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REMARKS ON DIRECT PRODUCTS OF COMMUTATIVE SEMIGROUPS

ALFRED TARSKI

The aim of this note is to show that the main results established in the extremely interesting paper of Hanf, [1], can be extended from Boolean algebras to commutative semigroups. We shall follow the notation adopted in that paper[1].

By a *commutative semigroup* we understand an algebraic system $\mathfrak{A}$ formed by an arbitrary set A and a binary operation $+$ under which this set is closed and which is assumed to satisfy the commutative, associative, and cancellation laws and to have a zero 0 (with $x+0=x$ for every $x \in A$). A familiar example of commutative semigroups is the *additive semigroup of non-negative integers*, which will be denoted here by $\mathfrak{N}$; it is formed by the set N of all non-negative integers and the ordinary arithmetical addition.

With every Boolean algebra $\mathfrak{B}$ we correlate a well determined semigroup $\mathfrak{B}^\star$ constructed in the following way. Let $P(\mathfrak{B})$ denote the set of all prime ideals of $\mathfrak{B}$. We first construct the direct power $N^{P(\mathfrak{B})}$; thus the elements of $N^{P(\mathfrak{B})}$ are arbitrary functions on $P(\mathfrak{B})$ to N, and the sum $f+g$ of two such functions f and g is the function h on $P(\mathfrak{B})$ to N defined by the condition

$$h(I) = f(I)+g(I) \qquad \text{for every} \quad I \in P(\mathfrak{B}) .$$

Obviously, $N^{P(\mathfrak{B})}$ is a commutative semigroup. Given any element x of $\mathfrak{B}$, let ch_x (the *characteristic function of* x) be the function f on $P(\mathfrak{B})$ to N such that, for every $I \in P(\mathfrak{B})$, $f(I)=0$ if $x \in I$, and $f(I)=1$ otherwise. We denote by $\mathfrak{B}^\star$ the subalgebra (sub-semigroup) of $N^{P(\mathfrak{B})}$ generated by all such functions ch_x. Thus the elements of $\mathfrak{B}^\star$ are all the function f on $P(\mathfrak{B})$ to N which are of the form

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[1] The results of this note are stated without proof in [2] and [5] (see the bibliography at the end of the note). The note was prepared for publication while the author was working on a research pro ject in the foundations of mathematics supported by the National Science Foundation.

$$f = ch_{x_1} + \ldots + ch_{x_n},$$

where $x_1, \ldots, x_n$ are arbitrary elements of $\mathfrak{B}$. Notice that, for any two elements x and y of $\mathfrak{B}$, we have

$$ch_{x+y} + ch_{x \cdot y} = ch_x + ch_y,$$

where $x+y$ and $x \cdot y$ are the Boolean sum (join) and the Boolean product (meet) of x and y. If $x \cdot y = 0$, that is, if the elements x and y are disjoint, then $ch_{x \cdot y}$ identically equals 0 and hence

$$ch_{x+y} = ch_x + ch_y.$$

The semigroups $\mathfrak{A}$ of the form $\mathfrak{A} = \mathfrak{B}^\star$ where $\mathfrak{B}$ is a Boolean algebra, and their isomorphic images may be called *Boolean semigroups*[1]. Some fundamental properties of the correlation $\mathfrak{B} \to \mathfrak{B}^\star$ are formulated in the following

THEOREM 1. *Let $\mathfrak{B}$ and $\mathfrak{C}$ be arbitrary Boolean algebras. Then*

(i) *$\mathfrak{B}^\star$ is always a commutative semigroup and, in fact, a subdirect power of the additive group $\mathfrak{N}$ of non-negative integers;*

(ii) *if $\mathfrak{B}$ is an at most denumerable algebra with at least two different elements, then $\mathfrak{B}^\star$ is denumerable; if, in particular, $\mathfrak{B}$ is a two-element algebra, then $\mathfrak{B}^\star \cong \mathfrak{N}$;*

(iii) *$\mathfrak{B}^\star \cong \mathfrak{C}^\star$ if and only if $\mathfrak{B} \cong \mathfrak{C}$;*

(iv) *$(\mathfrak{B} \times \mathfrak{C})^\star \cong \mathfrak{B}^\star \times \mathfrak{C}^\star$.*

PROOF. (i) and (ii) follow directly from the construction of $\mathfrak{B}^\star$.

To prove (iii) assume that $\mathfrak{B}^\star \cong \mathfrak{C}^\star$; let H be a function which maps $\mathfrak{B}^\star$ isomorphically onto $\mathfrak{C}^\star$. By definition $\mathfrak{B}^\star$ is generated by all the functions ch_x correlated with the elements x of $\mathfrak{B}$. We notice that these functions ch_x can be characterized intrinsically within the algebra $\mathfrak{B}^\star$ and that, in fact, they coincide with the multiple-free elements of $\mathfrak{B}^\star$. (An element a of a commutative semigroup is called multiple-free if there are no elements b and c with $a = b+b+c$ and $b \neq 0$.) The same applies, of course, to the functions ch_y in $\mathfrak{C}^\star$. We conclude that, for every element x in $\mathfrak{B}$ there is an element y in $\mathfrak{C}$ such that

$$H(ch_x) = ch_y.$$

This element y is uniquely determined by x. For, if y' is any element

[1] Another construction of Boolean semigroups which is equivalent (up to isomorphism) to the one just outlined and an intrinsic characterization of the semigroups can be found in [4, pp. 209f.]. This older construction is somewhat more involved than the present one, but it has the advantage that is enables us to prove Theorem 1 (below) without the help of any transfinite device, such as the prime ideal theorem for Boolean algebras.

of $\mathfrak{C}$ different from y, then, by the prime ideal theorem for Boolean algebras, there exists a prime ideal I of $\mathfrak{C}$ which contains just one of the two elements y and y'; therefore,

$$ch_y(I) \neq ch_{y'}(I) \qquad \text{and} \qquad ch_y \neq ch_{y'} \,.$$

Thus, with every element x of $\mathfrak{B}$, we have correlated a uniquely determined element y of $\mathfrak{C}$; we denote it by $g(x)$ and consequently we have

$$H(ch_x) = ch_{g(x)} \,. \tag{1}$$

Let x and z be any elements of $\mathfrak{B}$ such that $x \leq z$ (that is, $x+z=z$). By putting

$$y = \bar{x}\cdot z \,,$$

we obtain

$$x+y = z \qquad \text{and} \qquad x\cdot y = 0$$

whence

$$ch_x + ch_y = ch_z \,.$$

Therefore, since H is an isomorphism, we have

$$H(ch_x) + H(ch_y) = H(ch_z) \,. \tag{2}$$

Consequently, by (1) and (2),

$$ch_{g(x)} + ch_{g(y)} = ch_{g(z)} \,.$$

Hence, for every $I \in P(\mathfrak{C})$, $g(z) \in I$ implies $g(x) \in I$. By applying the prime ideal theorem again, we conclude that

$$g(x) \leq g(z) \,.$$

Thus $x \leq z$ always implies $g(x) \leq g(z)$.

By symmetry we prove, in an entirely analogous manner, that every element y of $\mathfrak{C}$ uniquely determines an element x of $\mathfrak{B}$ with $g(x)=y$ and that, for any elements x and z of $\mathfrak{B}$, the formula $g(x) \leq g(z)$ implies $x \leq z$. We conclude that g maps $\mathfrak{B}$ isomorphically onto $\mathfrak{C}$.

Thus $\mathfrak{B}^\star \cong \mathfrak{C}^\star$ leads to $\mathfrak{B} \cong \mathfrak{C}$. The implication in the opposite direction is obvious, and the proof of (iii) is complete.

Turning now to (iv), we consider two Boolean algebras $\mathfrak{B}$ and $\mathfrak{C}$ and their direct product $\mathfrak{B} \times \mathfrak{C}$. We notice that for a set I to be a prime ideal of $\mathfrak{B} \times \mathfrak{C}$ it is necessary and sufficient that I satisfy one of the following two conditions: either, for some $J \in P(\mathfrak{B})$,

(3) I consists of all ordered couples $\langle x, y\rangle$ where $x \in J$ and y is an element of $\mathfrak{C}$ or else, for some $J \in P(\mathfrak{C})$,

(4) I consists of all ordered couples $\langle x, y\rangle$ where $y \in J$ and x is an element of $\mathfrak{B}$.

Keeping this in mind, with every function f on $P(\mathfrak{B}\times\mathfrak{C})$ to N we correlate a function f_1 on $P(\mathfrak{B})$ to N and a function f_2 on $P(\mathfrak{C})$ to N by proceeding as follows: given an ideal $J\in P(\mathfrak{B})$, we construct the ideal $I\in P(\mathfrak{B}\times\mathfrak{C})$ satisfying (3), and we put

$$f_1(J) = f(I);$$

to define f_2 we consider an ideal $J\in P(\mathfrak{C})$ and use (4) instead of (3). Furthermore, we let

$$H(f) = \langle f_1, f_2\rangle .$$

As is easily seen, for every function g on $P(\mathfrak{B})$ to N and every function h on $P(\mathfrak{C})$ to N there is a uniquely determined function f on $P(\mathfrak{B}\times\mathfrak{C})$ to N such that

$$f_1 = g, f_2 = h, \qquad \text{and hence} \qquad H(f) = \langle g, h\rangle .$$

Also, if f, g and h are functions on $P(\mathfrak{B}\times\mathfrak{C})$ to N and

$$f = g + h ,$$

then

$$f_1 = g_1 + h_1, \qquad f_2 = g_2 + h_2 ,$$

and hence

$$H(f) = H(g) + H(h) .$$

Finally, if

$$f = ch_{x,y} ,$$

where x belongs to $\mathfrak{B}$ and y to $\mathfrak{C}$, then

$$f_1 = ch_x, \qquad f_2 = ch_y ,$$

so that

$$H(f) = \langle ch_x, ch_y\rangle .$$

Hence we conclude that H maps $(\mathfrak{B}\times\mathfrak{C})^\star$ isomorphically onto $\mathfrak{B}^\star\times\mathfrak{C}^\star$. This completes the proof of (iv).

It is easily seen that Theorem 1 (iv) can be extended to infinite direct products.

As an immediate consequence of Theorem 1 we obtain

THEOREM 2. *Theorems* 1, 2, 4, *and* 5 *of Hanf* [1] *remain valid if Boolean algebras are replaced in them throughout by commutative semigroups and if, in addition, the two-element Boolean algebra* $\mathfrak{T}$ *in Theorem* 4 *is replaced by the additive semigroup* $\mathfrak{N}$ *of non-negative integers.*

Thus Problems I–III formulated at the beginning of [1] admit a negative solution when applied to commutative semigroups.

Two classes of commutative semigroups deserve a special attention: (1) the semigroups in which for every element x there is a negative, that is, an element y with $x+y=0$; (2) the semigroups with just the opposite property,—in which no element $x \neq 0$ has a negative. The semigroups of the first class are simply the Abelian groups. The semigroups of the second class are the *partially ordered commutative semigroups*; the notion of order refers to the relation $\leq$ which holds between two elements x and z if and only if $x+y=z$ for some element y.

The question naturally arises what is the solution of Problems I–III restricted to semigroups of either of these two classes. As regards the Abelian groups, Problems I and III are still open, and only recently Jónsson has succeeded in showing that the solution of Problem II is negative (cf. [3]). On the other hand, from Theorem 1 (i) it is seen that all the commutative semigroups involved in Theorems 1 and 2 of our note are partially ordered; hence the solution of Problems I–III for this class of semigroups proves to be negative. A positive solution of these problems was previously obtained for some more special classes of semigroups. Thus, the solution of Problems I and III is known to be positive for all partially ordered commutative semigroups in which every infinite sequence of pairwise disjoint elements that is bounded above has a least upper bound, and the solution of Problem II is known to be positive for all partially ordered commutative semigroups in which every sequence of pairwise disjoint elements is bounded above.[1]

To conclude we want to mention some problems in the same general direction which still remain open. By Theorem 2, in its part concerning Theorem 4 of [1], there exists a (partially ordered) commutative semigroup $\mathfrak{A}$ such that $\mathfrak{A} \cong \mathfrak{A} \times \mathfrak{N}^2$ holds while $\mathfrak{A} \cong \mathfrak{A} \times \mathfrak{N}$ does not hold ($\mathfrak{N}$ being, as before, the additive group of non-negative integers). It is not known, however, whether there is a denumerable semigroup $\mathfrak{A}$ with these properties. By combining Theorem 3 (ii) of [1] (due to Vaught), with our Theorem 1 we can only conclude that no denumerable Boolean semigroup satisfies these conditions. Also it is not known whether there exists a denumerable partially ordered commutative semigroup $\mathfrak{A}$ such that $\mathfrak{A} \cong \mathfrak{A}^3$ but $\mathfrak{A}$ non-$\cong \mathfrak{A}^2$.

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[1] These are consequences of some results in [4], p. 288 (Theorem C. 4) and p. 291 (Theorem C. 8).

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UNIVERSITY OF CALIFORNIA, BERKELEY, CALF., U. S. A.