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ON ISOMORPHISM TYPES OF GROUPS AND OTHER ALGEBRAIC SYSTEMS

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In this note we shall use the results of Hanf [1] on Boolean algebras to obtain analogous results for groups.[1] Our method is similar to that used in Tarski [5] and, as we shall see later, this same method can be applied to arbitrary classes $\boldsymbol{K}$ of algebraic systems satisfying certain conditions of a very general nature.

While Hanf and Tarski work exclusively with outer direct products, it will be advantageous here to use inner direct products, and in this connection it is convenient not to distinguish between an algebraic system and the set of all its elements. With this exception, our notation and terminology will be the same as in the two papers referred to above.

Given a group H, we denote by $\mathfrak{F}(H)$ the set of all (direct) factors of H. It is known that if H is centerless, then $\mathfrak{F}(H)$ is a Boolean algebra whose inclusion relation and multiplication coincide with the set-theoretic inclusion and multiplication respectively. The Boolean sum of two factors H' and H'' of H is simply the subgroup which they generate. In particular, if H' and H'' have only the identity element of H in common, then the direct product $H' \times H''$ exists and is equal to the Boolean sum of H' and H''.

Our central result is the following.

THEOREM 1. *If A is an infinite Boolean algebra and G is a finite or denumerable indecomposable centerless group, then there exist a group H and a function F with the following properties:*

(i) *H is centerless.*
(ii) *F maps A isomorphically onto $\mathfrak{F}(H)$.*
(iii) *For every a, $b \in A$, $F(a) \cong F(b)$ if and only if $A[a] \cong A[b]$.*
(iv) *For every atom a of A, $F(a) \cong G$.*
(v) *The cardinal of H equals the cardinal of A.*

PROOF. Let Φ be a function which maps A isomorphically onto the

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[1] The results contained in this note were first stated in [2].

set-field $\mathfrak{A}$ consisting of all the open and closed subsets of a compact zero dimensional Hausdorff space S. With the usual definitions of the operations involved, the set G^S of all functions on S to G is a group, the product $f\cdot g$ of two functions f, $g \in G^S$, and the inverse f^{-1} of f are defined by the formulas

$$(f\cdot g)(p) = f(p)\cdot g(p) \qquad \text{and} \qquad f^{-1}(p) = f(p)^{-1}$$

for every $p \in S$, and the identity element of G^S is the function e such that $e(p)=\varepsilon$ for every $p \in S$, where ε is the identity element of G.

For each $f \in G^S$ and $\alpha \in G$, let $T(f, \alpha)$ be the set of all $p \in S$ such that $f(p)=\alpha$. We then define H to be the set of all $f \in G^S$ such that $T(f, \alpha) \in \mathfrak{A}$ for every $\alpha \in G$, and for each $a \in A$ we let $F(a)$ be the set of all $f \in H$ such that $f(p)=\varepsilon$ for every $p \in \Phi(\bar{a})$.

Clearly $e \in H$, and since

$$T(f^{-1}, \alpha) = T(f, \alpha^{-1})$$

for every $f \in G^S$ and $\alpha \in G$, we see that $f^{-1} \in H$ whenever $f \in H$. Observe that for fixed $f \in H$ the sets $T(f, \alpha)$ are pairwise disjoint open sets which cover S. By the compactness of S it follows that $T(f, \alpha)=\Lambda$ for all but finitely many elements $\alpha \in G$. In other words, each function $f \in H$ takes on only finitely many distinct values. Inasmuch as

$$T(f\cdot g, \alpha) = \bigcup_{\beta\in G} \big(T(f, \beta) \cap T(g, \beta^{-1}\alpha)\big)$$

for every f, $g \in G^S$ and $\alpha \in G$, we infer that $f\cdot g \in H$ whenever f, $g \in H$. Thus H is a subgroup of G^S.

For every $\alpha \in G$, let $\hat{\alpha}$ be the function on S such that

$$\hat{\alpha}(p) = \alpha \qquad \text{for every} \quad p \in S\,.$$

Clearly $\hat{\alpha} \in H$. Consequently H is a subdirect power of G, and from this it follows that H is centerless. For the correspondence $f \rightarrow f(p)$ maps H homomorphically onto G and therefore maps the center of H into the center of G. Since G is centerless, this implies that H is centerless.

If $a \in A$, then $F(a)$ is clearly a subgroup of H. Suppose $f \in F(a) \cap F(\bar{a})$. Since $\Phi(a) \cup \Phi(\bar{a})=S$, we have for each $p \in S$ either $p \in \Phi(a)$ or $p \in \Phi(\bar{a})$, and in either case it follows that $f(p)=\varepsilon$. Thus $F(a)$ and $F(\bar{a})$ have only the identity element of H in common. Next suppose $f \in F(a)$ and $g \in F(\bar{a})$. Again, for each $p \in S$, either $p \in \Phi(a)$ or $p \in \Phi(\bar{a})$, whence it follows that either $g(p)=\varepsilon$ or $f(p)=\varepsilon$. We therefore see that $f\cdot g=g\cdot f$. Thus the direct product $F(a) \times F(\bar{a})$ exists and is a subgroup of H. Now consider any $h \in H$. So define f, $g \in G^S$ that

$$f(p) = h(p) \quad \text{and} \quad g(p) = \varepsilon \qquad \text{for every} \quad p \in \Phi(a)\,,$$
$$f(p) = \varepsilon \quad \text{and} \quad g(p) = h(p) \quad \text{for every} \quad p \in \Phi(\bar{a})\,.$$

Inasmuch as

$$T(f, \alpha) = T(h, \alpha) \cap \Phi(a) \qquad \text{whenever} \quad \varepsilon \neq \alpha \in G\,,$$
$$T(f, \varepsilon) = T(h, \varepsilon) \cup \Phi(\bar{a})\,,$$

we see that $f \in H$ and, in fact, $f \in F(a)$. Similarly $g \in F(\bar{a})$. Furthermore, $f(p)\cdot g(p) = h(p)$ for every $p \in S$, so that $f\cdot g = h$. This proves that

(1) $$H = F(a) \times F(\bar{a}) \qquad \text{for every} \quad a \in A\,.$$

In particular we have shown that F does in fact map A into the Boolean algebra $\mathfrak{F}(H)$.

For any a, $b \in A$, if $a \leq b$, then $\Phi(\bar{b}) \subseteq \Phi(\bar{a})$ and therefore $F(a) \subseteq F(b)$. Conversely, assume that $F(a) \subseteq F(b)$. Choose an element $\alpha \in G$ with $\alpha \neq \varepsilon$, and let f be the function on S such that, for every $p \in S$,

$$f(p) = \alpha \quad \text{if} \quad p \in \Phi(a), \qquad f(p) = \varepsilon \quad \text{if} \quad p \in \Phi(\bar{a})\,.$$

Then $f \in F(a)$ and hence $f \in F(b)$, but this implies that $\Phi(\bar{b}) \subseteq \Phi(\bar{a})$, $\Phi(a) \subseteq \Phi(b)$, $a \leq b$. Thus we have shown that, for every a, $b \in A$,

(2) $$F(a) \subseteq F(b) \qquad \text{if and only if} \qquad a \leq b\,.$$

In order to complete the proof of (ii) it is sufficient to show that F maps A onto $\mathfrak{F}(H)$. More specifically, we are going to prove that if

(3) $$H = H' \times H''\,,$$

then there exists $a \in A$ such that

(4) $$H' = F(a) \qquad \text{and} \qquad H'' = F(\bar{a})\,.$$

Assuming that (3) holds, consider a fixed $p \in S$. Let G' be the set of all $\alpha' \in G$ such that $\alpha' = f'(p)$ for some $f' \in H'$, and let G'' be the set of all $\alpha'' \in G$ such that $\alpha'' = f''(p)$ for some $f'' \in H''$. Clearly G' and G'' are subgroups of G, and every element of G' commutes with every element of G''. Given $\alpha \in G$, we have $\hat{\alpha} = f'\cdot f''$ for some $f' \in H'$ and $f'' \in H''$. Consequently $\alpha = \alpha'\cdot\alpha''$ where $\alpha' = f'(p) \in G'$ and $\alpha'' = f''(p) \in G''$. Thus $G'\cdot G'' = G$. Since every element of $G' \cap G''$ commutes with both the elements of G' and the elements of G'', it follows that G' and G'' have only the element ε in common and that, therefore, $G = G' \times G''$. Since G is indecomposable, we infer that either G' or G'' consists of the identity element alone. We have therefore shown that, for every $p \in S$,

$$(5)\qquad \begin{array}{llll} \text{either} & f'(p) = \varepsilon & \text{for every} & f' \in H' \\ \text{or else} & f''(p) = \varepsilon & \text{for every} & f'' \in H'' . \end{array}$$

Let U' be the set of all $p \in S$ for which the second alternative in (5) holds, and let U'' be the set of all $p \in S$ for which the first alternative holds. Choose $\alpha \in G$ with $\alpha \neq \varepsilon$. Then $\hat{\alpha} = f' \cdot f''$ for some $f' \in H'$ and $f'' \in H''$, and it readily follows that $T(f', \alpha) = U'$ and $T(f'', \alpha) = U''$. Thus, U', $U'' \in \mathfrak{A}$, and there exists $a \in A$ such that $U' = \Phi(a)$ and $U'' = \Phi(\bar{a})$. By (5) and the definitions of U' and U'' it follows that

$$H' \subseteq F(a) \qquad \text{and} \qquad H'' \subseteq F(\bar{a}) ,$$

and we conclude with the aid of (1) and (3) that (4) must hold. This completes the proof of (ii).

Suppose a, $b \in A$, and assume that $F(a)$ and $F(b)$ are isomorphic. Then the Boolean algebras $\mathfrak{F}(F(a))$ and $\mathfrak{F}(F(B))$ are clearly isomorphic. But F maps $A[a]$ isomorphically onto $\mathfrak{F}(F(a))$ and maps $A[b]$ isomorphically onto $\mathfrak{F}(F(b))$. Hence $A[a]$ and $A[b]$ are isomorphic.

Conversely, assume that $A[a]$ and $A[b]$ are isomorphic. Then there exists a function φ which maps $\Phi(a)$ homeomorphically onto $\Phi(b)$. Let ψ be the inverse of φ, and for every $f \in F(a)$ and $g \in F(b)$ let f^{ϱ} and g^{σ} be the functions on S such that, for every $p \in S$,

$$\begin{array}{llll} f^{\varrho}(p) = f(\psi(p)) & \text{if} \quad p \in \Phi(b), & f^{\varrho}(p) = \varepsilon & \text{if} \quad p \in \Phi(\bar{b}) . \\ g^{\sigma}(p) = g(\varphi(p)) & \text{if} \quad p \in \Phi(a), & g^{\sigma}(p) = \varepsilon & \text{if} \quad p \in \Phi(\bar{a}) . \end{array}$$

If $f \in F(a)$ and $\varepsilon \neq \alpha \in G$, then $T(f^{\varrho}, \alpha)$ is the set of all $p \in S$ such that $\psi(p) \in T(f, \alpha)$. Since φ maps open and closed subsets of $\Phi(a)$ onto open and closed subsets of $\Phi(b)$, it follows that $f^{\varrho} \in F(b)$ for every $f \in F(a)$. Similarly $g^{\sigma} \in F(a)$ for every $g \in F(b)$. Since, clearly, $f^{\varrho\sigma} = f$ for every $f \in F(a)$ and $g^{\sigma\varrho} = g$ for every $g \in F(b)$, we see that the correlation $f \to f^{\varrho}$ is a one-to-one mapping of $F(a)$ onto $F(b)$. Furthermore, given f_1, $f_2 \in F(a)$, we have for any $p \in \Phi(a)$,

$$\begin{aligned} (f_1 \cdot f_2)^{\varrho}(p) &= (f_1 \cdot f_2)(\psi(p)) = f_1(\psi(p)) \cdot f_2(\psi(p)) \\ &= f_1{}^{\varrho}(p) \cdot f_2{}^{\varrho}(p) = (f_1{}^{\varrho} \cdot f_2{}^{\varrho})(p) , \end{aligned}$$

while for $p \in \Phi(\bar{a})$,

$$(f_1 \cdot f_2)^{\varrho}(p) = \varepsilon = f_1{}^{\varrho}(p) \cdot f_2{}^{\varrho}(p) = (f_1{}^{\varrho} \cdot f_2{}^{\varrho})(p) .$$

Consequently,

$$(f_1 \cdot f_2)^{\varrho} = f_1{}^{\varrho} \cdot f_2{}^{\varrho} \qquad \text{for every} \quad f_1, f_2 \in F(a) ,$$

and $F(a)$ and $F(b)$ are isomorphic. This completes the proof of (iii).

If a is an atom of A, then $\Phi(a)$ consists of a single isolated point

p of S, and $F(a)$ consists of all $f \in G^S$ such that $f(q) = \varepsilon$ whenever $p \neq q \in S$. This proves (iv).

Finally we prove (v). Consider a fixed element $\alpha \in G$ with $\alpha \neq \varepsilon$, and for each $a \in A$ let f_a be the function on S such that $f_a(p) = \alpha$ for every $p \in \Phi(a)$ and $f_a(p) = \varepsilon$ for every $p \in \Phi(\bar{a})$. Clearly $f_a \in H$ for every $a \in A$, and since the correspondence $a \to f_a$ is one-to-one, it follows that the cardinal of H is at least equal to the cardinal of A.

Each function $f \in H$ takes on only a finite number of values

$$\alpha_0, \alpha_1, \ldots, \alpha_n \in G ,$$

and f is completely determined by the elements α_j and the elements $a_j \in A$ such that $T(f, \alpha_j) = \Phi(a_j)$. Since there is at most a countable number of sequences $\alpha_0, \alpha_1, \ldots, \alpha_n \in G$, and since the number of sequences $a_0, a_1, \ldots, a_n \in A$ is equal to the cardinal of A, we see that the cardinal of H does not exceed the cardinal of A. Therefore (v) holds, and the proof of the theorem is complete.

With the aid of Theorem 1 we are able to obtain from Hanf's results on Boolean algebras analogous results for groups:

THEOREM 2. *Theorems* 1, 2, 4 *and* 5 *of Hanf* [1] *remain valid if Boolean algebras are replaced in them throughout by centerless groups and if, in addition, the two-element Boolean algebra T in Theorem* 4 *is replaced by an indecomposable centerless group.*

We shall now indicate briefly how Theorem 1, and hence also Theorem 2, can be generalized by considering, in the place of groups, certain other algebraic systems with a zero element 0. For the definitions of this and other concepts used below we refer the reader to Jónsson–Tarski [3] and Tarski [4]. As is shown in [4], the factor algebra $\mathfrak{F}(H)$ of a centerless algebra H is a Boolean algebra, and H therefore has the (strict) refinement property.

THEOREM 3. *If G is a finite or denumerable indecomposable centerless algebra with a zero element and with operations of finite rank, and if A is an infinite Boolean algebra, then there exist a subdirect power H of G and a function F which satisfy the conditions* (i)–(v) *of Theorem* 1.

OUTLINE OF PROOF. Let Φ, $\mathfrak{A}$ and S be as in the proof of Theorem 1, and define $T(f, \alpha)$, H and F in exactly the same way as was done there (with 0 replacing ε). It is easy to see that H is a subdirect power of G and is therefore centerless, and the proofs of the formulas (1) and (2) go through with only some minor changes.

Now suppose (3) holds, consider a fixed $p \in S$, and define G' and G'' as before. The proof that $G = G' \times G''$ easily reduces to showing that each $\alpha \in G$ has at most one representation $\alpha = \alpha' + \alpha''$ with $\alpha' \in G'$ and $\alpha'' \in G''$. Assume that also $\alpha = \beta' + \beta''$ with $\beta' \in G'$ and $\beta'' \in G''$. Then there exist $f', g' \in H'$ and $f'', g'' \in H''$ such that

$$\alpha' = f'(p), \qquad \alpha'' = f''(p), \qquad \beta' = g'(p), \qquad \beta'' = g''(p) .$$

The intersection of the sets $T(f', \alpha')$, $T(f'', \alpha'')$, $T(g', \beta')$ and $T(g'', \beta'')$ is a non-empty set belonging to $\mathfrak{A}$, and is therefore of the form $\Phi(a)$ with $0 \neq a \in A$. Since H has the strict refinement property,

$$F(a) = \big(F(a) \cap H'\big) \times \big(F(a) \cap H''\big) .$$

Let π be the projection of H onto $F(a)$ corresponding to the factorization (1). Then π maps H' onto $F(a) \cap H'$ and maps H'' onto $F(a) \cap H''$. Since

$$\pi(f') + \pi(f'') = \pi(g') + \pi(g'') ,$$

it follows that $\pi(f') = \pi(g')$ and $\pi(f'') = \pi(g'')$. Evaluating these functions at p we conclude that $\alpha' = \beta'$ and $\alpha'' = \beta''$. Therefore $G = G' \times G''$, and either G' or G'' must consist of 0 alone. We thus see that (5) holds for every $p \in S$, and we can now reason as before to show that (4) holds for some $a \in A$, thereby completing the proof of (ii). The proofs of (iii)–(v) require only trivial changes in the more general situation.

Theorem 3 clearly enables us to transfer most of Hanf's results for Boolean algebras to any class $\boldsymbol{K}$ of algebras with a zero element and with finitary operations, which contains a finite or denumerable indecomposable centerless algebra, and which is closed under the operation of taking subdirect products. Among such classes are the class of all groups, the class of all commutative partially ordered semigroups (cf. Tarski [5]), and the class of all rings, as well as various subclasses of these classes.

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