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ON AN ABSOLUTE CONSTANT FOR A CLASS OF POWER SERIES

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1. The following theorem is due partly to H. Bohr [1, § 4] and partly to A. Wintner [5].

THEOREM A. *Let $f(z)$ belong to a class $\mathscr{A}$ of functions satisfying the condition:*

$$\text{(A)} \qquad f(z) = \sum_0^\infty c_n z^n, \qquad |f(z)| < 1, \qquad \textit{for} \quad |z| < 1 .$$

Let $g(r)$ be the majorant of $f(z)$ defined as usual by

$$\text{(1)} \qquad g(r) = \sum_0^\infty |c_n| r^n, \qquad 0 \leqq r < 1 .$$

Then, for all functions $f(z)$ of the class $\mathscr{A}$,

$$\text{(2)} \qquad g(\tfrac{1}{3}) \leqq 1 ,$$

$$\text{(3)} \qquad \sup_{r<1} \big(r/g(r)\big) \geqq \tfrac{1}{3} .$$

In (2) *and* (3), $\frac{1}{3}$ *is the best* (*largest admissible*) *absolute constant, that is,* $\frac{1}{3}$ *cannot be replaced by* $\frac{1}{3}(1+\varepsilon)$ *for any absolute constant* $\varepsilon > 0$.

The theorem which follows is a companion to the above.

THEOREM B. *Theorem* A *is true also for the class $\mathscr{B}$ of functions $f(z)$ satisfying the condition:*

$$\text{(B)} \qquad f(z) = \sum_0^\infty c_n z^n \quad (c_n \geqq 0), \qquad \operatorname{Re} f(z) < 1, \qquad \textit{for} \quad |z| < 1 .$$

Theorem A can be deduced from Theorem B if we leave out the assertion of each theorem that $\frac{1}{3}$ is the best absolute constant. For, conclusion (2) of Theorem A can be deduced from the corresponding conclusion of Theorem B by applying the latter theorem to $f(z)\exp(-i \operatorname{am} c_0)$, and conclusion (3) readily follows thereafter.

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2. PROOF OF THEOREM B. To prove the conclusion of Theorem B corresponding to (2), we use, as in [4], the well-known inequality

$$|c_n| \leqq 2(1-\operatorname{Re} c_0) \quad \text{for} \quad n \geqq 1, \tag{4}$$

true for any $f(z)$ satisfying (B) even without the restriction $c_n \geqq 0$ [2, III Abschn., Nr. 235]. Then, for $f(z)$ satisfying (B), we have

$$g(\tfrac{1}{3}) \leqq c_0 + \sum_1^{\infty} 2(1-c_0)3^{-n} = 1. \tag{5}$$

To prove the conclusion of Theorem B corresponding to (3), we use the fact that, *if $g(r)$ is defined by* (1) *for any power series $f(z)$ in $|z|<1$, then*

$$\min_{0 \leqq r \leqq 1}\big(g(r)/r\big) \quad \textit{exists for} \quad r = r_0 \ (\textit{say}), \tag{6}$$

this minimum not exceeding $2|c_0|+|c_1|$ *in the cases* $r_0=0$ *and* $r_0=1$. This fact follows easily from Wintner's analysis [5, pp. 109–110] of the function

$$rg'(r) - g(r) = -|c_0| + \sum_2^{\infty} (n-1)|c_n| r^n. \tag{7}$$

If we exclude the case $c_2=c_3=\ldots=0$ in which the minimum in (6) is clearly $g(1)=|c_0|+|c_1|$, the function in (7) is strictly increasing for $0 \leqq r \leqq 1$ and has no more than one zero in this interval. Hence, depending on whether

$$|c_0| \geqq \sum_2^{\infty}(n-1)|c_n|, \qquad \textit{or} \qquad |c_0| < \sum_2^{\infty}(n-1)|c_n|,$$

the function in (7) is strictly negative in the interval $0 \leqq r < 1$ and $g(r)/r$ is strictly decreasing in this interval, the minimum in (6) being $g(1) < 2|c_0|+|c_1|$, *or else* the function in (7) has just one zero $r=r_0$, $0 \leqq r_0 < 1$, which gives the minimum in (5), the case $r_0=0$ corresponding to $c_0=0$ and $\min\big(g(r)/r\big)=|c_1|$. It now follows from (5) and (6) that

$$\min_{0 \leqq r \leqq 1}\big(g(r)/r\big) = g(r_0)/r_0 \leqq g(\tfrac{1}{3})/\tfrac{1}{3} \leqq \begin{cases} 3 & \text{if } 0 < r_0 < 1, \\ 2 & \text{if } r_0 = 0 \text{ or } 1. \end{cases}$$

This leads at once to the conclusion of Theorem B corresponding to (3).

It remains to show that $\frac{1}{3}$ is the best absolute constant in the two conclusions of Theorem B. With this end in view, we consider the function

$$f_k(z) = \frac{k}{1+k} + \frac{2}{1+k}\cdot\frac{z}{1+z}, \qquad k > 0,$$

which is of class $\mathscr{B}$ and has the majorant

$$g_k(r) = \frac{k}{1+k} + \frac{2}{1+k}\cdot\frac{r}{1-r},$$

possessing the easily verifiable properties:

$$g_k(0) > 0, \qquad g_k(1-0) = \infty ,$$

$$\min_{0<r<1} \bigl(g(r)/r\bigr) = \frac{(2^{\frac{1}{2}}+k^{\frac{1}{2}})^2}{1+k} \qquad \text{for} \qquad r = \frac{k^{\frac{1}{2}}}{2^{\frac{1}{2}}+k^{\frac{1}{2}}}.$$

Consequently $f_{\frac{1}{2}}(z)$ is a function of class $\mathscr{B}$, and such that

$$g_{\frac{1}{2}}(\tfrac{1}{3}) = 1, \quad g_{\frac{1}{2}}\bigl(\tfrac{1}{3}(1+\varepsilon)\bigr) > 1 \qquad \text{for any} \quad \varepsilon > 0 ,$$

$$\min_{0<r<1} \bigl(g_{\frac{1}{2}}(r)/r\bigr) = 3 \qquad \text{for} \qquad r = \tfrac{1}{3} .$$

This concludes the proof.

3. NOTE. There is a distinction between Theorems A and B which may be pointed out here. *In* (2) *and* (3) *of Theorem* A, *the inequality sign* $\leqq$ *is actually* $<$, *while, in the corresponding conclusions of Theorem* B, $\leqq$ *cannot be replaced by* $<$, as shown by the example of $f_{\frac{1}{2}}(z)$ given above. This is due to the fact that, for functions $f(z)$ of class $\mathscr{B}$, (4) can be an equality for all $n \geqq 1$, as in the case of

$$f_0(z) = \frac{2z}{1+z}.$$

However, the analogue of (4) for functions of class $\mathscr{A}$, which is obtained by changing $f(z)$ to $f(z)\exp(-i\,\mathrm{am}\,c_0)$ and required in the proof of Theorem A, namely,

$$|c_n| \leqq 2(1-|c_0|) \qquad \text{for} \qquad n \geqq 1 , \tag{4'}$$

is actually a strict inequality, with the result that the step corresponding to (5): $g(\frac{1}{3}) \leqq 1$, in the proof of Theorem A, is $g(\frac{1}{3}) < 1$. To show that (4′) is a strict inequality, we have only to consider (4′) in the amplified form (e.g. [4, II′]):

$$|c_n| \leqq (1-|c_0|^2) \leqq 2(1-|c_0|) .$$

From this it is clear that (4′) cannot reduce to an equality even for a single value of n as that would imply $|c_0| = 1$, a possibility ruled out by the second half of condition (A).

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