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BOUNDARY VALUES IN FUNCTION SPACES

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1. Introduction.

The present paper is a continuation of the article [2] by the author suggested by Lions, cf. [1].

We assume that on the interval $I=\{0<t\leqq 1\}$ the function ψ is locally (i.e. away from 0) in $\mathscr{L}^q$ and that ψ^{-1} is locally in $\mathscr{L}^{q'}$ (with $1/q+1/q'=1$). We consider the space $W=W^{(m)}(\psi,q,X)$ of those functions u on I with values in the Banach space X, for which $\psi u^{(m)} \in \mathscr{L}^q(I,X)$. Since $\psi \in \mathscr{L}^q_{\text{loc}}$, W contains the space $C_0^\infty(I,X)$ of infinitely differentiable functions from I into X which vanish near 0. Our main results are as follows:

The limit $u^{(j)}(0)=\lim\limits_{t\to 0} u^{(j)}(t)$ *exists for all* $u \in W$ *if and only if*

$$t^{m-j-1}\psi^{-1} \in \mathscr{L}^{q'}. \text{ (Theorem 3.3)}.$$

Let n be the largest $j<m$ for which the last condition of theorem 3.3 *holds. Then the closure of C_0^∞ in W consists of all $u \in W$ for which*

$$u(0) = u'(0) = \ldots = u^{(n)}(0) = 0. \text{ (Theorem 4.1)}.$$

This result may be of interest for the applications of Banach space and Hilbert space methods to partial differential equations, for in that context the property of "having boundary values 0" is generally substituted by "belonging to the closure of C_0^∞".

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2. Notation and definitions.

We denote by I the interval $I=\{t \mid 0<t\leqq 1\}$, and by $\bar I$ the interval $\bar I=\{t \mid 0\leqq t\leqq 1\}$.

We shall consider functions u defined on I or $\bar I$ and with values in a Banach space X; the norm of the element $x \in X$ being denoted $|x|$.

$C^k(I,X)$ will denote the space of k times continuously differentiable functions from I into X (one-sided derivatives at the endpoint).

$C^k(\bar I,X)$ is defined analogously.

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$C_0^k(I,X)$ (or just C_0^k) denotes the subspace of $C^k(I,X)$ consisting of functions with compact support in I (i.e. vanishing near 0).

As usual $C^\infty(I,X)$ (and analogously $C^\infty(\bar{I},X)$ and $C_0^\infty(I,X)$) is defined by

$$C^\infty(I,X) = \bigcap_{k=1}^{\infty} C^k(I,X) .$$

By $\mathscr{L}^q(X)$ we shall denote the Banach space of measurable functions u from I into X with

$$\|u\|_q = \begin{cases} \left(\int_0^1 |u(t)|^q dt\right)^{1/q} < \infty & \text{for} \quad 1 \leqq q < \infty \\ \operatorname*{ess\,sup}_{t \in I} |u(t)| < \infty & \text{for} \quad q = \infty . \end{cases}$$

The space $C^k(\bar{I},X)$ is a Banach space with the norm

$$\|u\|_\infty + \|u'\|_\infty + \ldots + \|u^{(k)}\|_\infty .$$

By ψ we shall denote a positive measurable function on I, and $W^{(m)}(\psi,q,X)$ (or just W) will denote the space of functions u from I into X with the properties

1) u is $m-1$ times continuously differentiable and $u^{(m-1)}$ is locally absolutely continuous in I;
2) $\psi u^{(m)} \in \mathscr{L}^q(X)$.

We shall assume that

$$\psi \in \mathscr{L}^q \text{ on every interval } \{\varepsilon \leqq t \leqq 1\} ,$$

so that $C_0^\infty(I,X) \subset W^{(m)}(\psi,q,X)$.

Since we are going to study the behavior of functions in W near 0, we do not want to allow singularities in points of I; it turns out that the relevant condition on ψ is

$$\psi^{-1} \in \mathscr{L}^{q'} \text{ on every interval } \{\varepsilon \leqq t \leqq 1\} ,$$

q' being determined by $1/q + 1/q' = 1$. The space $W^{(m)}(\psi,q,X)$ is a Banach space with the norm

$$\|u\|_W = \sum_{j=0}^{m-1} |u^{(j)}(1)| + \|\psi u^{(m)}\|_q .$$

$W_0^{(m)}(\psi,q,X)$ (or just W_0) will denote the closure in W of $C_0^\infty(I,X)$.

Definition 2.1. *We shall say that the function u from I into X has the boundary value u_0 at 0 if $u(t) \to u_0$ for $t \to 0$.*

REMARK 2.2. If u is a differentiable function from I into X, and if u' has a boundary value u'_0 at 0, then u has a boundary value u_0 at 0, and if u is extended by continuity to a function $\bar{u}$ defined on $\bar{I}$, then $\bar{u}$ is differentiable at 0 with the derivative u'_0. (Proof just as for real valued functions.)

In particular, if $u \in C'(I,X)$ and u' has a boundary value at 0, then $\bar{u} \in C'(\bar{I},X)$.

For convenience, we shall not distinguish between a function in $C^k(\bar{I},X)$ and its restriction to I (which is a function in $C^k(I,X)$).

3. On existence of boundary values.

LEMMA 3.1. *Let v be a locally absolutely continuous function from I into X, and assume that*

$$\int_0^1 t^\gamma |v'(t)|\,dt < \infty, \qquad \textit{where} \quad \gamma \geqq 0\,.$$

Then, if $\gamma > 0$,

$$t^\gamma v(t) \to 0 \qquad \textit{for} \quad t \to 0\,,$$

and

$$\int_0^1 t^{\gamma-1} |v(t)|\,dt < \infty\,,$$

while, if $\gamma = 0$,

$$v \in C^0(\bar{I},X)\,.$$

PROOF. First, let $\gamma > 0$. Then, for $0 < t \leqq a \leqq 1$,

$$t^\gamma |v(t)| \leqq t^\gamma \left| \int_t^a v'(s)ds \right| + t^\gamma |v(a)|$$

$$\leqq \int_0^a s^\gamma |v'(s)|\,ds + t^\gamma |v(a)|\,,$$

so that

$$\limsup_{t\to 0} t^\gamma |v(t)| \leqq \int_0^a s^\gamma |v'(s)|\,ds$$

for every $a > 0$. It follows that

$$t^\gamma |v(t)| \to 0 \qquad \text{for} \quad t \to 0\,.$$

Similarly, choosing $a = 1$,

$$t^{\gamma-1}|v(t)| \leqq t^{\gamma-1} \int_t^1 |v'(s)|\,ds + t^{\gamma-1}|v(1)|\,,$$

so that (integration by parts)

$$\int_\varepsilon^1 t^{\gamma-1}|v(t)|\,dt$$
$$\leqq \gamma^{-1}\left\{-\varepsilon^\gamma\int_\varepsilon^1 |v'(t)|\,dt + \int_\varepsilon^1 t^\gamma|v'(t)|\,dt + (1-\varepsilon^\gamma)|v(1)|\right\}$$
$$\leqq \gamma^{-1}\left\{\int_0^1 t^\gamma|v'(t)|\,dt + |v(1)|\right\}$$

for every $\varepsilon > 0$, and hence

$$\int_0^1 t^{\gamma-1}|v(t)|\,dt < \infty .$$

Secondly, if $\gamma = 0$, then v is absolutely continuous and hence uniformly continuous on I, and the assertion follows since X is complete.

Corollary 3.2. *If $v \in C^{k-1}(I,X)$ with $v^{(k-1)}$ locally absolutely continuous and*

$$\int_0^1 t^{k-1}|v^{(k)}(t)|\,dt < \infty ,$$

then $v \in C^0(\bar{I},X)$.

Theorem 3.3. *Assume $1 \leqq q \leqq \infty$ and $0 \leqq j < m$. Then the following assertions are equivalent:*

(i) $$W^{(m)}(\psi,q,X) \subset C^j(\bar{I},X)$$

(ii) $$t^{m-j-1}\psi^{-1} \in \mathscr{L}^{q'} ,$$

where $1/q + 1/q' = 1$.

Proof. (ii)$\Rightarrow$(i) is a consequence of Corollary 3.2. For if $u \in W$, then

$$\psi u^{(m)} \in \mathscr{L}^q(X)$$

and if (ii) holds, it follows by Hölder's inequality that

$$\int_0^1 t^{m-j-1}|u^{(m)}(t)|\,dt < \infty .$$

Corollary 3.2 gives the continuity of $u^{(j)}$ on $\bar{I}$, and Remark 2.2 shows that (i) holds.

(i)$\Rightarrow$(ii). It suffices to show that if

$$t^{m-j-1}\psi^{-1} \notin \mathscr{L}^{q'},$$

then there exists a *real valued* function $f \in W^{(m)}(\psi,q,R)$ such that $f^{(j)}$ is unbounded, for then the function u defined by

$$u(t) = f(t)v, \qquad 0 \neq v \in X ,$$

belongs to $W^{(m)}(\psi,q,X)$ and not to $C^j(\bar{I},X)$. Now, if $t^{m-j-1}\psi^{-1} \notin \mathscr{L}^{q'}$, there exists a non-negative function $g \in \mathscr{L}^q$ such that

$$\int_0^1 t^{m-j-1}\psi^{-1}(t)g(t)\,dt = \infty .$$

If we define (for $t>0$)

$$f(t) = \frac{(-1)^{m-1}}{(m-1)!}\int_t^1 (s-t)^{m-1}\psi^{-1}(s)g(s)\,ds ,$$

then $f^{(m)}=\psi^{-1}g$, so that $f \in W^{(m)}(\psi,q,R)$. On the other hand,

$$(-1)^{m-j-1}(m-j-1)!\,f^{(j)}(t) = \int_t^1 (s-t)^{m-j-1}\psi^{-1}(s)g(s)\,ds$$
$$\to \int_0^1 s^{m-j-1}\psi^{-1}(s)g(s)\,ds = \infty$$

for $t \to 0$.

Corollary 3.4. *If $t^{m-j-1}\psi^{-1} \in \mathscr{L}^{q'}$ then the imbedding mapping of $W^{(m)}(\psi,q,X)$ into $C^j(\bar{I},X)$ is continuous.*

Proof. For $0 \leqq i \leqq j$ and $0 \leqq t \leqq 1$ we have the estimate

$$\begin{aligned} |u^{(i)}(t)| &\leqq |u^{(i)}(1)| + \frac{1}{(m-i-1)!}\int_0^1 s^{m-i-1}|u^{(m)}(s)|\,ds \\ &\leqq |u^{(i)}(1)| + C\|s^{m-j-1}\psi^{-1}\|_{q'}\|\psi u^{(m)}\|_q \\ &\leqq C\|u\|_W . \end{aligned}$$

Remark 3.5. The imbedding mapping of $W^{(m)}(\psi,q,X)$ into $C^j(\bar{I},X)$ is not compact unless X is finite-dimensional (consider a bounded set of constant functions: it is not necessarily compact).

Corollary 3.6. *If $t^{m-j-1}\psi^{-1} \in \mathscr{L}^{q'}$ then $u(0)=u'(0)=\ldots=u^{(j)}(0)=0$ for $u \in W_0^{(m)}(\psi,q,X)$.*

Proof. Each of the mappings $u \to u^{(i)}(0)$ is a continuous function on $C^j(\bar{I},X)$ and hence on W, and it is 0 on $C_0^\infty(I,X)$, which is dense in W_0.

4. Characterization of $W_0^{(m)}(\psi, q, X)$.

In this section we shall prove a converse of Corollary 3.6, namely that (if $q < \infty$) W_0 is exactly the subspace of W consisting of those functions u for which all the boundary values $u^{(j)}(0)$ which exist by virtue of theorem 3.3 vanish.

THEOREM 4.1. *Assume that* $1 \leqq q < \infty$.

(i) *If, for some* n *with* $0 \leqq n < m$,

$$t^{m-j-1}\psi^{-1} \in \mathscr{L}^{q'} \qquad \text{for} \qquad 0 \leqq j \leqq n\,,$$

and

$$t^{m-j-1}\psi^{-1} \notin \mathscr{L}^{q'} \qquad \text{for} \qquad n < j < m\,,$$

then

$$W_0^{(m)}(\psi,q,X) = \{u \in W^{(m)}(\psi,q,X) \mid u^{(j)}(0) = 0 \text{ for } 0 \leqq j \leqq n\}\,.$$

(ii) *If* $t^{m-1}\psi^{-1} \notin \mathscr{L}^{q'}$ *then* $W_0^{(m)}(\psi,q,X) = W^{(m)}(\psi,q,X)$.

PROOF. Since W_0 is the closure in W of $C_0^\infty(I,X)$, it follows that W_0 is the annihilator in W of the annihilator in W^* (the dual of W) of C_0^∞; that is, if we define

$$A = \{L \in W^* \mid \langle L,u\rangle = 0 \text{ for all } u \in C_0^\infty\}\,,$$

then

$$W_0 = \{u \in W \mid \langle L,u\rangle = 0 \text{ for all } L \in A\}\,.$$

Thus the theorem follows (in view of Corollary 3.6) if we prove that in case (i) every functional $L \in A$ is of the form

$$\langle L,u\rangle = \sum_{j=0}^{n} \langle l_j, u^{(j)}(0)\rangle \qquad \text{with} \qquad l_j \in X^*\,,$$

and that in case (ii) the only functional in A is 0.

First, let us consider the case where X is one-dimensional (that is, X is the scalar field F, which is either the real or complex numbers). Then, since the norm in W is defined by

$$\|f\|_W = \|\psi f^{(m)}\|_q + \sum_{j=0}^{m-1} |f^{(j)}(1)|\,,$$

every continuous linear functional λ on W is of the form

$$\langle \lambda,f\rangle = \int_0^1 g(t)\psi(t)f^{(m)}(t)\,dt + \sum_{j=0}^{m-1} a_j f^{(j)}(1)\,,$$

where $g \in \mathscr{L}^{q'}$ and $a_j, j = 0, 1, \ldots, m-1$, are scalars.

Note that $\psi \in \mathscr{L}^q_{\text{loc}}$ by assumption, so that $g\psi \in \mathscr{L}^1_{\text{loc}}$ is a distribution in I. Now, if $\lambda \in A$, then the distribution $g\psi$ satisfies the equation

$$\left(\frac{d}{dt}\right)^m (g\psi) = 0 ,$$

that is

$$g(t)\psi(t) = \sum_{j=0}^{m-1} b_j t^j$$

or

$$g(t) = \sum_{j=0}^{m-1} b_j t^j \psi^{-1}(t) .$$

Since $\psi^{-1} \in \mathscr{L}^{q'}_{\mathrm{loc}}$, it follows from $g \in \mathscr{L}^{q'}$ that

$$\begin{aligned} b_j &= 0 \quad \text{for } j < m-n-1 \text{ in case (i)} , \\ b_j &= 0 \quad \text{for all } j \qquad\quad \text{in case (ii)} . \end{aligned}$$

Furthermore, if we consider $j \geqq m-n-1$ (in case (i)), then it follows from Lemma 3.1 that integration by parts is permissible in the integral

$$\int_0^1 t^j f^{(m)}(t)\,dt = f^{(m-1)}(1) - jf^{(m-2)}(1) + \ldots + \\ + (-1)^j j!\,\big(f^{(m-j-1)}(1) - f^{(m-j-1)}(0)\big) .$$

Inserting the expression for λ, we get

$$\langle \lambda, f \rangle = \sum_{j=0}^{n} l_j f^{(j)}(0) + \sum_{j=0}^{m-1} k_j f^{(j)}(1) ,$$

where l_j and k_j are scalars, and where the first sum should be interpreted as 0 in case (ii). It is clear, however, that all k_j must vanish since $\langle \lambda, f \rangle = 0$ for all $f \in C_0^\infty$. Thus we have proved that

$$\langle \lambda, f \rangle = \sum_{j=0}^{n} l_j f^{(j)}(0) ,$$

where the sum should be interpreted as 0 in the case (ii).

Now, let X be an arbitrary Banach space over the scalar field F. It is clear that if $f \in W^{(m)}(\psi, q, F)$ and $x \in X$, then $fx \in W^{(m)}(\psi, q, X)$, where fx is the function defined by

$$(fx)(t) = f(t)x \qquad \text{for} \quad t \in I .$$

Furthermore, for each x the mapping $f \to fx$ is continuous and maps $C_0^\infty(I,F)$ into $C_0^\infty(I,X)$. Consequently, if L is a continuous linear functional on $W^{(m)}(\psi, q, X)$, then $\lambda(x)$ defined by

$$\langle \lambda(x), f \rangle = \langle L, fx \rangle$$

is a continuous linear functional on $W^{(m)}(\psi,q,F)$, and if $L \in A$, then it follows from the result above that

$$\langle L,fx\rangle = \sum_{j=0}^{n} l_j(x) f^{(j)}(0)\,.$$

Since the mapping $x \to t^j x$ is a continuous mapping of X into $W^{(m)}$ for $0 \leqq j \leqq n$ it follows that the (obviously linear) functionals l_j are continuous, whence we can write

$$\langle L,fx\rangle = \sum_{j=0}^{n} \langle l_j,x\rangle f^{(j)}(0) = \sum_{j=0}^{n} \langle l_j, (fx)^{(j)}(0)\rangle$$

with $l_j \in X^*$. We then have

$$(*) \qquad \langle L,u\rangle = \sum_{j=0}^{n} \langle l_j, u^{(j)}(0)\rangle$$

for all $u \in W^{(m)}(\psi,q,X)$ which are finite linear combinations of functions of the form fx. Since these functions u are dense in W and both sides of (*) are continuous, it follows that (*) holds for all $u \in W$, and the theorem is proved.

Corollary 4.2. *The smooth functions are dense in* $W^{(m)}(\psi,q,X)$ — *more exactly* $C^\infty(\bar{I},X) \cap W^{(m)}(\psi,q,X)$ *is dense in* $W^{(m)}(\psi,q,X)$.

Proof. Let $u \in W$ and define

$$p(t) = \sum_{j=0}^{n} \frac{t^j}{j\,!} u^{(j)}(0)\,.$$

Then

$$p \in C^\infty(\bar{I},X) \cap W^{(m)}(\psi,q,X)\,,$$

and

$$v = u-p \in W_0^{(m)}(\psi,q,X)$$

by Theorem 4.1, that is, v can be approximated in W by functions in

$$C_0^\infty(I\ X) \subset C^\infty(\bar{I},X) \cap W^{(m)}(\psi,q,X)\,.$$

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