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ON SPECTRAL FUNCTIONS BELONGING TO AN ELLIPTIC DIFFERENTIAL OPERATOR WITH VARIABLE COEFFICIENTS

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In this paper we shall prove some theorems on the asymptotic properties of the spectral functions belonging to semi-bounded, self-adjoint extensions of an elliptic differential operator with variable coefficients.

Let R be the real line and let R^n be real n-space. We shall consider an open subset S of R^n when $n > 1$, and we shall denote by $C^k(S)$ all k times continuously differentiable functions defined on S. ($C^0(S)$ are the corresponding continuous functions.) Let $C_0{}^k(S)$ be the set of functions in $C^k(S)$ vanishing outside compact subsets of S, and put $C(S) = \bigcap C^k(S)$ and $C_0(S) = \bigcap C_0{}^k(S)$ for the corresponding sets of infinitely differentiable functions. Let a be a linear differential operator of order m, defined on S and with suitably differentiable coefficients:

$$a = a(x, D) = \sum a_\alpha(x) D^\alpha, \qquad |\alpha| \leqq m ,$$

$$D^\alpha = D_1{}^{\alpha_1} \ldots D_n{}^{\alpha_n}, \qquad D_k = i^{-1}\partial/\partial x_k .$$

Its principal part is

$$Pa = \sum a_\alpha(x) D^\alpha, \qquad |\alpha| = m ,$$

and its adjoint a^* with respect to the scalar product

$$(f, g) = \int_S f(x)\overline{g(x)}\,dx$$

is

$$a^* = \sum D^\alpha \bar{a}_\alpha(x) .$$

We say that a is symmetric if $a = a^*$ and that a is elliptic if the characteristic polynomial of Pa,

$$Pa(x, \xi) = \sum a_\alpha(x)\xi^\alpha, \qquad |\alpha| = m \quad (\xi \in R^n,\ \xi^\alpha = \xi_1{}^{\alpha_1} \ldots \xi_n{}^{\alpha_n}) ,$$

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never vanishes when $\xi \neq 0$. If a is both symmetric and elliptic, then Pa has real coefficients; and hence, since $n > 1$, m has to be even. Consider this case. Changing if necessary a to $-a$, we can also assume that $Pa \geqq 0$.

Let $H = H(S)$ be the Hilbert space of all square integrable functions on S with the scalar product (f, g). Consider a as an operator from H to H, with domain of definition $C_0^m(S)$. It has at least one self-adjoint, semi-bounded extension. Denote such extensions by $A, A_1, A_2, \ldots$ Let

$$A = \int \lambda dE_\lambda$$

be the spectral resolution of A. The spectral function $e(\lambda, x, y)$, belonging to A, is a kernel such that

$$(E_\lambda f)(y) = \int e(\lambda, x, y) f(x) dx$$

for almost all y, when $f \in H$. The existence and the properties of $e(\lambda, x, y)$ are discussed by Gårding [8]. It is a Borel function on $R \times S \times S$; for fixed λ it has any prescribed differentiability in x and y, provided that the coefficients a_α are sufficiently differentiable; it is hermitian and has the Carleman property

$$\int |e(\lambda, x, y)|^2 dx < \infty .$$

It vanishes when $\lambda < A$. Its total variation when λ varies in a finite interval is bounded on compact subsets of $S \times S$.

When the spectrum is discrete, then

$$e(\lambda, x, y) = \sum_{\lambda_k < \lambda} \overline{\varphi_k(x)} \varphi_k(y)$$

where φ_k are the orthonormalized eigenfunctions belonging to the eigenvalues λ_k. When $S = R^n$ and $a = a(D)$ has constant coefficients, then the self-adjoint extension is unique. It is diagonalized by the Fourier transform and its spectral function is

$$e_0(\lambda, x, y) = (2\pi)^{-n} \int_{a(\xi) < \lambda} e^{-i(x-y)\xi} d\xi .$$

Asymptotic formulas for spectral functions were first proved by Carleman [3]. A general version of Carleman's result is the following, proved by Gårding [8].

Let

$$Pa(z, D_x) = \sum a_\alpha(z) D^\alpha, \qquad |\alpha| = m ,$$

be the differential operator with the constant coefficients $a_\alpha(z)$. Then

$$e_z(\lambda, x, y) = (2\pi)^{-n} \int\limits_{Pa(z,\xi)<\lambda} e^{-i(x-y)\xi} d\xi$$

is the spectral function of the self-adjoint extension of $Pa(z, D_x)$, considered on the whole space. We have

THEOREM 1. *If e and e_z are the spectral functions defined above, then*

$$e(\lambda, x, y) = e_x(\lambda, x, y) + o(\lambda^{n/m}) , \tag{0.1}$$

uniformly on compact subsets of $S \times S$.

If the spectrum of A is discrete and if S is bounded, then by a formal integration we obtain Weyl's asymptotic law for the number $N(\lambda)$ of eigenvalues λ_k, smaller than λ,

$$N(\lambda) = \int\limits_S e_x(\lambda, x, x) dx + o(\lambda^{n/m}) .$$

Under suitable conditions the integration can be justified.

The estimate (0.1) is rather rough. In fact, Gårding has proved in [8] that if a has constant coefficients, then

$$I^k e(\lambda, x, y) = I^k e_0(\lambda, x, y) + O(\lambda^{(n-k)/m}) . \tag{0.2}$$

Here I^k is the Riesz mean of order k on the interval (λ_0, λ), defined by

$$I^k \sigma(\lambda) = \Gamma(k)^{-1}(\lambda-\lambda_0)^{1-k} \int_{\lambda_0}^{\lambda} (\lambda-\mu)^{k-1} d\sigma(\mu)$$

for any σ of bounded variation.

In this paper, we shall give a new proof of (0.1), and also in a sense generalize (0.2) to the case of variable coefficients. We shall show

THEOREM 2. *If A_1 and A_2 are semi-bounded self-adjoint extensions of a, if e_1 and e_2 are the corresponding spectral functions, and if $\lambda_0 < \min(A_1, A_2)$ and $k \geqq 1$, then*

$$I^k\big(e_1(\lambda, x, y) - e_2(\lambda, x, y)\big) = O(\lambda^{(n+1-k)/m}) , \tag{0.3}$$

uniformly on compact subsets of $S \times S$.

To get a complete analogue of (0.2) we have to find the analogue of e_0 when a has variable coefficients. The formula (0.1) shows that the dominating part of this function is $e_x(\lambda, x, y)$ and it might be possible to give an asymptotic series for e_0. A result in this direction is given by Avakumović [1]. Levitan has also studied the spectral function for equations with variable coefficients, see e.g. [10].

We shall prove our theorems by estimating Green's function

$$G(t, x, y) = \int e^{-t\lambda} de(\lambda, x, y)$$

for the parabolic operator

$$L = A - \partial/\partial t, \qquad t > 0 .$$

It has the property that

$$u(t, y) = \int G(t, x, y) f(x) dx, \qquad f \in C_0(S)$$

solves the equation

$$(A + \partial/\partial t)u = 0, \qquad u(0, .) = f .$$

When a has constant coefficients, then Green's function corresponding to A_0 is

$$G_0(t, x, y) = \int e^{-t\lambda} de_0(\lambda, x, y)$$
$$= (2\pi)^{-n} \int_{R^n} e^{-ta(\xi) - i(x-y)\xi} d\xi ,$$

and one has the estimate (Gårding [8])

$$G(t, x, y) = G_0(t, x, y) + O(1) \exp\{-Ct^{-\mu}\}, \qquad 0 < t < t_0 ,$$

where $\mu = m-1$. A Tauberian argument then implies (0.2).

In the case of variable coefficients we shall substitute for G_0 a fundamental solution Γ of the operator L, constructed by Ejdelman [5]. We shall prove that

$$G(t, x, y) = \Gamma(t, x, y) + O(1) \exp\{-Ct^{-\mu}\}, \qquad 0 < t < t_0 , \tag{0.4}$$

uniformly on compact subsets of $S \times S$.

An immediate consequence of (0.4) is

$$G_1(t, x, y) - G_2(t, x, y) = O(1) \exp\{-Ct^{-\mu}\} , \tag{0.5}$$

and this together with a Tauberian theorem gives Theorem 2.

The fundamental solution Γ is obtained as a Neumann series with the leading term

$$H(t, x, y) = (2\pi)^{-n} \int e^{-tPa(x,\xi) - i(x-y)\xi} d\xi$$
$$= \int e^{-t\lambda} de_x(\lambda, x, y) ,$$

and we have

$$\Gamma(t, x, y) - H(t, x, y) = O(1)\, t^{(1-n)/m} \exp\{-C|x-y|^{1+\mu} t^{-\mu}\}, \qquad 0 < t < t_0 . \tag{0.6}$$

Hence

$$(0.7) \qquad \int e^{-t\lambda} d\big(e(\lambda, x, y) - e_x(\lambda, x, y)\big) = O(t^{(1-n)/m}), \qquad 0 < t < t_0,$$

and now a Tauberian argument proves Theorem 1.

1. The fundamental solution. In this section we shall introduce the fundamental solution $\Gamma(t, x, y)$ of the parabolic differential operator $a(x, D) - \partial/\partial t$. Here a is not necessarily symmetric. We shall denote by $a^\star$ the algebraic adjoint of a, satisfying

$$\int_S af \cdot g = \int_S f \cdot a^\star g, \qquad f, g \in C_0(S).$$

By definition, the function $\Gamma(t, x, z)$, $t > 0$, is a fundamental solution of $a - \partial/\partial t$ if

$$(1.1) \qquad \int_t^\infty dt_1 \int_S \Gamma(t_1 - t, x, z) \{a^\star(z, D_z) - \partial/\partial t_1\} f(t_1, z) \, dz = f(t, x)$$

for all $f \in C_0(R \times S)$. Ejdelman constructed the fundamental solution Γ with the aid of the parametrix $H(t, x, z)$, following the method of E. E. Levi. (Already Bruk [2] constructed such a function Γ, but he did not obtain the important estimates (1.3) and (0.6).) Put

$$Z(t, x, z) = \{Pa(x, D_z) - a(z, D_z)\} H(t, x, z)$$

define $f \circ g$ by

$$(f \circ g)(t, x, z) = \int_0^t dt_1 \int_S f(t - t_1, x, y) g(t_1, y, z) \, dy$$

and put

$$f^{\circ n}(t, x, z) = (\underbrace{f \circ \ldots \circ f}_{n})(t, x, z).$$

Then

$$(1.2) \qquad \Gamma(t, x, z) = (H \circ \sum Z^{\circ n})(t, x, z), \qquad n = 0, 1, 2, \ldots$$

This series converges for all $t > 0$ and all $(x, z) \in S \times S$, and Γ is differentiable any prescribed number of times if the coefficients of a are sufficiently differentiable. Further, the estimates (0.6) and

$$(1.3) \qquad (\partial/\partial t)^k D_x^\alpha D_z^\beta \Gamma(t, x, z)$$
$$= O(1) t^{-(km + |\alpha| + |\beta| + n)/m} \exp\{-C|x - z|^{1+\mu} t^{-\mu}\}, \qquad 0 < t < t_0,$$

hold. An immediate consequence of (1.1) is

$$(1.4) \qquad \{a(z, D_z) + \partial/\partial t\} \Gamma(t, x, z) = 0 \quad \text{when} \quad x \neq z.$$

We shall show that

$$u(t, z) = \int_S \Gamma(t, x, z) f(x)\, dx, \qquad f \in C_0^0(S),$$

is a solution of the equation

$$(a + \partial/\partial t) u = 0 \tag{1.5}$$

with

$$u(0, z) = \lim_{t \to 0} u(t, z) = f(z). \tag{1.6}$$

It is clear from (1.3) and (1.4) that (1.5) holds. Let us compute

$$\lim_{t \to 0} \int \Gamma(t, x, z) f(x)\, dx .$$

The integral is equal to

$$(2\pi)^{-n} \int_{(R^n)^2} e^{-tPa(x,\xi) - i(x-z)\xi} f(x)\, dx\, d\xi + \int_S (\Gamma - H)(t, x, z) f(x)\, dx .$$

Here the first integral tends to

$$(2\pi)^{-n} \int_{R^n} e^{iz\xi} d\xi \int_{R^n} e^{-ix\xi} f(x)\, dx = f(z)$$

as $t \to 0$. For the second integral we use the estimate (0.6), and since f is bounded, the integral is

$$O(1) t^{(1-n)/m} \int \exp\{-C|x-z|^{1+\mu} t^{-\mu}\} dx .$$

The substitution $y = t^{-1/m}(x - z)$ shows that it is $O(1)t^{1/m}$, so that (1.6) is proved.

We shall need not only the fundamental solution Γ but also the corresponding fundamental solution $\Gamma^\star$ of $a^\star(x, D) - \partial/\partial t$. It is defined by a relation corresponding to (1.2), but it is convenient to require that it satisfies the relation

$$\int_t^\infty dt_1 \int_S \Gamma^\star(t_1 - t, z, x) \{a(z, D_z) - \partial/\partial t_1\} f(t_1, z)\, dz = f(t, x),$$

where z and x appear in the inverse order, as compared to (1.1).

The difference $\Gamma^\star(t, x, z) - \Gamma(t, x, z)$ has no singularities. This follows from the relation (1.7) below, which is easily proved in the same way as the corresponding statement for elliptic differential operators. (F. John [9], Gårding [7].)

Let U be a compact subset of S and let V_1 and V_2 be open sets such that
$$U \subset V_1 \subset \bar{V}_1 \subset V_2 \subset \bar{V}_2 \subset S .$$
Further, let $\chi \in C_0(V_2)$ with $\chi(x)=1$ for $x \in V_1$. Then

$$(1.7) \quad \Gamma^\star(t, x, z)-\Gamma(t, x, z) = \int_0^t dt_1 \int_{V_2-V_1} \Gamma(t-t_1, x, y)\{a^\star(y, D_y)+\partial/\partial t_1\}\,\chi(y)\,\Gamma^\star(t_1, y, z)\,dy$$

for all $(x, z) \in V_1 \times V_1$.

If we let $(x, z) \in U \times U$ then for the points x, y and z, occurring in this integral, the values of $|x-y|$ and $|y-z|$ are greater than a positive number. Hence the estimate (1.3), with $k=|\alpha|=|\beta|=0$ gives

$$(1.8) \qquad \Gamma^\star(t, x, z)-\Gamma(t, x, z) = O(1)\exp\{-Ct^{-\mu}\}, \qquad 0 < t < t_0 ,$$

uniformly on $U \times U$.

2. Estimates of Green's function. In this section we shall prove the estimate (0.4) under the assumptions stated in the introduction.

Since the spectral function is hermitian, (0.4) can be written as

$$G(t, z, x)-\overline{\Gamma(t, x, z)} = O(1)\exp\{-Ct^{-\mu}\} ,$$

where the left side is the kernel of the hermitian form

$$\tilde{V}(t, f, g) = (e^{-tA}f, g)-\int_{S\times S} \overline{\Gamma(t, x, z)}f(z)\overline{g(x)}\,dx\,dz, \qquad f, g \in C_0^0(S) .$$

In the following we shall also meet the form

$$V(t, f, g) = (e^{-tA}f, g)-\int \Gamma(t, z, x)f(z)\overline{g(x)}\,dx\,dz .$$

If the coefficients of a are constant, then $V=\tilde{V}$.

We observe that V and $\tilde{V}$ satisfy

$$(2.1) \qquad V(0, f, g) = \tilde{V}(0, f, g) = 0 ,$$

which is an immediate consequence of (1.6) and that the formula

$$(2.2) \qquad \frac{\partial}{\partial t} V(t, f, g(t, .)) = -V(t, f, Lg(t, .))$$

holds if $g \in C_0(R \times S)$. In fact, the left side of (2.2) is

$$-\left(e^{-tA}f, Lg(t,.)\right) - \int \Gamma(t,z,x)f(z)\frac{\partial}{\partial t}\overline{g(t,x)}\,dx\,dz -$$
$$- \int \frac{\partial}{\partial t}\Gamma(t,z,x)f(z)\overline{g(t,x)}\,dx\,dz$$

which equals

$$-\left(e^{-tA}f, Lg(t,.)\right) + \int \Gamma(t,z,x)f(z)\overline{Lg(t,x)}\,dx\,dz\,,$$

in view of (1.4).

We shall deduce the estimate (0.4) from a certain identity involving the form $\tilde{V}$. This identity appears below with the number (2.5) and was proved by Gårding [8] when a has constant coefficients. It exhibits the kernel of $\tilde{V}$ as an expression containing the form $\tilde{V}$ applied to certain judiciously chosen functions, which we now proceed to present.

Let $S_1 \subset S$ be open, $\bar{S}_1 \subset S$ compact, and let $h \in C_0(S)$ be equal to 1 on S_1. We shall denote the support of h by T. Put, for $x \in S_1$

$$b(t,x,z) = \{a(z,D_z) + \partial/\partial t\}\Gamma(t,x,z)\big(1-h(z)\big)\,.$$

Clearly, $b(t,x,z)$ is different from zero only when $z \in T - S_1$. If $a_\alpha \in C$, then $b \in C(R \times S_1 \times S)$. Let us assume this in the following. Because of (1.3), b and its derivatives vanish for $t=0$.

Let R^+ be the semi-axis $t>0$, and let $\psi \in C_0(R^+)$. Let φ, and later on also φ', belong to $C_0(S_1)$. Put

$$b(\psi,t,\varphi,z) = \int_t^\infty \psi(t_1)\,dt_1 \int_{S_1} \varphi(x)\,b(t_1-t,x,z)\,dx$$

and

$$\Gamma(\psi,t,\varphi,z) = \int_t^\infty \psi(t_1)\,dt_1 \int_{S_1} \varphi(x)\Gamma(t_1-t,x,z)\,dx\,.$$

We have

$$\begin{aligned}(2.3)\quad & b(\psi,t,\varphi,z)\\ &= \{a(z,D_z) - \partial/\partial t\}\int_t^\infty \psi(t_1)\,dt_1 \int_{S_1} \varphi(x)\,\Gamma(t_1-t,x,z)\big(1-h(z)\big)dx\\ &= \psi(t)\varphi(z) - Lf(t,z)\,,\end{aligned}$$

where

$$f(t,z) = h(z)\Gamma(\psi,t,\varphi,z)$$

belongs to $C^0(R\times S)$ and to $C_0^0(S)$ for fixed t. An estimate of Lf will be given in (2.10) below.

We shall now consider, for $t>0$ and $(y,y') \in S_1 \times S_1$,

$$\tilde{w}(t, y, y') = \int\limits_{t > t_1 > t_2 > 0} \tilde{V}(t_2, b(t-t_1, y, .), \; b(t_1-t_2, y', .))dt_1 dt_2 .$$

We also put

$$\tilde{w}(\psi, \varphi, \varphi') = \int\limits_{R^+ \times S \times S} \tilde{w}(t, y, y')\psi(t)\varphi(y)\overline{\varphi'(y')}\,dt\,dy\,dy' , \tag{2.4}$$

and we want to prove the identity

$$\int\limits_0^\infty \tilde{V}(t, \varphi, \varphi')\psi(t)dt = \tilde{w}(\psi, \varphi, \varphi') + Q , \tag{2.5}$$

where

$$Q = \int\limits_0^\infty dt \int\limits_{T \times S} \big(\Gamma(t, z, x) - \overline{\Gamma(t, x, z)}\big)\varphi(z)\overline{Lf'(t, x)}\,dx\,dz ,$$

with $f'(t, x) = h(x)\Gamma(\bar{\varphi}, t, \varphi', x)$.

Suppose that (2.5) is proved. We want an estimate for the kernel of $\tilde{V}(t, \varphi, \varphi')$. This kernel is the limit of the left side of (2.5) for sequences ψ_n, φ_n, and φ_n' such that

$$\begin{array}{llll} \psi_n \geqq 0, & \psi_n(t_1) \to 0 \quad \text{for} & t_1 \neq t, & \int \psi_n(t)dt = 1 , \\ \varphi_n \geqq 0, & \varphi_n(y_1) \to 0 \quad \text{for} & y_1 \neq y, & \int \varphi_n(y)dy = 1 , \\ \varphi_n' \geqq 0, & \varphi_n'(y_1') \to 0 \quad \text{for} & y_1' \neq y', & \int \varphi_n'(y')dy' = 1 . \end{array} \tag{2.6}$$

Hence we have to prove the identity (2.5) and then estimate the right side of it, uniformly for functions ψ, φ and φ' such as those in (2.6).

We change the order of integration in (2.4) and get

$$\tilde{w}(\psi, \varphi, \varphi') = \int\limits_{t_1 > t_2 > 0} \tilde{V}\big(t_2, b(\psi, t_1, \varphi, .), b(t_1-t_2, \varphi', .)\big)\,dt_1 dt_2$$

with an obvious notation, and this is, according to (2.3),

$$\int\limits_0^\infty \psi(t_1)dt_1 \int\limits_0^{t_1} \tilde{V}\big(t_2, \varphi, b(t_1-t_2, \varphi', .)\big)dt_2 - R , \tag{2.7}$$

where

$$\begin{aligned} R = & \int\limits_{t_1 > t_2 > 0} \big(e^{-t_2 A} L_1 f(t_1, .), \; b(t_1-t_2, \varphi', .)\big)\,dt_1 dt_2 - \\ & - \int\limits_{t_1 > t_2 > 0} dt_1 dt_2 \int\limits_{S \times S} \overline{\Gamma(t_2, x, z)}\,L_{1,z} f(t_1, z)\overline{b(t_1-t_2, \varphi', x)}\,dx\,dz . \end{aligned} \tag{2.8}$$

If we change the order of integration in the integral of (2.7), we get

$$\tilde{w}(\psi, \varphi, \varphi') = \int_0^\infty \tilde{V}\big(t_2, \varphi, b(\bar{\psi}, t_2, \varphi', .)\big)dt_2 - R ,$$

and hence, again by (2.3),

$$\tilde{w}(\psi, \varphi, \varphi') = \int_0^\infty \tilde{V}(t, \varphi, \varphi')\psi(t)\,dt - R - R' ,$$

with

$$R' = \int_0^\infty V\big(t, \varphi, Lf'(t, .)\big)dt + Q . \tag{2.9}$$

Hence (2.5) follows if we can show that the integral in (2.9) vanishes and that $R=0$. By virtue of (2.2), the integral of (2.9) is

$$-\int_0^\infty \frac{\partial}{\partial t} V\big(t, \varphi, f'(t, .)\big)dt ,$$

which is equal to zero, since $f'(t, 0)=0$ for large values of t and since $V\big(0, \varphi, f'(0, .)\big)=0$.

It remains to prove that $R=0$. Put $R=J_1-J_2$, where J_1 is the first and J_2 the second integral of (2.8). We have

$$J_1 = \int_0^\infty dt_2 \int_{t_2}^\infty \big(e^{-t_2 A}L_1 f(t_1, .),\ b(t_1-t_2, \varphi', .)\big)dt_1 ,$$

and since $b(0, \varphi, .)=0$ and $f(t_1, .)=0$ for large values of t_1, integration by parts gives

$$\int_0^\infty dt_2 \int_{t_2}^\infty \big(e^{-t_2 A}f(t_1, .),\ L_2 b(t_1-t_2, \varphi', .)\big)\,dt_1 ,$$

which by the arguments that gave (2.2) equals

$$\begin{aligned} -\int_0^\infty dt_2 \int_{t_2}^\infty \frac{\partial}{\partial t_2}\big(e^{-t_2 A}f(t_1, .),\ b(t_1-t_2, \varphi', .)\big)dt_1 & \\ = -\int_0^\infty dt_1 \int_0^{t_1} \frac{\partial}{\partial t_2}\big(e^{-t_2 A}f(t_1, .),\ b(t_1-t_2, \varphi', .)\big)dt_2 & \\ = \int_0^\infty \big(f(t_1, .),\ b(t_1, \varphi', .)\big)dt_1 . & \end{aligned}$$

On the other hand J_2 is equal to the same integral, since

$$J_2 = \int_0^\infty d\tau \int_0^\infty dt_2 \int_{S\times S} \overline{\Gamma(t_2, x, z)}\, L_{2,z} f(\tau + t_2, z)\, \overline{b(\tau, \varphi', x)}\, dx\, dz\,,$$

and this is, by (1.4), equal to

$$-\int_0^\infty d\tau \int_0^\infty dt_2 \frac{\partial}{\partial t_2} \int_{S\times S} \overline{\Gamma(t_2, x, z)} f(\tau + t_2, z)\, \overline{b(\tau, \varphi', x)}\, dx\, dz$$

$$= \int_0^\infty d\tau \int_{S\times S} \overline{\Gamma(0, x, z)} f(\tau, z)\, \overline{b(\tau, \varphi', x)}\, dx\, dz$$

$$= \int_0^\infty \big(f(\tau, .),\ b(\tau, \varphi', .)\big) d\tau$$

because of (1.6). Thus $R=0$.

Now we can turn to the estimates of the right side of (2.5). Let us begin with Q, which is an integral involving the difference

$$\varrho = \Gamma(t, z, x) - \overline{\Gamma(t, x, z)}\ .$$

Since $\overline{\Gamma(t, x, z)} = \Gamma^\star(t, z, x)$ and (x, z) belongs to a compact subset of $S \times S$, we have the estimate (1.8),

$$\varrho = O(1) \exp\{-Ct^{-\mu}\}\,,$$

uniformly in x and z. We shall show that the same estimate is valid for Q, when the functions ψ, φ and φ' vary as in (2.6). Let us further restrict ψ so that $\psi(t_1)=0$ for $t_1 \geqq 2t$.

In the formula

$$\text{(2.10)}\quad Lf'(t, x) = h(x)\varphi'(x)\psi(t) +$$

$$+\ O(1) \int_t^\infty dt_1 \int_{S_1} \sum D_x^\alpha \Gamma(t_1 - t, y, x)\psi(t_1)\varphi'(y)\, dy, \qquad |\alpha| \leqq m-1,$$

the integral is uniformly convergent for $x \in T$ according to (1.3). Hence we get, using (1.3) and (1.8)

$$Q = O(1) \int_0^{2t} \psi(t') \exp\{-Ct'^{-\mu}\} dt' \int_{S_1\times S_1} \varphi(z)\varphi'(x)\, dx\, dz\ +$$

$$+\ O(1) \int_0^{2t} \exp\{-Ct'^{-\mu}\} dt' \int_{t'}^{2t} (t_1 - t')^{-1+(1-n)/m} \psi(t_1)\, dt_1 \cdot$$

$$\cdot \int_{x\in T,\, y\in S_1,\, z\in S_1} \varphi'(y)\varphi(z) \exp\{-C|x-y|^{1+\mu}(t_1 - t')^{-\mu}\}\, dx\, dy\, dz\ .$$

Clearly, the first integral is

$$O(1)\exp\{-Ct^{-\mu}\}$$

and the second is easily reduced to

$$O(1)\exp\{-Ct^{-\mu}\}\int_0^{2t}\psi(t_1)\,dt_1\int_0^{t_1}(t_1-t')^{-1+(1-n)/m}dt'\cdot$$
$$\cdot\int_{R^n}\exp\{-C|x'|^{1+\mu}(t_1-t')^{-\mu}\}dx'\int_{R^n}\varphi'(x-x')\,dx$$

which equals

$$O(1)\exp\{-Ct^{-\mu}\}\int_0^{2t}\psi(t_1)\,dt_1\int_0^{t_1}(t_1-t')^{-1+1/m}dt' = O(1)\exp\{-Ct^{-\mu}\}\,.$$

Hence the identity (2.5) shows that

$$\begin{aligned} &G(t,y,y')-\overline{\Gamma(t,y',y)}\\ &\quad= \tilde{w}(t,y,y') + O(1)\exp\{-Ct^{-\mu}\}\\ &\quad= \int_{t>t_1>t_2>0}\tilde{V}(t_2, b(t-t_1,y,.),\ b(t_1-t_2,y',.))\,dt_1dt_2 + O(1)\exp\{-Ct^{-\mu}\}\,,\end{aligned}$$

uniformly on $S_1\times S_1$.

But this last integral can be estimated in a way similar to that in [8], and, like there, we find it to be

$$O(1)\exp\{-Ct^{-\mu}\}\,.$$

Thus the estimate (0.4) is valid, uniformly on compact subsets of $S\times S$.

3. The Tauberian argument. In this section we shall complete the proofs of our theorems. Let $\mathscr{C}$ be the closed unit circle in the complex plane and put

$$\sigma(\lambda) = e(\lambda,x,x) + 2\mathrm{Re}\,\gamma e(\lambda,x,y) + |\gamma|^2 e(\lambda,y,y)$$

and

$$\sigma_x(\lambda) = e_x(\lambda,x,x)(1+|\gamma|^2) + 2\mathrm{Re}\,\gamma e_x(\lambda,x,y)$$

for $\gamma\in\mathscr{C}$. In order to prove Theorem 1, we observe that to every compact subset $\mathscr{K}\subset\mathscr{C}\times S\times S$, there is a constant K such that

$$K\lambda^{n/m} - \sigma_x(\lambda)$$

is a non-decreasing function of λ when $\lambda\geqq 0$ and $(\gamma,x,y)\in\mathscr{K}$. (See [8].) Since $\sigma(\lambda)$ is non-decreasing, then certainly

$$\sigma(\lambda) - \sigma_x(\lambda) + K\lambda^{n/m}$$

is non-decreasing. Now (0.7) gives

$$\int e^{-t\lambda} d\big(\sigma(\lambda)-\sigma_x(\lambda)\big) = o(t^{n/m}), \qquad 0 < t < t_0 ,$$

and so a Tauberian theorem of Karamata (see [12, p. 197]) gives

$$\sigma(\lambda)-\sigma_x(\lambda) = o(\lambda^{n/m}), \qquad \lambda \to +\infty .$$

This is valid uniformly for all $\gamma \in \mathscr{C}$, so that Theorem 1 follows.

For the proof of Theorem 2, we write the estimate (0.5) as

$$\int e^{-t\lambda} d\big(\sigma_1(\lambda)-\sigma_2(\lambda)\big) = O(1)\exp\{-Ct^{-\mu}\} .$$

This estimate is uniform for $(\gamma, x, y) \in \mathscr{K}$. In order to apply a Tauberian theorem of Ganelius [6], we estimate

$$\Delta_i = \sigma_i(\lambda+\lambda^{1/(1+\varepsilon)}) - \sigma_i(\lambda), \qquad 0 < \varepsilon \leqq 1 .$$

Now Theorem 1 and an estimate of $e_x(\lambda, x, y)$ show that

$$\sigma_i(\lambda) = O(\lambda^{n/m})$$

uniformly on $\mathscr{K}$. Thus

$$\Delta_i \leqq \sigma_i(2\lambda) = O(\lambda^{n/m}) = O(^{\varkappa-\varepsilon/(1+\varepsilon)})$$

with $\varkappa = n/m + \varepsilon/(1+\varepsilon)$. We have to take

$$\varepsilon = \mu = 1/(m-1) ,$$

and so

$$\varkappa = (n+1)/m ,$$

and the corollary of [6] gives

$$I^k\big(\sigma_1(\lambda)-\sigma_2(\lambda)\big) = O(1)\,\lambda^{(n+1-k)/m} ,$$

so that Theorem 2 follows.

REFERENCES

1. V. G. Avakumović, *Über die Eigenfunktionen auf geschlossenen Riemannschen Mannigfaltigkeiten,* Math. Z. 65 (1956), 327–344.
2. S. Z. Bruk, *Fundamental solutions of systems of differential equations of parabolic type,* Dokl. Akad. Nauk SSSR 60 (1948), 9–12. (Russian).
3. T. Carleman, *Propriétés asymptotiques des fonctions fondamentales des membranes vibrantes,* C. R. du 8$^{\text{ième}}$ Congrès des Math. Scand. Stockholm 1934, Lund, 1935, 34–44.
4. S. D. Ejdelman, *On Cauchy's problem for parabolic systems,* Dokl. Akad. Nauk SSSR 98 (1954), 913–915. (Russian).
5. S. D. Ejdelman, *On fundamental solutions of parabolic systems,* Mat. Sbornik 38 (80) (1956), 51–92. (Russian).

6. T. Ganelius, *Un théorème taubérien pour la transformation de Laplace*, C. R. Acad. Sci. Paris 242 (1956), 719–721.
7. L. Gårding, *On the asymptotic distribution of the eigenvalues and eigenfunctions of elliptic differential operators*, Math. Scand. 1 (1953), 237–255.
8. L. Gårding, *On the asymptotic properties of the spectral function belonging to a self-adjoint semi-bounded extension of an elliptic differential operators*, Kungl. Fysiogr. Sällsk. i Lund Förh., Bd. 24, Nr. 21 (1954).
9. F. John, *General properties of solutions of linear elliptic partial differential equations*, Proc. of the Symposium on Spectral Theory and Differential Problems, Stillwater, Okla., 1951, 113–175.
10. B. M. Levitan, *On eigenfunction expansions for self-adjoint partial differential equations*, Trudy Moskov. Mat. Obšč. 5 (1956), 269–298. (Russian).
11. H. Weyl, *Das asymptotische Verteilungsgesetz der Eigenwerte linearer partieller Differentialgleichungen*, Math. Ann. 71 (1911), 441–471.
12. D. V. Widder, *The Laplace Transform*, Princeton, 1946.

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