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p-MEANS OF CONVEX BODIES

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1.

The setting of this discussion is Euclidean n-dimensional space. The letters u, v denote vectors; (u,v) denotes the inner product of two vectors. Convex bodies will be denoted by K with distinguishing marks. Such bodies will be assumed to contain the origin of the coordinate system as an interior point or, in the case of degenerate bodies, as an interior point with respect to the subspace of least dimension containing the convex body. λK denotes the body resulting from dilating K in the ratio $\lambda:1$ using the origin as a centre. Those standard results of the theory of convex bodies which are mentioned without reference are to be found in [1]. For the material concerning inequalities, see [3].

Since the work of Minkowski, sums of convex bodies have been considered which can be defined, among other ways, as follows: Let $h_i(u)$, $i=0, 1$, be the support functions of convex bodies K_i. These functions satisfy three conditions:

(a) $h_i(u) \geqq 0 \quad \text{for } (u,u) > 0, \quad h_i(0) = 0;$

(b) $h_i(\mu u) = \mu h_i(u) \quad \text{for } \mu > 0;$

(c) $h_i(u+v) \leqq h_i(u)+h_i(v)\,.$

These conditions are also sufficient for these functions to be support functions of unique convex bodies K_i of the sort considered here, whose supporting hyperplanes are described by $(x,u)=h_i(u)$. The function $\lambda_0 h_0(u)+\lambda_1 h_1(u)$, $\lambda_i \geqq 0$, satisfies conditions (a), (b), (c) and so is the support function of a convex body, denoted by $\lambda_0 K_0+\lambda_1 K_1$ called a weighted Minkowski sum of K_0 and K_1. In particular, we shall call

$$(1-\vartheta)K_0+\vartheta K_1, \qquad 0 \leqq \vartheta \leqq 1\,,$$

an arithmetic mean of K_0 and K_1.

It is possible to introduce other means of K_0 and K_1. For non-negative numbers a_i, let

$$M_p(a_0,a_1) = [(1-\vartheta)a_0{}^p+\vartheta a_1{}^p]^{1/p}\,.$$

We set

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$$M_\infty(a_0,a_1) = \lim_{p\to\infty} M_p(a_0,a_1) = \max(a_0,a_1)$$

for $0<\vartheta<1$. We assert that for $1\leqq p\leqq\infty$, whenever the functions $h_i(u)$ satisfy (a), (b) and (c), then so do $M_p(h_0(u),h_1(u))$. That the first two conditions are met is obvious. For (c), we have by Minkowski's inequality

$$M_p(a_0,a_1) \leqq M_p(b_0,b_1)+M_p(c_0,c_1) ,$$

where $a_i=b_i+c_i$ and $b_i\geqq 0$, $c_i\geqq 0$. Clearly the inequality remains valid if $a_i\leqq b_i+c_i$. Setting $a_i=h_i(u+v)$, $b_i=h_i(u)$ and $c_i=h_i(v)$ shows that $M_p(h_0(u),h_1(u))$ satisfies (c). Hence, whenever K_0 and K_1 have the origin as a common interior point (in the sense of the first paragraph), there is a unique convex body, which we denote by $K_\vartheta^{(p)}$ having $h_\vartheta^{(p)}(u)=M_p(h_0(u),h_1(u))$ as its support function. We call $K_\vartheta^{(p)}$ the weighted p-mean of K_0 and K_1. The convex body $2^{1/p}K_{1/2}^{(p)}$ whose support function is $[h_0{}^p(u)+h_1{}^p(u)]^{1/p}$ is called the p-sum of K_0 and K_1. We denote it by $S_p(K_0,K_1)$.

The following rules proceed immediately from the properties of the function $[a_0{}^p+a_1{}^p]^{1/p}$ for non-negative a_i applied to the appropriate support functions:

(i) $S_p(\lambda K_0,\lambda K_1) = \lambda S_p(K_0,K_1)$,

(ii) $S_p(K_0,K_1) = S_p(K_1,K_0)$,

(iii) $S_p(S_p(K_0,K_1),K_2) = S_p(K_0,S_p(K_1,K_2))$.

This last rule allows us to define inductively $S_p(K_0,\ldots,K_m)$ by

$$S_p(S_p(K_0,\ \ldots,\ K_{m-1}),\ K_m) .$$

In turn, set

$$S_p(w_0{}^{1/p}K_0,\ldots,w_m{}^{1/p}K_m) = M_p(K_0,\ldots,K_m)$$

if

$$\sum_{j=1}^{m} w_j = 1, \qquad w_j \geqq 0, \qquad 1 \leqq p < \infty .$$

Further, we define $M_\infty(K_0,\ldots,K_m)$ as that convex body whose support function is

$$\lim_{p\to\infty} M_p(h_0(u),\ldots,h_1(u)) = \max\{h_0(u),\ldots,h_m(u)\} .$$

With a similar definition for $S_\infty(K_0,K_1)$, we have $S_\infty(K_0,K_1)=M_\infty(K_0,K_1)$.

2.

In this section we examine the dependence of $K_\vartheta^{(p)}$ on its arguments p, ϑ, K_0 and K_1 for $1\leqq p<\infty$.

Let E signify the spherical body of radius one centred at the origin.

The p-deviation of two convex bodies K_0 and K_1 is defined as the greatest lower bound of non-negative numbers ϱ for which $S_p(K_0,\varrho E)\supseteq K_1$ and $S_p(K_1,\varrho E)\supseteq K_0$. This p-deviation, denoted by $\delta_p(K_0,K_1)$, is given by

$$\delta_p(K_0,K_1) = \max_{(u,u)=1} |h_0^p(u)-h_1^p(u)|^{1/p} .$$

It is positive definite, symmetric and satisfies a triangle inequality

$$\delta_p(K_0,K_2) \leqq \delta_p(K_0,K_1)+\delta_p(K_1,K_2) .$$

The deviation δ_1 is the usual deviation introduced by Blaschke.

For the range of p-values under consideration, these deviations are topologically equivalent. That is to say, if we call the sequence of convex bodies $\{K_m\}$ p-convergent to the limit K when $\lim_{m\to\infty}\delta_p(K_m,K)=0$, then p-convergence implies the usual 1-convergence and conversely. This depends on the fact that either sort of convergence implies that the corresponding sequence of support functions $\{h_m(u)\}$ is uniformly convergent to $h(u)$ and uniformly bounded over $(u,u)=1$. Thus, in discussing convergence of sequences of convex bodies and questions of continuity any one of the p-deviations may be used; we shall use $\delta(K_0,K_1)$ to denote such a deviation.

Let $\mathscr{S}_P$ be the space of elements (p,ϑ,K_0,K_1) where

$$1 \leqq p \leqq P < \infty, \qquad 0 \leqq \vartheta \leqq 1$$

and K_0, K_1 are any two convex bodies. Take as the distance $d(e,e')$ between elements $e=(p,\vartheta,K_0,K_1)$ and $e'=(p',\vartheta',K_0',K_1')$ the number

$$d(e,e') = |p-p'|+|\vartheta-\vartheta'|+\delta(K_0,K_0')+\delta(K_1,K_1') .$$

We set $K_\vartheta^{(p)}=K(e)$; then $K(e)$ is continuous in e in the sense that, if $\{e_m\}$ is any sequence of elements of $\mathscr{S}_p$ such that $\lim_{m\to\infty}d(e_m,e)=0$, we have $\lim_{m\to\infty}\delta\big(K(e_m),K(e)\big)=0$.

The algebraic function $f(p,\vartheta,a_0,a_1)=[(1-\vartheta)a_0^p+\vartheta a_1^p]^{1/p}$ is uniformly continuous for $1\leqq p\leqq P<\infty$, $0\leqq a_i\leqq A<\infty$, $0\leqq\vartheta\leqq 1$. Consequently if $\{h_{im}(u)\}$, $i=0$, 1, are two uniformly bounded and uniformly convergent sequences over $(u,u)=1$, the sequence $\big\{f\big(p_m,\vartheta_m,h_{0m}(u),h_{1m}(u)\big)\big\}$ has these same properties, when $1\leqq p_m\leqq P$, $0\leqq\vartheta_m\leqq 1$. The condition that $\lim_{m\to\infty}d(e_m,e)=0$ implies

$$\lim_{m\to\infty}\delta(K_{0m},K_0) = \lim_{m\to\infty}\delta(K_{1m},K_1) = 0 .$$

These in turn, as remarked earlier, imply that the sequences $\{h_{im}(u)\}$ of support functions corresponding to the sequences of convex bodies $\{K_{im}\}$

converge uniformly over $(u,u)=1$ to $h_i(u)$. Thus the convergence of $\{e_m\}$ to e entails the convergence of the bounded sequence

$$\{f(p_m,\vartheta_m,h_{0m}(u),h_{1m}(u))\}$$

to $f(p,\vartheta,h_0(u),h_1(u))$ uniformly over $(u,u)=1$ when $\{p_m\}$ and $\{\vartheta_m\}$ lie in the allowed intervals and tend to p and ϑ. This gives

$$\lim_{m\to\infty} \delta(K(e_m),K(e)) = 0$$

as asserted.

Since $2^{1/p}K_{1/2}^{(p)}=S_p(K_0,K_1)$ it follows that this family is also continuous in its arguments. These results will be included in Theorem 1 of Section 3.

3.

We now consider certain inclusion relations among the means. Here we allow $p=+\infty$. In this case the p-means $K_\vartheta^{(\infty)}$ and the sums $S_\infty(K_0,K_1)$ coincide when $0<\vartheta<1$. For

$$\lim_{p\to\infty}[(1-\vartheta)a_0{}^p+\vartheta a_1{}^p]^{1/p} = \lim_{p\to\infty}[a_0{}^p+a_1{}^p]^{1/p} = \max(a_0,a_1)\,.$$

Consequently the support function of $K_\vartheta^{(\infty)}$ and $S_\infty(K_0, K_1)$ is $\max(h_0(u), h_1(u))$ when ϑ is neither zero nor one. Hence, if we let $\bar{L}$ mean the convex closure of L, the bodies in question are

$$\overline{K_0\cup K_1},\ K_0^{(\infty)} = K_0, \qquad K_1^{(\infty)} = K_1\,.$$

Thus $K_\vartheta^{(\infty)}$ is continuous in ϑ and K_0, K_1 for $0<\vartheta<1$. At $\vartheta=0$, the mean is continuous in ϑ from the right if and only if $K_1\subseteq K_0$; at $\vartheta=1$, the mean is continuous in ϑ from the left if and only if $K_0\subseteq K_1$.

The inclusion relations to follow are based on the well-known theorems that

$$M_p(a_0,a_1) \leqq M_q(a_0,a_1) \qquad \text{and} \qquad S_p(a_0,a_1) \geqq S_q(a_0,a_1)$$

if $1\leqq p<q\leqq\infty$. In the first inequality, there is equality, for $0<\vartheta<1$, if and only if $a_0=a_1$; in the second there is equality if and only if one of the $a_i=0$. Applying these inequalities to the support functions

$$[(1-\vartheta)h_0{}^p(u)+\vartheta h_1{}^p(u)]^{1/p} \qquad \text{and} \qquad [h_0{}^p(u)+h_1{}^p(u)]^{1/p}$$

shows that

$$K_\vartheta^{(p)} \subseteq K_\vartheta^{(q)} \qquad \text{and} \qquad S_p(K_0,K_1) \supseteq S_q(K_0,K_1)\,.$$

In the first inclusion there is identity, for $0<\vartheta<1$, if and only if K_0 and K_1 coincide. In the second inclusion there is identity if and only if, in each direction u, one of the support functions $h_0(u)$ and $h_1(u)$

vanishes. Since we have imposed the supposition that the origin is a common interior point (relative to the containing subspaces of least dimension) of the bodies K_0 and K_1, this implies that $S_p(K_0, K_1) = S_q(K_0, K_1)$ if and only if one of the bodies is a point.

In particular: for ϑ fixed, all p-means are subsets of $\overline{K_0 \cup K_1}$ and contain the arithmetic mean $K_\vartheta^{(1)}$; all p-sums contain $\overline{K_0 \cup K_1}$ and are subsets of the Minkowski sum $K_0 + K_1$. From the continuous dependence of $K_\vartheta^{(p)}$ and $S_p(K_0, K_1)$ on p we see that, through any point of $K_0 + K_1$ which is an exterior point for $K_\vartheta^{(1)}$ there passes one and only one hypersurface of the family of bounding hypersurfaces of $K_\vartheta^{(p)}$ and $S_p(K_0, K_1)$.

We next consider inclusion relations among the means $K_\vartheta^{(p)}$ for fixed p and varying ϑ. For $p = \infty$, it is geometrically obvious that the convex bodies $K_\vartheta^{(p)}$ form a concave family by which is meant that

$$K_{\vartheta'}^{(p)} \supseteq (1-\vartheta) K_{\vartheta_0}^{(p)} + \vartheta K_{\vartheta_1}^{(p)} \qquad \text{where} \qquad \vartheta' = (1-\vartheta)\vartheta_0 + \vartheta\vartheta_1 \,.$$

But more generally this is true for $1 \leqq p \leqq \infty$. We have

$$M_p(a_0, a_1) \geqq M_1(a_0, a_1) \,,$$

where these means are formed with weights $(1-\vartheta)$ and ϑ. Set

$$h_{\vartheta_i}(u) = [(1-\vartheta_i) h_0{}^p(u) + \vartheta_i h_1{}^p(u)]^{1/p}, \qquad i = 1, 2;$$

a direct computation then shows that

$$M_p\big(h_{\vartheta_0}(u), h_{\vartheta_1}(u)\big) = [(1-\vartheta') h_0{}^p(u) + \vartheta' h_1{}^p(u)]^{1/p} = h_{\vartheta'}(u)$$

and the preceding inequality gives

$$h_{\vartheta'}{}^p(u) \geqq (1-\vartheta) h_{\vartheta_0}{}^p(u) + \vartheta h_{\vartheta_1}{}^p(u)$$

which proves the assertion. For $p > 1$, there is equality if and only if $K_{\vartheta_0}^{(p)}$ and $K_{\vartheta_1}^{(p)}$ are identical, which means either $\vartheta_0 = \vartheta_1$ or $K_0 = K_1$.

We summarize the results obtained so far.

THEOREM 1. *The families $K_\vartheta^{(p)}$ and $S_p(K_0, K_1)$ depend continuously on (p, ϑ, K_0, K_1) for $1 \leqq p < \infty$, $0 \leqq \vartheta \leqq 1$ and at $p = \infty$ for $0 < \vartheta < 1$. They are monotonic increasing and decreasing respectively in p over $1 \leqq p \leqq \infty$. In the first case, the monotonicity fails to be strict if and only if $K_0 = K_1$. In the second case, the monotonicity fails to be strict if and only if either K_0 or K_1 degenerates to a point. The family $K_\vartheta^{(p)}$ is strictly concave with respect to ϑ, for $1 < p < \infty$, $0 \leqq \vartheta \leqq 1$, if K_0 and K_1 are not identical, and linear if $K_0 = K_1$ or $p = 1$.*

The connection between the processes of projection into lower dimensional linear subspaces and the formation of p-means is the same as that for Minkowski addition. Let K^* denote the projection of K upon an m-dimensional linear subspace L_m, $m<n$. The support function of K^* in L_m is the restriction of $h(u)$ to vectors u in L_m. Hence the support function of $S_p{}^*(K_0,K_1)$ is the same restriction of $[h_0{}^p(u)+h_1{}^p(u)]^{1/p}$ which is identical with the p-sum of the restrictions of $h_0(u)$ and $h_1(u)$ to vectors u in L_m. Therefore $S_p{}^*(K_0,K_1)=S_p(K_0{}^*,K_1{}^*)$. A similar statement holds for the p-means.

4.

Let $W_{(s)}(K)$ denote the s^{th} mean cross-sectional measure of K, that is the mixed volume

$$V(\underbrace{K,\ldots,K}_{n-s},\underbrace{E,\ldots,E}_{s}),$$

where $s=0,1,\ldots,n-1$. These measures are well-known to be continuous and monotonic in K in the sense that $K\subseteq K'$ implies $W_{(s)}(K)\leqq W_{(s)}(K')$. Coupling this with the monotonicity and continuity assertions of theorem 1 shows that $W_{(s)}(K_\vartheta^{(p)})$ is continuously dependent on its arguments (p,ϑ,K_0,K_1) and monotonic increasing in p for $1\leqq p\leqq\infty$. In particular then, for fixed K_0, K_1 and ϑ, the arithmetic mean has minimal mean cross-sectional measures and $\overline{K_0\cup K_1}$ has maximal.

It is possible to derive a slightly modified Brunn-Minkowski theorem for the means $K_\vartheta^{(p)}$. Let $V(K)$ denote the volume of K. We shall prove that

$$V^{1/n}(K_\vartheta^{(p)}) \geqq [(1-\vartheta)V^{p/n}(K_0)+\vartheta V^{p/n}(K_1)]^{1/p}.$$

In case both bodies are degenerate and lie in a common linear subspace, there is a trivial equality since all the volumes vanish. Suppose that one of the bodies, say K_0, is degenerate and that K_1 is either not degenerate or at least does not lie in a subspace containing K_0. Then $K_\vartheta^{(p)}$ contains proper interior points and we must show that $V(K_\vartheta^{(p)})\geqq\vartheta^n V(K_1)$. But the support function of $K_\vartheta^{(p)}$ is

$$[(1-\vartheta)h_0{}^p(u)+\vartheta h_1{}^p(u)]^{1/p} \geqq \vartheta^{1/p}h_1(u) \geqq \vartheta h_1(u)$$

with equality in the last case if and only if $\vartheta=1$. Hence $K_\vartheta^{(p)}\supseteq\vartheta K_1$ and the assertion follows in this case.

Finally we consider the case in which neither body is degenerate. If $p=\infty$ and $0<\vartheta<1$, our assertion reads

$$V^{1/n}(\overline{K_0\cup K_1}) \geqq \max\left(V^{1/n}(K_0), V^{1/n}(K_1)\right).$$

This follows directly from the inclusion relations $K_i\subseteq\overline{K_0\cup K_1}$, $i=0,1$. There is identity in at least one of these inclusions if and only if one of the bodies K_i is a subset of the other. Hence there is equality between the volumes $V(\overline{K_0 \cup K_1})$ and $\max\big(V(K_0), V(K_1)\big)$ if and only if one of the K_i is a subset of the other.

For the remaining case in which $1 \leqq p < \infty$ and neither of the convex bodies K_i is degenerate, we make the usual reduction to the special case in which the convex bodies K_i have unit volumes. Thus we let $\lambda_i = V^{1/n}(K_i)$ and take

$$\vartheta' = \vartheta\lambda_1{}^p/[(1-\vartheta)\lambda_0{}^p + \vartheta\lambda_1{}^p] .$$

M_p and $M_p{}'$ are to denote means formed with weights $(1-\vartheta)$, ϑ and $(1-\vartheta')$, ϑ' respectively. The support function of $M_p{}'(K_0/\lambda_0, K_1/\lambda_1)$ is

$$\left[(1-\vartheta')\frac{h_0{}^p(u)}{\lambda_0{}^p} + \vartheta'\frac{h_1{}^p(u)}{\lambda_1{}^p}\right]^{1/p} = \left[\frac{(1-\vartheta)h_0{}^p(u) + \vartheta h_1{}^p(u)}{(1-\vartheta)\lambda_0{}^p + \vartheta\lambda_1{}^p}\right]^{1/p} .$$

In turn, this is the support function of $M_p(K_0, K_1)/\mu = K_\vartheta^{(p)}/\mu$ where

$$\mu = [(1-\vartheta)V^{p/n}(K_0) + \vartheta V^{p/n}(K_1)]^{1/p} .$$

Since the volumes of $K_i{}' = K_i/\lambda_i$ are one, it is enough to show that the volume of $M_p{}'(K_0/\lambda_0, K_1/\lambda_1)$ is greater than or equal to one.

By the Brunn-Minkowski theorem, $(p=1)$, $V\big(M_1{}'(K_0{}', K_1{}')\big) \geqq 1$. From theorem 1, $M_p{}'(K_0{}', K_1{}') \supseteq M_1{}'(K_0{}', K_1{}')$. These results, together with the monotonic increasing character of the volume functional, gives $V\big(M_p{}'(K_0{}', K_1{}')\big) \geqq 1$. In the set inclusion there is identity if and only if $K_0{}' = K_1{}'$, that is $K_0 = \varrho K_1$ where $\varrho = \lambda_0/\lambda_1$. Since these are sufficient (as well as necessary) conditions for equality in the Brunn–Minkowski theorem, they are the necessary and sufficient conditions for the linearity of $V^{p/n}(K_\vartheta^{(p)})$ in ϑ.

As a consequence of the inequalities of Fenchel and A.D. Alexandrov, cf. [2,p. 49], it is known that $W_{(s)}{}^{1/(n-s)}(K_\vartheta^{(1)})$ is a concave function of ϑ and is linear if $K_0 = \sigma K_1$ for some $\sigma > 0$, or $s = n-1$. Aside from a difference in the cases of degeneracy, the preceding discussion remains valid if, for $V^{1/n}$, we write $W_{(s)}{}^{1/(n-s)}$. Since $W_{(s)}(K)$ vanishes if and only if K lies in an $n-s-1$ dimensional subspace, $W_{(s)}{}^{p/(n-s)}(K_\vartheta^{(p)})$ is linear in ϑ if and only if K_0 and K_1 lie in a common $n-s-1$ dimensional subspace or $K_0 = \sigma K_1$ or $p = n-s = 1$.

The following theorem summarizes the results of this section.

THEOREM 2. *For* $1 \leqq p \leqq \infty$, $s = 0, 1, \ldots, n-1$ *and* $0 < \vartheta < 1$:

$$[(1-\vartheta)W_{(s)}{}^{p/(n-s)}(K_0) + \vartheta W_{(s)}{}^{p/(n-s)}(K_1)]^{1/p} \leqq W_{(s)}{}^{1/(n-s)}(K_\vartheta^{(p)}) \leqq W_{(s)}{}^{1/(n-s)}(\overline{K_0 \cup K_1}) .$$

In the first inequality there is equality if and only if at least one of the following conditions is satisfied:

(i) *both convex bodies* K_0 *and* K_1 *lie in a common* $n-s-1$ *dimensional linear subspace*;

(ii) *there is a* $\sigma>0$ *for which* $K_0=\sigma K_1$;

(iii) $p=\infty$ *and one of the convex bodies* K_0, K_1 *is a subset of the other*;

(iv) $p=n-s=1$.

In the second inequality there is equality if and only if either condition (i) *holds or* $K_0=K_1$ *or* $p=\infty$.

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