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ON THE EIGENVALUES OF GENERALIZED TOEPLITZ MATRICES

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1. Introduction.

The class of generalized Toeplitz matrices which we will consider was introduced by M. Kac, W. L. Murdock and G. Szegö [3] and is defined in the following way.

Let $f(x,\theta)$ be a complex valued function defined for $0 \leqq x \leqq 1$, $0 \leqq \theta \leqq 2\pi$, and with the property that the Fourier coefficients

$$c_\nu(x) = \frac{1}{2\pi}\int_0^{2\pi} f(x,\theta)\, e^{-i\nu\theta}\, d\theta, \qquad \nu = 0, \pm 1, \pm 2, \ldots ,$$

are defined. For each positive integer n we associate with f the $(n+1)$ by $(n+1)$ matrix

$$(1) \qquad T_n(f) = \left(c_{j-i}\left(\frac{i+j}{2n+2}\right)\right), \qquad i,j = 0,1,\ldots,n .$$

We denote the $n+1$ eigenvalues of $T_n(f)$ by

$$\lambda_{n0}, \lambda_{n1}, \ldots, \lambda_{nn} .$$

It is the purpose of our paper to obtain information about the behavior of these eigenvalues as n becomes infinite, and like M. Kac, W. L. Murdock and G. Szegö [3] we will do this by studying the behavior of the determinants

$$D_n(f) = \det T_n(f) = \lambda_{n0}\lambda_{n1} \ldots \lambda_{nn} ,$$

and the traces ($p \geqq 0$ an integer)

$$\operatorname{tr}\left([T_n(f)]^p\right) = \sum_{j=0}^{n} \lambda_{nj}{}^p ,$$

as n becomes infinite. The result by M. Kac, W. L. Murdock and G. Szegö [3] can be stated in the form that if $f(x,\theta)$ satisfies certain conditions we have

$$(2) \qquad \lim_{n\to\infty} [D_n(f)]^{1/(n+1)} = G(f) ,$$

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where

$$G(f) = \exp\left\{\frac{1}{2\pi}\int_0^1\int_0^{2\pi} \log f(x,\theta)\, d\theta\, dx\right\},$$

and for each fixed integer $p \geqq 0$ we have

$$\lim_{n\to\infty} \frac{1}{n+1}\sum_{j=0}^{n} \lambda_{nj}^{p} = \frac{1}{2\pi}\int_0^1\int_0^{2\pi} [f(x,\theta)]^p\, d\theta\, dx\,. \tag{3}$$

If $f=f(\theta)$ is a function of θ only, the matrices $T_n(f)$, $n=1,2,\ldots$, reduce to the ordinary Toeplitz matrices associated with a function defined and integrable in $(0,2\pi)$, in which case G. Szegö [5] and M. Kac [2] have sharpened (2) and (3) respectively for a large class of functions $f(\theta)$ by proving that the limits

$$\lim_{n\to\infty} \frac{D_n(f)}{[G(f)]^{n+1}} \qquad \text{and} \qquad \lim_{n\to\infty}\left\{\sum_{j=0}^{n} \lambda_{nj}^{p} - \frac{n+1}{2\pi}\int_0^{2\pi} [f(\theta)]^p\, d\theta\right\}$$

exist, and by evaluating these limits. It is natural to try to find corresponding sharpenings of (2) and (3) also in the general case. In a previous note [4] we have done so, but with very restrictive conditions on the function $f(x,\theta)$. In this paper we will extend the results from [4]. For sufficiently nice functions $f(x,\theta)$ we will show that, if $p>0$ is a fixed integer, then

$$\lim_{n\to\infty}\left\{\sum_{j=0}^{n} \lambda_{nj}^{p} - \frac{n+1}{2\pi}\int_0^1\int_0^{2\pi} [f(x,\theta)]^p\, d\theta\, dx\right\}$$

exists, and we will evaluate that limit. The precise statement and the proof are given in Section 2.

This sharpening of (3) implies in a natural way a certain sharpening of (2) too. If the function $f(x,\theta)$ is "small", we have

$$D_n(1-f) = \exp\left\{-\sum_{p=1}^{\infty}\frac{1}{p}\sum_{j=0}^{n}\lambda_{nj}^{p}\right\},$$

which enables us to prove that the limit

$$\lim_{n\to\infty} \frac{D_n(1-f)}{[G(1-f)]^{n+1}}$$

exists, and to evaluate that limit. The precise statement and the proof are given in Section 3. Regrettably we have not been able to remove the requirement that f should be "small".

We want to express our gratitude to professor M. Kac, who introduced us to the problems considered in this paper.

2. Asymptotic behavior of the traces of generalized Toeplitz matrices. We consider the class of complex valued functions of the type

$$f(x,\theta) = \sum_{\nu=-\infty}^{\infty} c_\nu(x)\, e^{i\nu\theta} ,$$

satisfying the following condition.

CONDITION A. (i) *The Fourier coefficients* $c_\nu(x)$, $\nu = 0, \pm 1, \pm 2, \ldots,$ *are twice continuously differentiable in the interval* $0 \leqq x \leqq 1$.

(ii) *Let*

$$c_\nu = \max_{0\leqq x\leqq 1} |c_\nu(x)|, \qquad c_\nu' = \max_{0\leqq x\leqq 1} |c_\nu'(x)|, \qquad c_\nu'' = \max_{0\leqq x\leqq 1} |c_\nu''(x)| ,$$

for $\nu = 0, \pm 1, \pm 2, \ldots$. *Then*

$$\sum_{\nu=-\infty}^{\infty} c_\nu' < \infty \qquad \text{and} \qquad \sum_{\nu=-\infty}^{\infty} c_\nu'' < \infty ,$$

and there exists a number $\alpha > 2$ *such that*

$$\sum_{\nu=-\infty}^{\infty} |\nu|^\alpha c_\nu < \infty .$$

We now state our main theorem.

THEOREM 1. *Let* $f(x,\theta)$ *satisfy Condition* A, *and let* $\lambda_{n0}, \lambda_{n1}, \ldots, \lambda_{nn}$ *be the eigenvalues of the matrix* $T_n(f)$ *defined by* (1). *Then for every integer* $p \geqq 0$

$$\lim_{n\to\infty} \left\{ \sum_{j=0}^{n} \lambda_{nj}{}^p - \frac{n+1}{2\pi} \int_0^1 \int_0^{2\pi} [f(x,\theta)]^p \, d\theta\, dx \right\}$$
$$= \frac{1}{4\pi} \int_0^{2\pi} [f(0,\theta)]^p \, d\theta - \frac{1}{4\pi} \int_0^{2\pi} [f(1,\theta)]^p \, d\theta \; -$$
$$- \sum_{\substack{-\infty < l_1, \ldots, l_p < \infty \\ l_1 + \ldots + l_p = 0}} [c_{l_1}(0) \ldots c_{l_p}(0) + c_{l_1}(1) \ldots c_{l_p}(1)] \cdot$$
$$\cdot \max(0, l_1, l_1 + l_2, \ldots, l_1 + \ldots + l_{p-1}) .$$

Furthermore there exists a constant C *such that for all* $n, p = 1, 2, \ldots,$

$$\left| \sum_{j=0}^{n} \lambda_{nj}{}^p - \frac{n+1}{2\pi} \int_0^1 \int_0^{2\pi} [f(x,\theta)]^p \, d\theta\, dx \right| \leqq C p^4 \left(\sum_{\nu=-\infty}^{\infty} c_\nu \right)^p .$$

It is interesting to observe that the limit occurring in Theorem 1 only depends on the two boundary functions $f(0,\theta)$ and $f(1,\theta)$.

If $f=f(\theta)$ is a function of θ only, Theorem 1 with Condition A replaced by the condition

$$\sum_{\nu=-\infty}^{\infty} |\nu|\,|c_\nu| < \infty ,$$

has been proved by M. Kac [2], as already mentioned in the introduction.

The proof of Theorem 1 is quite elementary, but rather lengthy. It is divided into a number of steps, and to simplify the exposition we carry out the details for $p=3$, which case is representative of the general situation.

We introduce the constants

$$M = \sum_{\nu=-\infty}^{\infty} c_\nu, \qquad M' = \sum_{\nu=-\infty}^{\infty} c_\nu', \qquad M'' = \sum_{\nu=-\infty}^{\infty} c_\nu'' ,$$

and for notational convenience we also introduce $\varepsilon=\alpha^{-1}$.

2.1. The first step is to show that we can replace the matrix $T_n(f)$ by the matrix obtained by substituting 0 for the (i,j)'th element in $T_n(f)$ whenever $|j-i|\geqq n^\varepsilon$. This matrix is $T_n(f_n)$, the n'th generalized Toeplitz matrix associated with the function

$$f_n(x,\theta) = \sum_{|\nu|<n^\varepsilon} c_\nu(x)\, e^{i\nu\theta}, \qquad n=1,2,\ldots .$$

We have

$$\begin{aligned}
\sum_{j=0}^{n} \lambda_{nj}^{\,3} &= \operatorname{tr}\big([T_n(f)]^3\big) \\
&= \sum_{j=0}^{n} \sum_{\substack{l_1+l_2+l_3=0\\ -j\leqq l_1,\, l_1+l_2\leqq n-j}} c_{l_1}\left(\frac{2j+l_1}{2n+2}\right) c_{l_2}\left(\frac{2j+2l_1+l_2}{2n+2}\right) c_{l_3}\left(\frac{2j+2l_1+2l_2+l_3}{2n+2}\right) \\
&= \sum_{j=0}^{n} \sum_{\substack{l_1+l_2+l_3=0\\ -j\leqq l_1,\, l_1+l_2\leqq n-j\\ |l_1|,|l_2|,|l_3|<n^\varepsilon}} + \sum_{j=0}^{n} \sum_{\substack{l_1+l_2+l_3=0\\ -j\leqq l_1,\, l_1+l_2\leqq n-j\\ \max(|l_1|,|l_2|,|l_3|)\geqq n^\varepsilon}} \\
&= \operatorname{tr}\big([T_n(f_n)]^3\big) + R_{1n}^{(3)} ,
\end{aligned}$$

where

$$\begin{aligned}
|R_{1n}^{(3)}| &\leqq \sum_{j=0}^{n} \sum_{\max(|l_1|,|l_2|,|l_3|)\geqq n^\varepsilon} c_{l_1}c_{l_2}c_{l_3} \\
&\leqq 3(n+1)\Big(\sum_{|\nu|\geqq n^\varepsilon} c_\nu\Big)\Big(\sum_{\nu=-\infty}^{\infty} c_\nu\Big)^2 \\
&\leqq 3\,\frac{n+1}{n}\Big(\sum_{|\nu|\geqq n^\varepsilon} |\nu|^\alpha c_\nu\Big) M^2 .
\end{aligned}$$

By Condition A, (ii), the last quantity tends to zero as n becomes infinite. Hence we have proved

$$\sum_{j=0}^{n} \lambda_{nj}^3 = \operatorname{tr}\left([T_n(f_n)]^3\right) + R_{1n}^{(3)}, \tag{4}$$

where

$$\lim_{n\to\infty} R_{1n}^{(3)} = 0 . \tag{5}$$

If we do the calculations for a general $p>0$, we find

$$|R_{1n}^{(p)}| \leqq p\frac{n+1}{n}\left(\sum_{|\nu|\geqq n^\varepsilon} |\nu|^\alpha c_\nu\right) M^{p-1},$$

and consequently there exists a constant C_1 such that for all n and p

$$|R_{1n}^{(p)}| \leqq C_1 p M^p . \tag{6}$$

2.2. Consider the j'th diagonal element of the matrix $[T_n(f_n)]^3$, $j=0,1,\ldots,n$,

$$\left([T_n(f_n)]^3\right)_{jj} = \sum_{\substack{l_1+l_2+l_3=0 \\ -j\leq l_1,\, l_1+l_2\leq n-j \\ |l_1|,|l_2|,|l_3|<n^\varepsilon}} c_{l_1}\left(\frac{2j+l_1}{2n+2}\right) c_{l_2}\left(\frac{2j+2l_1+l_2}{2n+2}\right) c_{l_3}\left(\frac{2j+2l_1+2l_2+l_3}{2n+2}\right),$$

and let r_{jn} be determined by

$$\left([T_n(f_n)]^3\right)_{jj} = \sum_{\substack{l_1+l_2+l_3=0 \\ -j\leq l_1,\, l_1+l_2\leq n-j \\ |l_1|,|l_2|,|l_3|<n^\varepsilon}} c_{l_1}\left(\frac{j}{n+1}\right) c_{l_2}\left(\frac{j}{n+1}\right) c_{l_3}\left(\frac{j}{n+1}\right) + r_{jn} . \tag{7}$$

For every set of indices occurring in the above summation we introduce the function (depending on n and j)

$$g_{l_1l_2l_3}(t) = c_{l_1}\left(\frac{j}{n+1}+t\frac{l_1}{2n+2}\right)c_{l_2}\left(\frac{j}{n+1}+t\frac{2l_1+l_2}{2n+2}\right)c_{l_3}\left(\frac{j}{n+1}+t\frac{2l_1+2l_2+l_3}{2n+2}\right),$$

which is defined in an interval containing $0\leqq t\leqq 1$. Then

$$r_{jn} = \sum_{\substack{l_1+l_2+l_3=0 \\ -j\leq l_1,\, l_1+l_2\leq n-j \\ |l_1|,|l_2|,|l_3|<n^\varepsilon}} \{g_{l_1l_2l_3}(1)-g_{l_1l_2l_3}(0)\},$$

and by the mean value theorem we get

$$|r_{jn}| \leqq \sum_{\substack{l_1+l_2+l_3=0 \\ -j\leq l_1,\, l_1+l_2\leq n-j \\ |l_1|,|l_2|,|l_3|<n^\varepsilon}} \max_{0\leq t\leq 1} |g'_{l_1l_2l_3}(t)| \leqq \frac{3n^\varepsilon}{n+1} 3M'M^2 . \tag{8}$$

Now let $3n^\varepsilon < j < n - 3n^\varepsilon$ and let $T_n^*(f_n)$ be the transpose of $T_n(f_n)$. Then

$$\begin{aligned}
&([T_n^*(f_n)]^3)_{jj} \\
&= \sum_{\substack{l_1+l_2+l_3=0 \\ |l_1|,|l_2|,|l_3|<n^\varepsilon}} c_{-l_1}\left(\frac{2j+l_1}{2n+2}\right) c_{-l_2}\left(\frac{2j+2l_1+l_2}{2n+2}\right) c_{-l_3}\left(\frac{2j+2l_1+2l_2+l_3}{2n+2}\right) \\
&= \sum_{\substack{l_1+l_2+l_3=0 \\ |l_1|,|l_2|,|l_3|<n^\varepsilon}} c_{l_1}\left(\frac{2j-l_1}{2n+2}\right) c_{l_2}\left(\frac{2j-2l_1-l_2}{2n+2}\right) c_{l_3}\left(\frac{2j-2l_1-2l_2-l_3}{2n+2}\right) \\
&= \sum_{\substack{l_1+l_2+l_3=0 \\ |l_1|,|l_2|,|l_3|<n^\varepsilon}} g_{l_1l_2l_3}(-1)\,,
\end{aligned}$$

and hence

$$\begin{aligned}
([T_n(f_n)]^3)_{jj} &= \tfrac{1}{2}\{([T_n(f_n)]^3)_{jj} + ([T_n^*(f_n)]^3)_{jj}\} \\
&= \tfrac{1}{2} \sum_{\substack{l_1+l_2+l_3=0 \\ |l_1|,|l_2|,|l_3|<n^\varepsilon}} \{g_{l_1l_2l_3}(-1) + g_{l_1l_2l_3}(1)\}\,,
\end{aligned}$$

from which follows

$$r_{jn} = \tfrac{1}{2} \sum_{\substack{l_1+l_2+l_3=0 \\ |l_1|,|l_2|,|l_3|<n^\varepsilon}} \{g_{l_1l_2l_3}(-1) + g_{l_1l_2l_3}(1) - 2g_{l_1l_2l_3}(0)\}\,.$$

By Taylors formula we get

$$\begin{aligned}
(9) \qquad |r_{jn}| &\leqq \tfrac{1}{2} \sum_{\substack{l_1+l_2+l_3=0 \\ |l_1|,|l_2|,|l_3|<n^\varepsilon}} \max_{-1\leqq t\leqq 1} |g''_{l_1l_2l_3}(t)| \\
&\leqq \tfrac{1}{2}\left(\frac{3n^\varepsilon}{n+1}\right)^2 (3M''M^2 + 6M'^2M)\,.
\end{aligned}$$

Note that while the estimate (8) holds for all $j = 0, 1, \ldots, n$, the estimate (9) only holds for $3n^\varepsilon < j < n - 3n^\varepsilon$.

Summing the equation (7) over $j = 0, 1, \ldots, n$, we get

$$(10)\ \operatorname{tr}([T_n(f_n)]^3) = \sum_{j=0}^{n} \sum_{\substack{l_1+l_2+l_3=0 \\ -j\leqq l_1,\, l_1+l_2 \leqq n-j \\ |l_1|,|l_2|,|l_3|<n^\varepsilon}} c_{l_1}\left(\frac{j}{n+1}\right) c_{l_2}\left(\frac{j}{n+1}\right) c_{l_3}\left(\frac{j}{n+1}\right) + R_{2n}^{(3)}\,,$$

where

$$R_{2n}^{(3)} = \sum_{j=0}^{n} r_{jn} = \sum_{j\leqq 3n^\varepsilon} r_{jn} + \sum_{n-j\leqq 3n^\varepsilon} r_{jn} + \sum_{3n^\varepsilon<j<n-3n^\varepsilon} r_{jn}\,,$$

and hence by (8) and (9)

$$|R_{2n}^{(3)}| \leqq 2(3n^\varepsilon+1)\frac{3n^\varepsilon}{n+1} 3M'M^2 + \tfrac{1}{2}\frac{(3n^\varepsilon)^2}{n+1}(3M''M^2 + 6M'^2M)\,,$$

which, since $2\varepsilon < 1$, implies

$$\lim_{n\to\infty} R_{2n}^{(3)} = 0 . \tag{11}$$

If we do the calculations for a general $p > 0$, we find

$$|R_{2n}^{(p)}| \leqq 2(pn^\varepsilon + 1)\frac{pn^\varepsilon}{n+1} pM'M^{p-1} + + \tfrac{1}{2}\frac{(pn^\varepsilon)^2}{n+1}\left(pM''M^{p-1} + p(p-1)M'^2M^{p-2}\right) .$$

Consequently there exists a constant C_2 such that for all n and p

$$|R_{2n}^{(p)}| \leqq C_2 p^4 M^p . \tag{12}$$

2.3. Let

$$\sum_{j=0}^{n} \sum_{\substack{l_1+l_2+l_3=0 \\ -j\leqq l_1,\, l_1+l_2 \leqq n-j \\ |l_1|,|l_2|,|l_3|<n^\varepsilon}} c_{l_1}\left(\frac{j}{n+1}\right) c_{l_2}\left(\frac{j}{n+1}\right) c_{l_3}\left(\frac{j}{n+1}\right) = \sum_{j=0}^{n} \sum_{\substack{l_1+l_2+l_3=0 \\ -j\leqq l_1,\, l_1+l_2 \leqq n-j}} c_{l_1}\left(\frac{j}{n+1}\right) c_{l_2}\left(\frac{j}{n+1}\right) c_{l_3}\left(\frac{j}{n+1}\right) + R_{3n}^{(3)} . \tag{13}$$

We can then estimate $R_{3n}^{(3)}$ in exactly the same way as we did $R_{1n}^{(3)}$ (see Section 2.1). Hence

$$\lim_{n\to\infty} R_{3n}^{(3)} = 0 , \tag{14}$$

and by doing the calculations for a general $p > 0$, we find

$$|R_{3n}^{(p)}| \leqq C_1 p M^p \tag{15}$$

for all n and p.

2.4. By use of

$$\frac{1}{2\pi}\int_0^{2\pi} \left[f\left(\frac{j}{n+1}, \theta\right)\right]^3 d\theta = \sum_{l_1+l_2+l_3=0} c_{l_1}\left(\frac{j}{n+1}\right) c_{l_2}\left(\frac{j}{n+1}\right) c_{l_3}\left(\frac{j}{n+1}\right)$$

we can write

$$\sum_{j=0}^{n} \sum_{\substack{l_1+l_2+l_3=0 \\ -j\leqq l_1,\, l_1+l_2 \leqq n-j}} c_{l_1}\left(\frac{j}{n+1}\right) c_{l_2}\left(\frac{j}{n+1}\right) c_{l_3}\left(\frac{j}{n+1}\right) = \sum_{j=0}^{n} \frac{1}{2\pi}\int_0^{2\pi} \left[f\left(\frac{j}{n+1}, \theta\right)\right]^3 d\theta + S_{1n}^{(3)} + S_{2n}^{(3)} , \tag{16}$$

where

$$S_{1n}^{(3)} = -\sum_{j=0}^{n} \sum_{\substack{l_1+l_2+l_3=0 \\ \min(l_1,l_1+l_2)<-j \\ \max(l_1,l_1+l_2)\leq n-j}} c_{l_1}\left(\frac{j}{n+1}\right) c_{l_2}\left(\frac{j}{n+1}\right) c_{l_3}\left(\frac{j}{n+1}\right),$$

$$S_{2n}^{(3)} = -\sum_{k=0}^{n} \sum_{\substack{l_1+l_2+l_3=0 \\ \max(l_1,l_1+l_2)>k}} c_{l_1}\left(\frac{n-k}{n+1}\right) c_{l_2}\left(\frac{n-k}{n+1}\right) c_{l_3}\left(\frac{n-k}{n+1}\right).$$

The sum defining $S_{1n}^{(3)}$ is dominated by the sum

$$\sum_{j=0}^{\infty} \sum_{\substack{l_1+l_2+l_3=0 \\ \min(l_1,l_1+l_2)<-j}} c_{l_1}c_{l_2}c_{l_3} = -\sum_{l_1+l_2+l_3=0} \min(0,l_1,l_1+l_2)\, c_{l_1}c_{l_2}c_{l_3}$$
$$\leq \sum_{l_1+l_2+l_3=0} (|l_1|+|l_2|)\, c_{l_1}c_{l_2}c_{l_3} \leq 2\left(\sum_{\nu=-\infty}^{\infty} |\nu|\, c_\nu\right) M^2 .$$

Hence we can conclude

$$\lim_{n\to\infty} S_{1n}^{(3)} = -\sum_{j=0}^{\infty} \sum_{\substack{l_1+l_2+l_3=0 \\ \min(l_1,l_1+l_2)<-j}} c_{l_1}(0)\, c_{l_2}(0)\, c_{l_3}(0)$$
$$= \sum_{l_1+l_2+l_3=0} \min(0,l_1,l_1+l_2)\, c_{l_1}(0)\, c_{l_2}(0)\, c_{l_3}(0) .$$

Using the fact that if $l_1+l_2+l_3=0$ then

$$\min(0,l_1,l_1+l_2) = -\max(0,l_3,l_3+l_2)$$

we get

$$\lim_{n\to\infty} S_{1n}^{(3)} = -\sum_{l_1+l_2+l_3=0} \max(0,l_1,l_1+l_2)\, c_{l_1}(0)\, c_{l_2}(0)\, c_{l_3}(0) . \tag{17}$$

In the same way we prove

$$\lim_{n\to\infty} S_{2n}^{(3)} = -\sum_{l_1+l_2+l_3=0} \max(0,l_1,l_1+l_2)\, c_{l_1}(1)\, c_{l_2}(1)\, c_{l_3}(1) . \tag{18}$$

Furthermore, if we do the calculations for a general $p>0$, we obtain the estimate

$$|S_{1n}^{(p)} + S_{2n}^{(p)}| \leq 2(p-1)\left(\sum_{\nu=-\infty}^{\infty} |\nu|\, c_\nu\right) M^{p-1} \tag{19}$$

for all n and p.

2.5. Finally we write

$$\sum_{j=0}^{n} \frac{1}{2\pi}\int_0^{2\pi} \left[f\left(\frac{j}{n+1},\theta\right)\right]^3 d\theta = \frac{n+1}{2\pi}\int_0^1\int_0^{2\pi} [f(x,\theta)]^3\, d\theta dx + S_{3n}^{(3)} . \tag{20}$$

Introducing the function

$$F(x) = \frac{1}{2\pi}\int_0^{2\pi} [f(x,\theta)]^3\, d\theta ,$$

we have

$$S_{3n}^{(3)} = \sum_{j=0}^{n} F\left(\frac{j}{n+1}\right) - (n+1)\int_0^1 F(x)\, dx ,$$

and it is elementary to prove that under Condition A we have

$$\lim_{n\to\infty} S_{3n}^{(3)} = \tfrac{1}{2}[F(0)-F(1)] \tag{21}$$
$$= \frac{1}{4\pi}\int_0^{2\pi} [f(0,\theta)]^3\, d\theta - \frac{1}{4\pi}\int_0^{2\pi} [f(1,\theta)]^3\, d\theta .$$

By doing the calculations for a general $p>0$, we find that for all n and p

$$|S_{3n}^{(p)}| \leqq pM'M^{p-1} . \tag{22}$$

2.6. Addition of the equations (4), (10), (13), (16) and (20) gives

$$\sum_{j=0}^{n} \lambda_{nj}^{\;3} - \frac{n+1}{2\pi}\int_0^1\int_0^{2\pi} [f(x,\theta)]^3\, d\theta dx = R_{1n}^{(3)} + R_{2n}^{(3)} + R_{3n}^{(3)} + S_{1n}^{(3)} + S_{2n}^{(3)} + S_{3n}^{(3)} ,$$

and the first half of Theorem 1 now follows from (5), (11), (14), (17), (18) and (21). The second half of Theorem 1 follows from (6), (12), (15), (19) and (22).

3. Asymptotic behavior of the determinants of generalized Toeplitz matrices.

Let $f(x,\theta)$ be a function satisfying Condition A of the preceding section and let $D_n(1-f)$, $n=1,2,\ldots$, be the generalized Toeplitz determinants associated with the function $1-f(x,\theta)$. That is, let for each positive integer n

$$D_n(1-f) = \det\left(\delta_{ij} - c_{j-i}\left(\frac{i+j}{2n+2}\right)\right), \qquad i,j = 0,1,\ldots,n ,$$

where δ_{ij} is the Kronecker symbol. As a simple consequence of Theorem 1 we will now prove the following theorem about the asymptotic behavior of $D_n(1-f)$ as n becomes infinite.

THEOREM 2. *Let $f(x,\theta)$ satisfy Condition A and let*

$$M = \sum_{\nu=-\infty}^{\infty} c_\nu < 1 .$$

Then

$$\lim_{n\to\infty}\frac{D_n(1-f)}{[G(1-f)]^{n+1}} = \exp\left\{\tfrac{1}{2}\left[h_0(0)-h_0(1)+\sum_{\nu=1}^{\infty}\nu h_\nu(0)h_{-\nu}(0)+\sum_{\nu=1}^{\infty}\nu h_\nu(1)h_{-\nu}(1)\right]\right\}$$

where

$$G(1-f) = \exp\left\{\frac{1}{2\pi}\int_0^1\int_0^{2\pi}\log\left[1-f(x,\theta)\right]d\theta\, dx\right\},$$

and where $h_\nu(x)$, $0\leqq x\leqq 1$, $\nu=0,\pm 1,\pm 2,\ldots$, *are defined by*

$$\log\left[1-f(x,\theta)\right] = \sum_{\nu=-\infty}^{\infty} h_\nu(x)e^{i\nu\theta}.$$

Remark. Theorem 2 generalizes trivially to the determinants associated with a function

$$g(x,\theta) = \sum_{\nu=-\infty}^{\infty} a_\nu(x)e^{i\nu\theta}$$

satisfying Condition A and

$$\min|a_0(x)| > \sum_{\substack{\nu=-\infty\\ \nu\neq 0}}^{\infty}\max|a_\nu(x)|,$$

because such a function can be written in the form $a[1-f(x,\theta)]$, where f satisfies the conditions of Theorem 2, and because

$$\frac{D_n\big(a(1-f)\big)}{\big[G\big(a(1-f)\big)\big]^{n+1}} = \frac{D_n(1-f)}{[G(1-f)]^{n+1}}$$

for any complex number $a\neq 0$.

If $f=f(\theta)$ is a function of θ only, Theorem 2 has been proved by M. Kac [2] with Condition A replaced by the condition

$$\sum_{\nu=-\infty}^{\infty}|\nu|\,|c_\nu| < \infty.$$

Hence if, for any fixed x in the interval $0\leqq x\leqq 1$, we consider the function $f_x(\theta)=f(x,\theta)$, then we have the relation

$$\lim_{n\to\infty}\frac{D_n(1-f_x)}{[G(1-f_x)]^{n+1}} = \exp\left\{\sum_{\nu=1}^{\infty}\nu h_\nu(x)h_{-\nu}(x)\right\}, \tag{23}$$

which we shall make use of in a moment.

Now let $\lambda_{n0},\lambda_{n1},\ldots,\lambda_{nn}$ be the eigenvalues of the matrix $T_n(f)$ defined in (1). Then

$$D_n(1-f) = \prod_{j=0}^{n}(1-\lambda_{nj}).$$

It is easy to see that for all n and j

$$|\lambda_{nj}| \leqq M < 1 ,$$

and hence we can write

$$\frac{D_n(1-f)}{[G(1-f)]^{n+1}} = \exp\left\{-\sum_{p=1}^{\infty}\frac{1}{p}\left[\sum_{j=0}^{n}\lambda_{nj}{}^p - \frac{n+1}{2\pi}\int_0^1\int_0^{2\pi}[f(x,\theta)]^p d\theta dx\right]\right\}.$$

By the estimate in Theorem 1, the series occurring in the above expression is majorized by the convergent series

$$\sum_{p=1}^{\infty} C p^3 M^p ,$$

and hence by the first part of Theorem 1

$$\lim_{n\to\infty}\frac{D_n(1-f)}{[G(1-f)]^{n+1}} = \exp\left\{\frac{1}{4\pi}\int_0^{2\pi}\log[1-f(0,\theta)]\,d\theta - \frac{1}{4\pi}\int_0^{2\pi}\log[1-f(1,\theta)]\,d\theta + \right.$$

$$+\sum_{p=1}^{\infty}\frac{1}{p}\sum_{l_1+\ldots+l_p=0}\max[0,l_1,l_1+l_2,\ldots,l_1+\ldots+l_{p-1}]c_{l_1}(0)\ldots c_{l_p}(0) +$$

$$\left. +\sum_{p=1}^{\infty}\frac{1}{p}\sum_{l_1+\ldots+l_p=0}\max[0,l_1,l_1+l_2,\ldots,l_1+\ldots+l_{p-1}]c_{l_1}(1)\ldots c_{l_p}(1)\right\}.$$

Using the same technique on the function f_x we get a similar result, which combined with (23) gives

$$\exp\left\{2\sum_{p=1}^{\infty}\frac{1}{p}\sum_{l_1+\ldots+l_p=0}\max[0,l_1,l_1+l_2,\ldots,l_1+\ldots+l_{p-1}]c_{l_1}(x)\ldots c_{l_p}(x)\right\}$$

$$= \exp\left\{\sum_{\nu=1}^{\infty}\nu h_\nu(x)h_{-\nu}(x)\right\},$$

and from the continuity of the functions involved it follows that

$$\exp\left\{\sum_{p=1}^{\infty}\frac{1}{p}\sum_{l_1+\ldots+l_p=0}\max[0,l_1,l_1+l_2,\ldots,l_1+\ldots+l_{p-1}]\,c_{l_1}(0)\ldots c_{l_p}(0) + \right.$$

$$\left. +\sum_{p=1}^{\infty}\frac{1}{p}\sum_{l_1+\ldots+l_p=0}\max[0,l_1,l_1+l_2,\ldots,l_1+\ldots+l_{p-1}]\,c_{l_1}(1)\ldots c_{l_p}(1)\right\}$$

$$= \exp\left\{\tfrac{1}{2}\sum_{\nu=1}^{\infty}\nu h_\nu(0)h_{-\nu}(0) + \tfrac{1}{2}\sum_{\nu=1}^{\infty}\nu h_\nu(1)h_{-\nu}(1)\right\}.$$

This concludes the proof of Theorem 2.

REFERENCES

1. U. Grenander and G. Szegö, *Toeplitz forms and their applications*, University of California Press, 1958.
2. M. Kac, *Toeplitz matrices, translation kernels and a related problem in probability theory*, Duke Math. J. 21 (1954), 501–509.
3. M. Kac, W. L. Murdock and G. Szegö, *On the eigenvalues of certain Hermitian forms*, J. Rational Mech. Anal. 2 (1953), 767–800.
4. L. C. Mejlbo and P. F. Schmidt, *On the determinants of certain Toeplitz matrices*, Bull. Amer. Math. Soc. 67 (1961), 159–162.
5. G. Szegö, *On certain Hermitian forms associated with the Fourier series of a positive function*, Communications du Séminaire Mathématique de l'Université de Lund, tome supplémentaire (1952) dédie à Marcel Riesz, 228–238.

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