MATH. SCAND. 5 (1957), 236–240

THE THEOREM OF ENESTRÖM AND THE EXTREMAL FUNCTIONS OF LANDAU-SCHUR

AUREL WINTNER

1. Let $f \in E$ mean that $f(z)$ is a function which is regular, and does not exceed 1 in absolute value, on the circle $|z| < 1$ and, if $s_n(z)$ denotes the n-th partial sum,

$$s_n(z) = c_0 + \ldots + c_n z^n ,$$

of

$$f(z) = c_0 + c_1 z + \ldots ,$$

denote by G_n the least absolute constant which, for a fixed n, has the property that the maximum of $|s_n(z)|$ on the closure $|z| \leqq 1$ does not exceed G_n for any $f \in E$. Thus

$$(1) \qquad G_n = \sup_{f\in E} \max_{|z|\leqq 1} |s_n(z)|, \qquad \text{hence} \qquad G_n = \sup_{f\in E} \max_{|z|=1} |s_n(z)| ,$$

by the maximum principle, and so, since $f(e^{i\varphi}z) \in E$ if $f(z) \in E$ and $0 \leqq \varphi < 2\pi$,

$$(2) \qquad G_n = \sup_{f\in E} |s_n(1)| .$$

It is known that if C_m is defined by

$$\sum_{m=0}^{\infty} C_m z^m = (1-z^2)^{-\frac{1}{2}} \qquad \text{for} \qquad |z| < 1$$

(this function fails to be of class E), that is to say,

$$(3) \qquad C_m = \prod_{k=1}^{m} (2k-1)/(2k) \qquad (C_0 = 1) ,$$

then

$$(4) \qquad G_n = \sum_{m=0}^{n} (C_m)^2$$

(which implies that $G_n \to \infty$ as $n \to \infty$, and that the absolute constant (1) is the n-th partial sum of

Received October 15, 1957.

$$\sum_{m=0}^{\infty} (C_m)^2 z^m = F(\tfrac{1}{2}, \tfrac{1}{2}, 1; z) = (\pi/2)^{-1} \int_0^1 (1-t^2)^{-\frac{1}{2}} (1-zt^2)^{-\frac{1}{2}} dt$$

at $z=1$). In fact, it was shown by Landau ([3, pp. 26–28]) that the sup occurring in (2) is a max, since, for every n, there exists in the class E a function, say

$$f_n^*(z) = \sum_{m=0}^{\infty} c_m^*(n) z^m , \tag{5}$$

satisfying

$$G_n = s_n^*(1) , \tag{6}$$

where

$$s_n^*(z) = \sum_{m=0}^{n} c_m^*(n) z^m ; \tag{7}$$

and he has also shown that these extremal functions f_n^* (functions which are unique up to constant factors of absolute value 1) are the rational functions

$$f_n^*(z) = \left\{ \sum_{m=0}^{n} C_{n-m} z^m \right\} \left\{ \sum_{m=0}^{n} C_m z^m \right\}^{-1} , \tag{8}$$

where the coefficients C are those defined by (3). The existence and the (substantial) uniqueness of the extremal functions $f_0^*(z), f_1^*(z), \ldots$, along with their explicit form (8), was accounted for by Schur within the framework of a general theory ([4, pp. 122–124]; cf. also [1, pp. 12–14]).

In what follows, it will be shown that

$$c_m^*(n) > 0 \qquad \text{for} \qquad 0 \leqq m \leqq n ; \tag{9}$$

in other words, that *the* $n+1$ *coefficients of the n-th partial sum* (7) *of the expansion* (5) *of the n-th extremal function* (8) *of class E are positive for every n.* Note that (9) is an essential refinement of the inequality

$$\sum_{m=0}^{n} c_m^*(n) > 0 , \tag{10}$$

whereas (10) is an obvious consequence of (7) and (6) (since the absolute constant (1) must be positive).

2. Actually, (9) proves to be a corollary of a more general fact, applying not only to Landau's functions (8) but to the entire class of rational functions ("Blaschke products") which go back to Jacobi [2], and which occur as solutions of the extremal problem considered by Carathéodory-Fejér in connection with the Carathéodory–Toeplitz problem (concerning harmonic functions which are positive within the unit circle) and its well-known ramifications.

If $p(z)$ is a polynomial of degree n and if every coefficient a_m of

$$(11) \qquad p(z) = \sum_{m=0}^{n} a_m z^m$$

is real, then, as observed by Jacobi, the rational function

$$(12) \qquad q(z) = z^n p(1/z)/p(z)$$

will have the property that $|q(z)| = 1$ holds at every point z of the circumference $|z| = 1$ (with an obvious interpretation of $|q(z)|$ at a point of the circumference $|z| = 1$ if the denominator, $p(z)$, of (12) vanishes at that point; an interpretation made possible by continuity unless $p(z)$ vanishes identically, and so (12) becomes undefined for every z). In view of the maximum principle, this implies that the power series

$$(13) \qquad q(z) = \sum_{m=0}^{\infty} b_m z^m$$

(converges and) satisfies the inequality $|q(z)| \leqq 1$ for $|z| < 1$, if $p(z)$ has no zero in the circle $|z| < 1$. According to the theorem of Eneström (see [3, p. 26]; the first edition of [3] attributed the theorem to Kakeya whose publication is however of a later date than that of Eneström), this will, in particular, be the case if

$$(14) \qquad a_0 > a_1 > \dots > a_n > 0 .$$

It follows for general reasons (reasons which apply also if $q(z)$ is derived from $p(z)$ in a way more general than (12); cf. the Hilfssatz in [3, p. 29]) that, by virtue of (11), (14) and (12), the partial sums of (13) at $z = 1$ must satisfy the inequalities

$$(15) \qquad \sum_{k=0}^{m} b_k > 0 .$$

In what follows, it will be shown that (15) can be refined to

$$(16) \qquad b_m > 0 \quad \text{if} \quad 0 \leqq m \leqq n .$$

Note that the proviso of (16), the limitation of m by the degree of (11), is not needed in (15).

The inequalitites (9) for the first $n+1$ coefficients of the expansion (5) of the extremal function (8) are immediate consequences of the preceding assertion, according to which (16) *holds for the first* $n+1$ *coefficients of the expansion* (13) (*for* $|z| < 1$) *of the rational function defined* *by* (12), *whenever the coefficients of the polynomial* (11) *satisfy the inequalities* (14). For, on the one hand, it is clear that the case $a_m = C_m$ of (11) reduces the function (12) to the function (8) and, on the other hand, (5) implies that (14) is satisfied if $a_m = C_m$. Correspondingly, (10) is nothing but the trivial inequality (15) belonging to the particular case (5)–(8) of (11)–(14).

3. The proof of (16) proceeds as follows:

If both sides of (12) are multiplied by $p(z)$, substitution of (11) and (13) shows that

$$\left\{\sum_{m=0}^{n} a_m z^m\right\} \sum_{m=0}^{\infty} b_m z^m = \sum_{m=0}^{n} a_{n-m} z^m . \tag{17}$$

Hence, comparison of those powers of z which belong to exponents not exceeding n leads to

$$\sum_{k=0}^{m} a_k b_{m-k} = a_{n-m}, \qquad \text{where} \qquad 0 \leqq m \leqq n . \tag{18}$$

If the latter relation is subtracted from what results if m is replaced by $m+1$, it follows that

$$a_0 b_{m+1} + \sum_{k=0}^{m} (a_{m+1-k} - a_{m-k}) b_k = a_{n-m-1} - a_{n-m} ,$$

where $0 \leqq m \leqq n-1$ (so that $n > 0$; this restriction is allowed, since there is nothing to be proved if $n = 0$). But (4) shows that the difference on the right and the coefficient $(= a_0)$ of b_{m+1} on the left are positive, and that the differences which multiply the numbers b_k in the sum on the left (a sum in which $0 \leqq k \leqq m$) are negative. Consequently, b_{m+1} must be positive if b_k is positive for $0 \leqq k \leqq m$.

This means that (16) follows, by induction, if $b_0 > 0$ is granted. But (17) reduced at $z = 0$ to $a_0 b_0 = a_n$, and so $b_0 > 0$ is clear from (14).

4. The restriction imposed in (16) on m cannot be omitted, since, *under the assumptions of the last italicized assertion,*

$$b_{n+1} < 0 \qquad (n > 0) . \tag{19}$$

A corollary is that, *besides* (9),

$$c_{n+1}{}^*(n) < 0 \qquad (n > 0) \tag{20}$$

holds for the coefficients of the expansion (5) *of the extremal function* (8). In fact, (20) follows from (19) in the same way in which (9) followed from (16).

The proof of (16) depended on (18), and (18) resulted by equating in (17) the coefficients of z^m for every $m \leqq n$. But it is also seen from (17) that

$$\sum_{k=0}^{n} a_k b_{m-k} = 0 \quad \text{if} \quad m > n \,, \tag{21}$$

and it is clear that (14), (16) and the case $m = n+1$ of (21) imply (19).

REFERENCES

1. C. Carathéodory, *Funktionentheorie*, vol. 2, Basel, 1950.
2. G. Kowalewski, *Ein funktionentheoretischer Satz Jacobis*, Jber. Deutsch. Math. Ver. 27 (1918), 53–55.
3. E. Landau, *Darstellung und Begründung einiger neuerer Ergebnisse der Funktionentheorie*, 2. Auflage, Berlin, 1929.
4. I. Schur, *Über Potenzreihen, die im Innern des Einheitskreises beschränkt sind*, J. Reine Angew. Math. 147 (1917), 205–232, and 148 (1918), 122–145.

THE JOHNS HOPKINS UNIVERSITY, BALTIMORE, MD., U.S.A.