MATH. SCAND. 5 (1957), 271–275

THE PINCHERLE BASIS PROBLEM AND A THEOREM OF BOAS

MAYNARD G. ARSOVE

1. Introduction. The successive powers of z can be regarded as a basis for the linear space $\mathscr{F}$ of all functions analytic on a neighborhood $N_R(0)$ of radius R about the origin. That is to say, if we interpret linear combinations in the infinite-series sense, then the successive powers of z are linearly independent functions which span the space[1].

Perhaps the most fundamental problem in basis theory is that of determining when a given sequence of functions comprises a basis for the space. No satisfactory solution has yet been found for this general basis problem. However, for certain restricted classes of sequences, which are, moreover, of interest in themselves, progress has been made in this direction.

We shall consider here one such class, namely the class of all sequences[2] $\{\alpha_n\}$ of the form

$$\alpha_n(z) = z^n[1+\lambda_n(z)], \tag{1.1}$$

where each λ_n is a function in $\mathscr{F}$ vanishing at the origin. In recognition of the fact that Pincherle [6] was the first to examine the possibility of expanding analytic functions in series of such functions, we shall refer to bases $\{\alpha_n\}$ of this sort as *Pincherle bases.*

The Pincherle basis problem—that of determining conditions on the perturbation functions λ_n under which $\{\alpha_n\}$ is a basis—has been studied in a number of papers[3], culminating in the work of Boas [3]. Starting with the Paley-Wiener basis approximation theorem [5, p. 100] and using orthogonal expansions in L^2, Boas has derived a general theorem [3, p. 477, Theorem 4.1] furnishing a partial solution to this problem.

Our aim here is to present a different approach to the problem, based upon the inversion of an operator $I+T$ by means of a geometric series in

Received November 25, 1957.

[1] An explicit presentation of the linear space terminology which we employ is given in [2].

[2] The sequences arising in the present paper will be indexed by $n=0, 1, \ldots$.

[3] For a bibliography, see Boas [3].

T. This device, now classical in operator theory, yields an elementary proof of the theorem of Boas.[1] Moreover, we obtain en route a corresponding partial solution of the Pincherle basis problem in certain Banach spaces of analytic functions.

2. A linear operator defined by $\{\lambda_n\}$. We fix $0 < R \leqq +\infty$ and agree to topologize the space $\mathscr{F}$ by the metric of uniform convergence on compact sets. As is well known, $\mathscr{F}$ is then complete but non-normable.

For $0 < r < R$ and

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

it is obvious that

$$\|f\|_r = \sum_{n=0}^{\infty} |a_n| r^n$$

defines a norm on $\mathscr{F}$. Since $\mathscr{F}$ is not complete under this norm, we shall examine also the linear subspace $\mathscr{A}$ of $\mathscr{F}$ consisting of those functions f for which

$$\|f\| = \lim_{r \to R} \|f\|_r < +\infty .$$

Discarding the trivial case of $R = +\infty$, we infer from Abel's theorem [4, pp. 177, 178] that

$$\|f\| = \sum_{n=0}^{\infty} |a_n| R^n .$$

This defines a valid norm, under which $\mathscr{A}$ is a Banach space [2][2].

Let us, for the moment, allow $\{\lambda_n\}$ to be any sequence of functions in $\mathscr{A}$ for which $\{\|\lambda_n\|\}$ is bounded. In terms of $\{\lambda_n\}$ we introduce on $\mathscr{A}$ the linear transformation T defined by

$$Tf(z) = \sum_{n=0}^{\infty} a_n z^n \lambda_n(z) . \tag{2.1}$$

That this series converges uniformly on compact sets to a function in $\mathscr{A}$ is apparent by using the power series expansions for the functions λ_n and taking account of the boundedness of $\{\|\lambda_n\|\}$. The same reasoning yields the inequality

$$\|Tf\| \leqq \sum_{n=0}^{\infty} |a_n| R^n \|\lambda_n\| ,$$

---

[1] Actually, we do not prove Theorem 4.1 of Boas [3] in its entirety. The assertion corresponding to condition (4.5) of this theorem is in many respects the most natural of the three assertions listed there, and it is this result that occupies our attention here. The assertion corresponding to condition (4.7) then follows directly, but that corresponding to (4.6) seems to require the L^2 techniques developed by Boas.

[2] When $R = 1$, it is clear that $\mathscr{A} = l^1$; and for all other $R < \infty$, $\mathscr{A}$ is easily seen to be isomorphic to l^1.

so that T is a bounded linear operator on $\mathscr{A}$. It is important to note also that if f has a zero of order m or greater at the origin, then

$$\|Tf\| \leqq \|f\| \sup_{n \geqq m} \|\lambda_n\| . \tag{2.2}$$

3. Application of the operator T to the Pincherle basis problem. In conformity with the requirements for Pincherle bases we now suppose that each λ_n vanishes at the origin. From this it is readily seen that $T^m f$ has a zero of order at least m at the origin, regardless of the choice of f in $\mathscr{A}$.

Let us suppose further that

$$\limsup_{n \to \infty} \|\lambda_n\| < 1 .$$

Then, fixing δ as any number such that

$$\limsup_{n \to \infty} \|\lambda_n\| < \delta < 1$$

and invoking (2.2), we infer without difficulty that there exist constants m_0 and K (depending only on the operator T) such that the inequality

$$\|T^m f\| \leqq K \|f\| \delta^m$$

holds for $m > m_0$ and all f in $\mathscr{A}$. Comparison with the corresponding geometric series in δ thus ensures convergence of the operator series

$$U = \sum_{n=0}^{\infty} (-T)^n , \tag{3.1}$$

and U, so defined, is a continuous linear operator on $\mathscr{A}$.

Moreover,

$$U = (I + T)^{-1} , \tag{3.2}$$

where I signifies the identity operator. That is, for an arbitrary function f in $\mathscr{A}$ the corresponding function g in $\mathscr{A}$ determined by $g = Uf$ satisfies the relation

$$f = (I + T)g . \tag{3.3}$$

If g has the power series expansion

$$g(z) = \sum_{n=0}^{\infty} c_n z^n ,$$

then (3.3) can be expressed as

$$f(z) = \sum_{n=0}^{\infty} c_n z^n [1 + \lambda_n(z)] . \tag{3.4}$$

The functions α_n of (1.1) therefore span $\mathscr{A}$. Since an elementary argument [1, p. 45] shows that functions of this form are linearly independent, it follows that $\{\alpha_n\}$ is a Pincherle basis in $\mathscr{A}$. Furthermore, the coefficient sequences $\{c_n\}$ for which $\Sigma c_n \alpha_n$ converges in $\mathscr{A}$ are exactly the sequences of Taylor coefficients of functions in $\mathscr{A}$. In the terminology of [2] this states that $\{\alpha_n\}$ is a *proper* basis in $\mathscr{A}$.

Collecting the above information, we have

THEOREM 1. *Let $\{\lambda_n\}$ be a sequence of functions in $\mathscr{A}$ such that*

(i) *each λ_n vanishes at the origin, and*
(ii) $\limsup_{n\to\infty} \|\lambda_n\| < 1$.

Then the sequence $\{\alpha_n\}$ defined by

$$\alpha_n(z) = z^n[1+\lambda_n(z)]$$

is a proper Pincherle basis in $\mathscr{A}$.

It is an easy matter to go from this result to a corresponding partial solution of the Pincherle basis problem, as originally formulated for $\mathscr{F}$.

THEOREM 2. *Let $\{\lambda_n\}$ be a sequence of functions in $\mathscr{F}$ such that*

(i) *each λ_n vanishes at the origin, and*
(ii) $\lim_{r\to R} (\limsup_{n\to\infty} \|\lambda_n\|_r) < 1$.

Then the sequence $\{\alpha_n\}$ defined by

$$\alpha_n(z) = z^n[1+\lambda_n(z)]$$

is a proper Pincherle basis in $\mathscr{F}$.

PROOF. To show that $\{\alpha_n\}$ spans $\mathscr{F}$, let f be an arbitrary function analytic on $N_R(0)$ and fix $r < R$. The functions f and λ_n are plainly all in $\mathscr{A}$ relative to $N_r(0)$. Theorem 1 therefore yields a sequence $\{c_n\}$ of complex numbers for which (3.4) holds, the convergence being uniform on compact subsets of $N_r(0)$. But, r can be taken arbitrarily close to R, and the coefficients c_n are independent of r. It follows that $\{\alpha_n\}$ is a basis in $\mathscr{F}$. That this basis is proper can be inferred without difficulty from the previous work. The simplest method, however, is to use condition (α) of [2]:

$$\limsup_{n\to\infty} (\|\alpha_n\|_r)^{1/n} < R \qquad (\text{all } r < R)\,.$$

Since $\|\alpha_n\|_r < 2r^n$ for large n, the above condition is obviously satisfied in the present case. This completes the proof.

Reasoning of a similar nature shows that for f in $\mathscr{F}$ and $\{\lambda_n\}$ taken as in Theorem 2, the development centering around equations (2.1), (3.1), (3.2), and (3.3) carries over unchanged to the space $\mathscr{F}$.

Thus, for the space $\mathscr{F}$, just as for the space $\mathscr{A}$, the operator $I+T$ is an automorphism[1] mapping z^n into $\alpha_n(z)$ for $n=0, 1, \ldots$. It is of interest to compare this with Theorems 2 and 4 of [2], which assert that the existence of such an automorphism is necessary and sufficient for $\{\alpha_n\}$ to be a proper basis in the respective spaces.

REFERENCES

1. M. G. Arsove, *Proper Pincherle bases in the space of entire functions*, Quart. J. Math. Oxford Ser. (2) 9 (1958), 40–54.
2. M. G. Arsove, *Proper bases and linear homeomorphisms in spaces of analytic functions*. Math. Ann. (to appear).
3. R. P. Boas, Jr., *Expansions of analytic functions*, Trans. Amer. Math. Soc. 48 (1940), 467–487.
4. K. Knopp, *Theory and Application of Infinite Series*, London, 1928.
5. R. E. A. C. Paley and N. Wiener, *Fourier Transforms in the Complex Domain*, New York, 1934.
6. S. Pincherle, *Sopra alcuni sviluppi in serie per funzioni analitiche*, Mem. Reale Acc. Sci. Bologna (4) 3 (1881), 151–180.

[1] By an *automorphism* we mean a linear homeomorphic mapping of the space onto itself.

INSTITUT HENRI POINCARÉ, PARIS, FRANCE
AND
UNIVERSITY OF WASHINGTON, SEATTLE, WASH., U. S. A.