2.

It is well known that Boolean algebras can be considered as special cases of partially ordered systems. C. C. Chang has shown that Theorem 4 leads to the following theorem concerning such systems.

THEOREM 6 (C. C. Chang). *There exist partially ordered systems* $\mathfrak{A}$ *and* $\mathfrak{B}$ *and a finite partially ordered system* $\mathfrak{C}$ *such that* $\mathfrak{A} \times \mathfrak{C} \cong \mathfrak{B} \times \mathfrak{C}$ *but* $\mathfrak{A}$ non-$\cong \mathfrak{B}$.[1]

PROOF. Let $\mathfrak{U}$ be a one element partially ordered system whose element is distinct from each of the two elements of the Boolean algebra $\mathfrak{T}$. Take $\mathfrak{A} \cong \mathfrak{M}_2$, $\mathfrak{B} \cong \mathfrak{M}_2 \times \mathfrak{T}$, and $\mathfrak{C} \cong \mathfrak{U} + \mathfrak{T}$ (the cardinal sum of $\mathfrak{U}$ and $\mathfrak{T}$). By applying Lemma 1 (and various elementary formal laws involving $\times$ and $+$), we obtain

$$
\begin{aligned}
\mathfrak{A} \times \mathfrak{C} &\cong \mathfrak{M}_2 \times (\mathfrak{U} + \mathfrak{T}) \\
&\cong (\mathfrak{M}_2 \times \mathfrak{U}) + (\mathfrak{M}_2 \times \mathfrak{T}) \\
&\cong \mathfrak{M}_2 + (\mathfrak{M}_2 \times \mathfrak{T}) \\
&= (\mathfrak{M}_2 \times \mathfrak{T}^2) + (\mathfrak{M}_2 \times \mathfrak{T}) \\
&\cong [(\mathfrak{M}_2 \times \mathfrak{T}) \times \mathfrak{T}] + (\mathfrak{M}_2 \times \mathfrak{T}) \times \mathfrak{U}] \\
&\cong (\mathfrak{M}_2 \times \mathfrak{T}) \times (\mathfrak{T} + \mathfrak{U}) \\
&\cong (\mathfrak{M}_2 \times \mathfrak{T}) \times (\mathfrak{U} + \mathfrak{T}) \\
&\cong \mathfrak{B} \times \mathfrak{C}.
\end{aligned}
$$

And by Lemma 2, $\mathfrak{A}$ non-$\cong \mathfrak{B}$.

[1] Before Chang made the observation formulated in Theorem 5, it was not even known whether there are three reflexive relational systems $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ (each formed by a set of elements and a reflexive relation) such that $\mathfrak{C}$ is finite and $\mathfrak{A} \times \mathfrak{C} \cong \mathfrak{B} \times \mathfrak{C}$ but $\mathfrak{A}$ non-$\cong \mathfrak{B}$. On the other hand, an example found in [11, pp. 310f.] provides three finite and, in fact, two-element relational systems with these properties; in this example, however, the systems $\mathfrak{B}$ and $\mathfrak{C}$ (which actually coincide) are not reflexive.

By a well-known theorem of Stone (see [9]), theorems about Boolean algebras can be translated into theorems about topological spaces and, conversely, certain theorems about topological spaces yield results concerning Boolean algebras. As a matter of fact, the result of Kinoshita mentioned above and Theorems 1 and 2 of this note were originally stated and proved in terms of topological spaces. As examples of the results obtained in this way, we give here the immediate topological consequences of Theorems 2, 3 (i), and 4.

THEOREM 2′ (A. Tarski). *There exists a zero-dimensional compact topological space S with two subspaces S_1 and S_2 such that* (i) *each of the subspaces S_1 and S_2 is both closed and open in S,* (ii) *each of the subspaces S_1 and S_2 is homeomorphic to its complement* (*with respect to S*), *and* (iii) *S_1 and S_2 are not homeomorphic.*

THEOREM 3 (i)′ (R. L. Vaught). *Every zero-dimensional, compact, and separable topological space, with infinitely many isolated points, is homeomorphic to any of its subspaces obtained by removing one isolated point.*

Actually, Vaught has shown that Theorem 2′ can be strengthened in several ways by supplying an independent topological proof. For example, the conclusion follows for any Hausdorff space which contains a convergent sequence of distinct isolated points.

THEOREM 4′. *There exists a zero-dimensional compact topological space, with infinitely many isolated points, which is homeomorphic to each of its subspaces obtained by removing n isolated points, but is not homeomorphic to any subspace obtained by removing k isolated points for $k=1, 2, \ldots, n-1$.*

In particular, Theorem 4′ provides an example of a compact topological space, with infinitely many isolated points, which is not homeomorphic to any subspace obtained by removing one isolated point. The first example of such a topological space was obtained recently by B. Jónsson (his result is unpublished). The space he constructed is, however, not zero-dimensional.

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