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SOME REMARKS ON ONE-SIDED APPROXIMATION

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1. Problems concerning one-sided approximation have previously been considered by the authors (Freud [2], Ganelius [5]). In the present paper we are going to study one-sided approximation by trigonometrical polynomials to periodic functions with certain differentiability properties. By the methods used in [5] we shall construct approximation polynomials with some local properties which are of interest for several applications (Freud [3], Ganelius [4]).

Let $r \geqq 1$ be an integer. If a function h of period 2π is $r-1$ times continuously differentiable and if $h^{(r-1)}$ is the integral of a function h_r of bounded variation over a period, we shall say that $h \in H_r$. In addition H_0 consists of the periodic functions of bounded variation.

Let the periodic functions $\{b_m\}_1^\infty$ be defined by the conditions

$$b_m' = m b_{m-1}, \qquad \int_0^{2\pi} b_m(t)\,dt = 0 ,$$

and

$$b_1(\theta) = \theta - \pi \qquad \text{if} \qquad 0 < \theta < 2\pi .$$

(If $\bar{B}_m$ denotes that function of period 1 which in the interval (0,1) coincides with the Bernoulli polynomial B_m, then $b_m(\theta) = \bar{B}_m(\theta/(2\pi))$.)

Now, if $h \in H_r$, it holds that

$$(1.1) \qquad (r+1)!\left\{h(\theta) - (2\pi)^{-1}\int_0^{2\pi} h(t)\,dt\right\} = b_{r+1} * dh_r ,$$

where the expression on the right is defined by

$$b_{r+1} * dh_r = -(2\pi)^{-1}\int_0^{2\pi} b_{r+1}(\theta - t)\,dh_r(t) .$$

We shall also use the notations

$$f * g = (2\pi)^{-1}\int_0^{2\pi} f(\theta - t) g(t)\,dt$$

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and

$$\|f\| = (2\pi)^{-1}\int_0^{2\pi} |f(t)|\,dt\,.$$

2. For the proof of our fist theorem we need some results on the one-sided approximation to b_m.

As was proved in [5, sect. 1] there is a trigonometrical polynomial $U_{1,n}$ of order n with the following properties ($U_{1,n}$ is the $T_{1,n}$ of [5]).

If we put

$$K_{1,n} = U_{1,n} - b_1\,,$$

then

$$0 \leqq K_{1,n}(\theta) = O(1)\min\left(1, (n \sin\tfrac{1}{2}\theta)^{-2}\right), \tag{2.1}$$

$$\|K_{1,n}\| = O(n^{-1})\,, \tag{2.2}$$

$$\operatorname{var}_{[0,2\pi]} K_{1,n} = O(1)\,. \tag{2.3}$$

The constant symbolized by the O-sign is always independent of n and θ.

If we put $k_{1,n} = b_1 - u_{1,n}$, where the polynomial $u_{1,n}$ is defined by $u_{1,n}(\theta) = -U_{1,n}(-\theta)$, then $k_{1,n}$ satisfies exactly the same inequalities as $K_{1,n}$ since $b_1(\theta) = -b_1(-\theta)$.

We need another estimate for $U_{1,n}$, viz.

$$|U_{1,n}'(\theta) - 1| = O(n)\min\left(1, (n \sin\tfrac{1}{2}\theta)^{-2}\right). \tag{2.4}$$

In fact, such an inequality is true for every polynomial of order n satisfying a formula corresponding to (2.1). To prove that, we consider the function

$$g(\theta) = b_1(\theta)\sin^2\tfrac{1}{2}\theta\,.$$

It is easily seen that $g \in H_2$, and hence there is a trigonometrical polynomial G_n of order n, such that

$$|G_n - g| = O(n^{-2})\,, \tag{2.5}$$

$$|G_n' - g'| = O(n^{-1})\,. \tag{2.6}$$

(Put, for example, $G_n = -b_1 * W_n$, where W_n is the polynomial belonging to g' according to theorem III in [5].)

But from (2.1) and (2.5) we infer that

$$|\sin^2\tfrac{1}{2}\theta\; U_{1,n}(\theta) - G_n(\theta)| = O(n^{-2})\,,$$

and by the Bernstein inequality

$$|\sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta\; U_{1,n}(\theta) + \sin^2\tfrac{1}{2}\theta\; U_{1,n}'(\theta) - G_n'(\theta)| = O(n^{-1})\,.$$

By aid of (2.6) we obtain

$$\left|\sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta\left(U_{1,n}(\theta)-b_1(\theta)\right)+\sin^2\tfrac{1}{2}\theta\left(U_{1,n}'(\theta)-1\right)\right| = O(n^{-1}),$$

and using (2.1) we get the estimate $O(n)\,(n\sin\frac{1}{2}\theta)^{-2}$ of (2.4). The estimate $O(n)$ is a simple consequence of $U_{1,n}(\theta)=O(1)$ and Bernstein's inequality.

3. We now define polynomials $U_{m,n}$ for $m>1$ by

$$U_{m,n}-b_m = m(U_{1,n}-b_1)*(U_{m-1,n}-b_{m-1})$$

and similarly

$$b_m-u_{m,n} = m(b_1-u_{1,n})*(U_{m-1,n}-b_{m-1}).$$

We observe that $U_{m,n}-b_m\geqq 0$, $b_m-u_{m,n}\geqq 0$. That $U_{m,n}$ and $u_{m,n}$ are polynomials of order n follows from the construction since $b_m=-mb_1*b_{m-1}$. In conformity with our previous notations we put

$$K_{m,n} = U_{m,n}-b_m \quad\text{and}\quad k_{m,n} = b_m-u_{m,n}.$$

We observe that

$$K_{m,n} = mK_{1,n}*K_{m-1,n}, \tag{3.1}$$

and hence

$$\|K_{m,n}\| = m\|K_{1,n}\|\|K_{m-1,n}\|.$$

Since $\|K_{1,n}\|=O(n^{-1})$ it follows that

$$\|K_{m,n}\| = O(n^{-m}), \tag{3.2}$$

and the same is true for $k_{m,n}$.

Lemma. *The functions $K_{m,n}$ and $k_{m,n}$ defined above satisfy the inequalities*

$$|K_{m,n}^{(q)}(\theta)| = O(n^{q-m+1})\min\left(1,(n\sin\tfrac{1}{2}\theta)^{-2}\right), \tag{3.3}$$

$$|k_{m,n}^{(q)}(\theta)| = O(n^{q-m+1})\min\left(1,(n\sin\tfrac{1}{2}\theta)^{-2}\right), \tag{3.4}$$

for all θ, if $0\leqq q\leqq m-1$.

It is sufficient to prove (3.3) since (3.4) follows in a similar way.

We first derive the estimate $O(n^{q-m+1})$ given in (3.3). From (3.1) we see that

$$\sup|K_{m,n}| \leqq m\|K_{m-1,n}\|\sup|K_{1,n}|,$$

and application of (2.1) and (3.2) yields

$$\sup|K_{m,n}| = O(n^{-m+1}), \tag{3.5}$$

i.e. the first half of (3.3) in the case $q=0$.

If $0<q\leqq m-1$, we obtain from (3.1) that

$$K_{m,n}^{(q)} = mK_{m-1,n}^{(q-1)}*dK_{1,n}, \tag{3.6}$$

and hence $\sup|K_{m,n}{}^{(q)}| = O(1)\sup|K_{m-1,n}{}^{(q-1)}|\operatorname{var}_{[0,2\pi]}K_{1,n}$.

Using (2.3) and iterating we get

$$\sup|K_{m,n}{}^{(q)}| = O(1)\sup|K_{m-q,n}| = O(n^{q-m+1})$$

by (3.5).

It remains to prove that

$$|K_{m,n}{}^{(q)}(\theta)| = O\big(n^{q-m-1}(\sin\tfrac{1}{2}\theta)^{-2}\big). \tag{3.7}$$

It is no restriction to suppose that $|\theta|\leqq\pi$. We rewrite (3.1) as

$$(2\pi/m)K_{m,n}(\theta) = \int_0^{\pi} K_{1,n}(\tfrac{1}{2}\theta-t)\,K_{m-1,n}(\tfrac{1}{2}\theta+t)\,dt + \int_{-\pi}^{0} K_{1,n}(\tfrac{1}{2}\theta-t)\,K_{m-1,n}(\tfrac{1}{2}\theta+t)\,dt$$

and find

$$m^{-1}|K_{m,n}(\theta)| \leqq \|K_{1,n}\|\sup_{\frac{1}{2}|\theta|\leqq t\leqq 2\pi-\frac{1}{2}|\theta|}|K_{m-1,n}(t)| + \|K_{m-1,n}\|\sup_{\frac{1}{2}|\theta|\leqq t\leqq 2\pi-\frac{1}{2}|\theta|}|K_{1,n}(t)|.$$

By aid of (2.1), (2.2) and (3.2) we obtain

$$|K_{m,n}(\theta)| = O(n^{-1})\sup_{\frac{1}{2}|\theta|\leqq t\leqq 2\pi-\frac{1}{2}|\theta|}|K_{m-1,n}(t)| + O\big(n^{-m-1}(\sin\tfrac{1}{2}\theta)^{-2}\big).$$

Since $K_{1,n}(\theta)=O\big((n\sin\frac{1}{2}\theta)^{-2}\big)$, it follows by repeated use of this inequality that

$$|K_{m,n}(\theta)| \leqq O\big(n^{-m-1}(\sin\tfrac{1}{2}\theta)^{-2}\big). \tag{3.8}$$

Hence (3.7) is proved if $q=0$.

If $0<q\leqq m-1$, we use (3.6) written in the form

$$(2\pi/m)K_{m,n}{}^{(q)}(\theta) = \int_0^{\pi} K_{m-1,n}{}^{(q-1)}(\tfrac{1}{2}\theta-t)\,dK_{1,n}(\tfrac{1}{2}\theta+t) + \int_{-\pi}^{0} K_{m-1,n}{}^{(q-1)}(\tfrac{1}{2}\theta-t)\,dK_{1,n}(\tfrac{1}{2}\theta+t),$$

and hence

$$m^{-1}|K_{m,n}{}^{(q)}(\theta)| \leqq \sup_{\frac{1}{2}|\theta|\leqq t\leqq 2\pi-\frac{1}{2}|\theta|}|K_{m-1,n}{}^{(q-1)}|\operatorname{var}_{[0,2\pi]}K_{1,n} + \sup_{\frac{1}{2}|\theta|\leqq t\leqq 2\pi-\frac{1}{2}|\theta|}|K_{1,n}{}'|\,\|K_{m-1,n}{}^{(q-1)}\|. \tag{3.9}$$

By aid of (3.6), (2.3) and (3.2) we get

$$\|K_{m-1,n}{}^{(q-1)}\| = O(1)\,\|K_{m-q,n}\| = O(n^{q-m}).$$

Insertion of this result in (3.9) and application of (2.3) and (2.4) yield

$$|K_{m,n}^{(q)}(\theta)| = O(1) \sup_{\frac{1}{2}|\theta| \leqq t \leqq 2\pi-\frac{1}{2}|\theta|} |K_{m-1,n}^{(q-1)}(\theta)| + O\big(n^{q-m-1}(\sin\tfrac{1}{2}\theta)^{-2}\big).$$

By repeated application of this formula we obtain

$$|K_{m,n}^{(q)}(\theta)| = O(1) \sup_{\frac{1}{2}|\theta| \leqq t \leqq 2\pi-\frac{1}{2}|\theta|} |K_{m-q,n}(\theta)| + O\big(n^{q-m-1}(\sin\tfrac{1}{2}\theta)^{-2}\big),$$

and with the help of (3.8) we get (3.7).

Hence the lemma is proved.

THEOREM 1. *Let $h \in H_r$ and let q be an integer, $0 \leqq q \leqq r$.*

To every positive integer n there is a trigonometrical polynomial W_n of order n, such that

$$W_n - h \geqq 0,$$

and

$$\|W_n - h\| \leqq C_r \operatorname{var} h_r\, n^{-r-1}, \tag{4.1}$$

where C_r depends only on r.

If γ is a point of continuity of h_r, then

$$W_n^{(q)}(\gamma) - h^{(q)}(\gamma) = o(n^{q-r}), \tag{4.2}$$

and for all θ it holds uniformly that

$$W_n^{(q)}(\theta) - h^{(q)}(\theta) = O(n^{q-r}). \tag{4.3}$$

REMARK. Comparing this theorem with theorem III in [5], we see that new results are given only in (4.2) and (4.3). That the polynomials W_n of theorem 1 have the same property,

$$\operatorname{var}_{[0,2\pi]}\{W_n - h\} = O(n^{-r}),$$

as the corresponding polynomials in [5], is an easy consequence of [5, theorem III] and the L_1-version of Bernstein's inequality (cf. the proof of (5.8) below).

PROOF OF THEOREM 1. We construct W_n in the following way. Denoting the positive and negative variations of h_r by h_r^+ and h_r^-, we put

$$(r+1)!\,\{W_n - h\} = (b_{r+1} - u_{r+1,n})\,dh_r^+ + (U_{r+1,n} - b_{r+1}) * dh_r^-. \tag{4.4}$$

Since $h = h^+ - h^- + \text{const.}$ it follows from (1.1) that W_n is a trigonometrical polynomial of order n, and it is evident that $W_n - h \geqq 0$.

Introducing our previous notations and integrating (4.4) we find

$$\|W_n - h\| = \tfrac{1}{2} \operatorname{var} h_r \,\{\|k_{r+1,n}\| + \|K_{r+1,n}\|\} = O(n^{-r-1}) \operatorname{var} h_r,$$

and hence (4.1) is proved.

Differentiating (4.4) we get

$$(r+1)!\,\{W_n^{(q)}(\theta)-h^{(q)}(\theta)\}$$
$$= k_{r+1,n}^{(q)} * dh_r^+ + K_{r+1,n}^{(q)} * dh_r^-$$
$$= \int_{\theta-\varepsilon}^{\theta+\varepsilon} + \int_{\theta+\varepsilon}^{2\pi+\theta-\varepsilon} \big(k_{r+1,n}^{(q)}(\theta-t)\,dh_r^+(t) + K_{r+1,n}^{(q)}(\theta-t)\,dh_r^-(t)\big)\,.$$

Hence

$$|W_n^{(q)}(\theta)-h^{(q)}(\theta)| = O(1)\,\mathrm{var}_{[\theta-\varepsilon,\theta+\varepsilon]}h_r\,\{\sup|K_{r+1,n}^{(q)}|+\sup|k_{r+1,n}^{(q)}|\} +$$
$$+ O(1)\,\mathrm{var}_{[0,2\pi]}h_r\left\{\sup_{\varepsilon\leqq t\leqq 2\pi-\varepsilon}|K_{r+1,n}^{(q)}(t)| + \sup_{\varepsilon\leqq t\leqq 2\pi-\varepsilon}|k_{r+1,n}^{(q)}(t)|\right\},$$

and application of the lemma yields

$$(4.5)\quad |W_n^{(q)}(\theta)-h^{(q)}(\theta)| = O(n^{q-r})\,\{\mathrm{var}_{[\theta-\varepsilon,\theta-\varepsilon]}h_r + n^{-2}\varepsilon^{-2}\,\mathrm{var}_{[0,2\pi]}h_r\}\,.$$

This result immediately implies (4.3). If θ is a point of continuity of h_r, then $\mathrm{var}_{[\theta-\varepsilon,\theta+\varepsilon]}h_r$ is arbitrarily small with ε and (4.2) follows.

Remark. Formula (4.5) shows that the following sharper formulation of our result holds. *If γ is a point of continuity of h_r, then to every $\varepsilon>0$ there are numbers δ and N so that*

$$|W_n^{(q)}(\theta)-h^{(q)}(\theta)| \leqq \varepsilon n^{q-r}$$

for all θ and N satisfying $|\theta-\gamma|<\varepsilon$, $n>N$.

5. Our second theorem will be derived from theorem III in [5] with the help of some special polynomials Q_n.

If φ and ψ are two numbers, $0<\psi-\varphi<2\pi$, the polynomial Q_n belonging to the interval $[\varphi,\psi]$ is defined by

$$Q_n(\theta) = \int_\varphi^\psi \big(\cos\tfrac12(\theta-t)\big)^{2n}dt\left\{\int_0^{2\pi}(\cos\tfrac12 t)^{2n}dt\right\}^{-1}.$$

Since $\big(\cos\frac12(\theta-t)\big)^{2n}$ is a trigonometrical polynomial of order n in θ, the same holds for Q_n. Evidently $0\leqq Q_n\leqq 1$. Since

$$\left\{\int_0^{2\pi}(\cos\tfrac12 t)^{2n}dt\right\}^{-1} = O(n^{\frac12})\,,$$

we find that

$$Q_n(\theta) = O(e^{-kn}) \qquad \text{if} \qquad \psi+\varepsilon \leqq \theta \leqq \varphi+2\pi-\varepsilon\,,$$

where the positive number k depends on $\varepsilon > 0$. By considering $1 - Q_n$ we find in the same way that

$$1 - Q_n(\theta) = O(e^{-kn}) \qquad \text{if} \qquad \varphi + \varepsilon \leqq \theta \leqq \psi - \varepsilon .$$

THEOREM 2. *Let $h \in H_r$ and suppose that there is an interval I such that $h(\theta) = \eta(\theta)$ if $\theta \in I$, where $\eta \in H_s$, $s > p$. Let I^* be a closed interval in the interior of I. Then to every positive integer n there is a trigonometrical polynomial W_n of order n satisfying*

$$V_n = W_n - h \geqq 0 , \tag{5.1}$$

$$\|V_n\| = O(n^{-p-1}) , \tag{5.2}$$

$$\mathrm{var}_{[0,2\pi]} V_n = O(n^{-p}) , \tag{5.3}$$

$$\int_{I^*} V_n(t)\, dt = O(n^{-s-1}) , \tag{5.4}$$

$$\mathrm{var}_{I^*} V_n = O(n^{-s}) . \tag{5.5}$$

If there are several disjoint intervals $\{I_k\}$ such that $h(\theta) = \eta_k(\theta)$ if $\theta \in I_k$, and $\eta_k \in H_{s_k}$, $s_k > p$, then inequalities corresponding to (5.4–5) can be simultaneously obtained on closed subintervals $I_k^* \subset I_k$.

PROOF. On account of our assumptions on h and η we know by theorem III in [5] that there are trigonometrical polynomials F_n and Φ_n of order n such that

$$F_n - h \geqq 0, \qquad \Phi_n - \eta \geqq 0 , \tag{5.6}$$

$$\|F_n - h\| = O(n^{-p-1}), \qquad \|\Phi_n - \eta\| = O(n^{-s-1}) , \tag{5.7}$$

$$\mathrm{var}_{[0,2\pi]}(F_n - h) = O(n^{-p}), \qquad \mathrm{var}_{[0,2\pi]}(\Phi_n - \eta) = O(n^{-s}) . \tag{5.8}$$

Let us now consider the trigonometrical polynomial of order n defined by

$$U_n = (1 - Q_n) F_n + Q_n \Phi_n ,$$

where Q_n is the polynomial constructed above for the interval $[\varphi, \psi]$ which we assume to belong to the interior of the interval I. We find that

$$U_n - h = (1 - Q_n)(F_n - h) + Q_n(\Phi_n - h) , \tag{5.9}$$

and since $h = \eta$ in the interval I, we infer that $U_n - h \geqq 0$ in I. Outside this interval the term $Q_n(\Phi_n - h)$ can be negative, but since $Q_n = O(e^{-kn})$, there is a constant $c_n = O(e^{-kn})$ such that

$$V_n = U_n + c_n - h_p \geqq 0 \qquad \text{for all} \quad \theta . \tag{5.10}$$

We shall now prove that with this choice, $W_n = U_n + c_n$, the formulas (5.2–5) are true. That W_n is of order $2n$ is, of course, immaterial.

Let us first consider $\|V_n\|$. We find by aid of (5.10) and (5.9) that

$$\int_0^{2\pi} V_n dt \leqq 2\pi c_n + \int_0^{2\pi} (F_n - h)dt + \int_\varphi^\psi (\Phi_n - h)dt + \int_\psi^{\varphi+2\pi} |\Phi_n - h|\, dt \max_{\theta \in I} Q_n(\theta)$$
$$= O(e^{-kn}) + O(n^{-p-1}) + O(n^{-s-1}) + O(e^{-kn})$$
$$= O(n^{-p-1}) ,$$

and hence V_n satisfies (5.2).

(5.3) can be deduced by aid of Bernstein's theorem on the derivative of a trigonometrical polynomial in its L_1-formulation (Zygmund [6, p. 23]) in the following way. According to (5.6–8) the trigonometrical polynomial F_n satisfies

$$\|F_n - h\| = O(n^{-p-1})$$

and

$$\|F_n' - h'\| = (2\pi)^{-1} \operatorname{var}(F_n - h) = O(n^{-p}) .$$

The polynomial W_n of order $2n$ satisfies $\|W_n - h\| = O(n^{-p-1})$. Hence

$$\|W_n - F_n\| = O(n^{-p-1})$$

and by Bernstein's inequality

$$\|W_n' - F_n'\| = O(n^{-p}) .$$

If this result is combined with $\|F_n' - h'\| = O(n^{-p})$, we get

$$\|W_n' - h'\| = O(n^{-p}) ,$$

and (5.3) is proved. If $p=0$, the function h is not necessarily differentiable but a simple approximation argument works.

Let I^* be a closed interval in the interior of (φ, ψ). Then

$$\int_{I^*} V_n d\theta \leqq 2\pi c_n + \int_{I^*} (F_n - h)dt \max_{I^*}(1 - Q_n) + \int_{I^*} (\Phi_n - h)dt$$
$$= O(e^{-kn}) + \int_0^{2\pi} (\Phi_n - \eta)dt$$
$$= O(n^{-s-1}) ,$$

and (5.4) is proved.

It remains to prove (5.5). Since $h = \eta$ in I^* it follows from (5.4) and (5.7) that

$$\int_{I^*} |\Phi_n - W_n|\, dt = O(n^{-s-1}) . \tag{5.11}$$

Now N. K. Bari [1] has proved a localized version of Bernstein's theorem, so that it follows from (5.11) that

$$\operatorname*{var}_{I^{**}}(\Phi_n - W_n) = \int_{I^{**}} |\Phi_n' - W_n'|\, dt = O(n^{-s})$$

if I^{**} is an interval in the interior of I^*. If this result is combined with (5.8) we get

$$\operatorname*{var}_{I^{**}}(W_n - \eta) = O(n^{-s}) .$$

But then (5.5) is proved, since we can change the notations for the different sub-intervals of I.

An examination of the proof reveals that the process can be repeated for another interval I_k^* with conservation of the result just obtained, and the proof of the theorem is finished.

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