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LINEAR FUNCTIONALS ON BOUNDED POLYNOMIALS OF GIVEN ORDER

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Introduction.

1. Given are n functions $\varphi_k(t)$, $1 \leqq k \leqq n$, defined on some set T of a *parameter* t. By a *polynomial* we understand any linear combination

$$(0.1)\qquad p(t) = \sum_{k=1}^{n} a_k \varphi_k(t), \qquad t \in T ,$$

of the $\varphi_k(t)$. For a *real polynomial* the $\varphi_k(t)$ and the a_k are supposed to be real; for a *complex polynomial* they are complex. We assume that the $\varphi_k(t)$ are *linearly independent on* T. This implies that T contains at least n different elements. Otherwise, the $\varphi_k(t)$ and T may be (at present) quite arbitrary.

2. The φ_k span the linear manifold $\mathscr{P}$ of polynomials p, and $\mathscr{P}$ is of dimension n, real or complex. We consider linear functionals L on $\mathscr{P}$. They are of the form

$$(0.2)\qquad L(p) = \sum_{1}^{n} a_k y_k \qquad \text{where} \qquad y_k = L(\varphi_k) .$$

Here the y_k are to be real, or complex, according to whether $\mathscr{P}$ is real or complex. These L form the dual linear manifold $\mathscr{L}$ also of dimension n. We write $L = \{y_k\}$ and consider the y_k as *co-ordinates* of the point L in $\mathscr{L}$.

If $\tau \in T$, then L_τ, where $L_\tau(p) = p(\tau)$, is a linear functional on $\mathscr{P}$, the so-called *spotting functional*. We have $L_\tau = \{\varphi_k(\tau)\}$. If the φ_k and T are given, then the set Φ of all spotting functionals L_τ is given in $\mathscr{L}$.

No hyperplane (and hence no proper subspace) of $\mathscr{L}$ contains Φ. For, by the linear independence of the φ_k,

$$\sum_{k=1}^{n} A_k \varphi_k(t) \equiv 0$$

for all $t \in T$ implies $A_k \equiv 0$. Hence Φ spans $\mathscr{L}$, so that every L admits *some* representation of the form

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(0.3)
$$L = \sum_{i=1}^{m} \mu_i L_{\tau_i}, \qquad \mu_i \neq 0;$$

i.e.
$$y_k = \sum_{i=1}^{m} \mu_i \varphi_k(\tau_i), \qquad 1 \leqq k \leqq n, \qquad \tau_i \in T,$$

where the τ_i are different. In fact, there exist such representations with $m \leqq n$. The μ_i are real, or complex, according to whether $\mathscr{P}$ is real or complex.

3. We assume, from now on, that the $\varphi_k(t)$ are *bounded on* T, normalizing this assumption by

(0.4)
$$|\varphi_k(t)| \leqq 1 .$$

On normalizing $\mathscr{P}$ through the uniform norm (Tchebychef norm)

(0.5)
$$\|p\| = \sup_{t \in T} |p(t)| ,$$

$\mathscr{P}$ becomes a normed n-dimensional linear vector space on which every linear functional L is bounded (continuous, cf. [1, p. 245]); and $\mathscr{L}$ itself becomes a normed n-dimensional linear vector space under the norm

(0.6)
$$\|L\| = \sup_{\|p\|=1} |L(p)| .$$

We also note that the sup in (0.6) is always attained; there exists at least one *maximal polynomial* P for which

(0.7)
$$\|P\| = 1 \qquad \text{and} \qquad L(P) = \|L\| .$$

4. In this paper we shall first determine $\|L\|$, for given L in $\mathscr{L}$, in a 'geometrical' fashion, by means of the given set Φ. This determination is, at least for real polynomials, hardly new. Nor, unfortunately, seems it to lend itself easily to actual calculation in the classical cases of algebraic polynomials ($T = [-1, 1]$) and trigonometric polynomials ($T = [-\pi, \pi]$) where many interesting results are known, obtainable by various ad hoc methods (cf. [3, II. Abschn. 6]). However, we shall derive from this determination some remarkable theoretical insight regarding the general case.

Next, we find a second equivalent definition of $\|L\|$, namely

(0.8)
$$\|L\| = \inf \sum_{i=1}^{m} |\mu_i|$$

where the inf is taken with respect to all possible representations (0.3) of L. Here the inf need not be attained for a given L. Any representation (0.3) for which the inf is attained is called a *minimal representation* for L. The investigation into the existence of such minimal representations throws additional light on the general case, revealing curious interrelations (*direction theorems*) between the maximal polynomials and the (possible) minimal representations for a given L. These results have interesting applications in various directions. It should be pointed out that the methods used in this investigation are completely n-dimensional. No use, apart from the definition of a norm, is made of further functional analysis. Even the simple Hahn–Banach extension theorem is avoided.

I. The general theory.

1. We shall assume that the $\varphi_k(t)$ and the polynomials $p(t)$ are complex. The real case can be treated in the same way, but is somewhat simpler. It seems that neither of the two cases can easily be deduced from the other. We prove our results in the complex case and state them in the real case.

In the following $\mathscr{C}$ will be a non-empty *symmetrically convex* set in the n-dimensional complex space $\mathscr{L}$; i.e., it has the following two properties:

(i) if $L \in \mathscr{C}$, then $\varepsilon L \in \mathscr{C}$ for every ε with $|\varepsilon| = 1$.

(ii) if $L_1, L_2 \in \mathscr{C}$ then $\varrho L_1 + (1-\varrho)L_2 \in \mathscr{C}$ where $0 \leqq \varrho \leqq 1$.

It follows that the origin $O = \frac{1}{2}[L + (-L)] \in \mathscr{C}$.

For the real space $\mathscr{L}$, ε is restricted to ± 1.

We use vector notation and write x for the point $\{x_k\}$. The scalar product of x and y is denoted by

$$\langle x,y \rangle = \sum_{k=1}^{n} x_k \bar{y}_k ,$$

so that

$$\langle x,x \rangle = \sum_{k=1}^{n} |x_k|^2 = |x|^2$$

where $|x|$ is the "Euclidean norm" in $\mathscr{L}$. Clearly, $\langle y,x \rangle = \overline{\langle x,y \rangle}$.

Any hyperplane $\mathscr{H}$ in $\mathscr{L}$ can be written in the form

$$\mathscr{H}: \ \langle x,a \rangle = \sum_{k=1}^{n} x_k \bar{a}_k = \alpha, \qquad \alpha \geqq 0, \quad |a| = 1 . \tag{1.1}$$

It divides $\mathscr{L}$ into two parts:

$$\mathscr{H}_+: \ |\langle x,a \rangle| \leqq \alpha, \qquad \mathscr{H}_-: \ |\langle x,a \rangle| > \alpha . \tag{1.2}$$

Clearly, $O \in \mathscr{H}_+$ for every $\mathscr{H}$. We say that $\mathscr{H}$ *bounds* $\mathscr{C}$ if $\mathscr{C} \subset \mathscr{H}_+$, and that $\mathscr{H}$ *supports* $\mathscr{C}$ if

$$\sup_{x \in \mathscr{C}} |\langle x,a \rangle| = \alpha . \tag{1.3}$$

We note that, if $\alpha = 0$, then $\mathscr{H}$ is a genuine subspace of $\mathscr{L}$ and cannot support $\mathscr{C}$ unless $\mathscr{C} \subset \mathscr{H}$.

2. We require the following

LEMMA 1. *If y is an Euclidean frontier point of $\mathscr{C}$, then there exists a hyperplane $\mathscr{H}$ of support for $\mathscr{C}$ with $y \in \mathscr{H}$.*

REMARK. In real space this Lemma is familiar (for convex sets). The present proof is modelled after that for the real case (compare [2, Vol. I, pp. 397–98]). Because of the "circularity" of $\mathscr{C}$ it *could* be derived from the real case.

PROOF OF THE LEMMA. Clearly, we may assume that $\mathscr{C}$ is closed.

(i) Let, first, y be exterior to $\mathscr{C}$. Then there exists an $x_0 \in \mathscr{C}$ such that, for all $x \in \mathscr{C}$,

$$|x-y| \geqq |x_0-y| = \delta > 0 . \tag{1.4}$$

For any $x \in \mathscr{C}$ also $x_0+\varrho(x-x_0) \in \mathscr{C}$ for $0 \leqq \varrho \leqq 1$. Hence

$$|x_0+\varrho(x-x_0)-y|^2 \geqq |x_0-y|^2 ,$$
$$2\varrho \operatorname{Re}\langle x-x_0, x_0-y\rangle + \varrho^2|x-x_0|^2 \geqq 0 .$$

On letting $\varrho \to 0$ we find

$$\operatorname{Re}\langle x-x_0, y-x_0\rangle \leqq 0 ,$$

or

$$\operatorname{Re}\langle x, y-x_0\rangle \leqq \operatorname{Re}\langle x_0, y-x_0\rangle = \operatorname{Re}\langle x_0-y+y, y-x_0\rangle$$
$$= -\delta^2 + \operatorname{Re}\langle y, y-x_0\rangle < \operatorname{Re}\langle y, y-x_0\rangle .$$

Here we can replace x by εx, $|\varepsilon| = 1$, and obtain

$$|\langle x, y-x_0\rangle| < \operatorname{Re}\langle y, y-x_0\rangle .$$

Hence, putting $a = (y-x_0)/|y-x_0|$ we have $|a| = 1$ and

$$|\langle x, a\rangle| < \operatorname{Re}\langle y, a\rangle = \alpha , \tag{1.5}$$

say, for all $x \in \mathscr{C}$.

(ii) Now let y be a frontier point of $\mathscr{C}$. Then there exists a sequence of exterior points $y^{(\nu)}$ tending to y (in the Euclidean metric). By (1.5), there exists a corresponding sequence of $a^{(\nu)}$ with $|a^{(\nu)}| = 1$ such that, for all $x \in \mathscr{C}$,

$$|\langle x, a^{(\nu)}\rangle| < \operatorname{Re}\langle y^{(\nu)}, a^{(\nu)}\rangle = \alpha^{(\nu)} ,$$

say. There exists also a subsequence $a^{(\nu')} \to a$, say, where $|a| = 1$. Hence, letting $\nu' \to \infty$, we obtain

$$|\langle x, a\rangle| \leqq \operatorname{Re}\langle y, a\rangle = \alpha . \tag{1.6}$$

Since $\mathscr{C}$ is closed, $y \in \mathscr{C}$, so that

$$|\langle y,a\rangle| = \operatorname{Re}\langle y,a\rangle = \alpha . \tag{1.7}$$

This proves the Lemma.

Corollary. *If $\mathscr{C}$ is closed (and not the whole of $\mathscr{L}$) then it is the intersection $\mathscr{S}$ of all $\mathscr{H}_+$ where $\mathscr{H}$ supports $\mathscr{C}$.*

Proof. To avoid irrelevant complications we shall assume that $\mathscr{C}$ does not lie in any genuine subspace of $\mathscr{L}$. Anyhow, this will be the case in our application.

Clearly, $\mathscr{C} \subset \mathscr{S}$. Next O is not a frontier point of $\mathscr{C}$, since otherwise a hyperplane of support through O would have $\alpha = 0$, and hence would contain $\mathscr{C}$. Let $y_0 \in \mathscr{S}$. If $y_0 \notin \mathscr{C}$ then $y_0 \neq O$, and there exists a frontier point y of $\mathscr{C}$ of the form $y = \varrho y_0$ where $0 < \varrho < 1$. Let $\langle x,a\rangle = \alpha > 0$ be a hyperplane $\mathscr{H}$ of support through y. Then

$$\langle y_0,a\rangle = \langle y,a\rangle/\varrho = \alpha/\varrho > \alpha ,$$

so that $y_0 \in \mathscr{H}_-$. This contradicts $y_0 \in \mathscr{S}$. Hence $y_0 \in \mathscr{C}$ and $\mathscr{S} \subset \mathscr{C}$.

3. The symmetric convex hull $\mathscr{K}$ of Φ is defined as the set of all $L \in \mathscr{L}$ that admit some finite representation of the form

$$L = \sum_{i=1}^{m} \mu_i L_{\tau_i}, \qquad \mu_i \neq 0, \quad \sum_{i=1}^{m} |\mu_i| = 1 ; \tag{1.8}$$

i.e.

$$y_k = \sum_{i=1}^{m} \mu_i \varphi_k(\tau_i), \qquad 1 \leqq k \leqq n .$$

Clearly, $\mathscr{K}$ is symmetrically convex. Equivalently, $\mathscr{K}$ is the smallest convex set in $\mathscr{L}$ containing all points εL_t where $|\varepsilon| = 1$. We note that O is an interior point of $\mathscr{K}$.

For the real space $\mathscr{P}$ the μ_i are all real and the ε are restricted to $\varepsilon = \pm 1$.

Let $\mathscr{H}$ be a hyperplane of support for $\mathscr{K}$. We can write it in the form

$$\mathscr{H}: \sum_{k=1}^{n} x_k A_k = 1 . \tag{1.9}$$

For, since $\alpha > 0$, we can put $A_k = \bar{a}_k/\alpha$ in (1.1). All $L_t \in \mathscr{H}_+$, that is

$$\left|\sum_{k=1}^{n} A_k \varphi_k(t)\right| = |P_{\mathscr{H}}(t)| \leqq 1 . \tag{1.10}$$

Also, by (1.3) we must have

(1.11) $$\sup_{t\in T}|P_{\mathscr{H}}(t)| = 1; \qquad \text{that is,} \qquad \|P_{\mathscr{H}}\| = 1 .$$

Conversely, if $P(t)=\sum_{k=1}^{n}A_k\varphi_k(t)$ is any polynomial with $\|P\|=1$, then, by (1.8), for every $L\in\mathscr{K}$,

$$\left|\sum_1^n y_k A_k\right| = \left|\sum_{i=1}^{m}\mu_i\sum_{k=1}^{n}A_k\varphi_k(\tau_i)\right| \leqq \max|P(\tau_i)| \leqq 1 .$$

On choosing in (1.8) the τ_i and μ_i so that $|P(\tau_i)|>1-\varepsilon$ and $\mu_i P(\tau_i)>0$, we see that $|\sum_1^n y_k A_k|>1-\varepsilon$, for any given $\varepsilon>0$.

Hence, in the dual space $\mathscr{L}$, the hyperplanes of support for $\mathscr{K}$ are exactly the $\mathscr{H}_P$ generated by the polynomials P with $\|P\|=1$.

Similarly, the polynomials p with $\|p\|\leqq 1$ generate all the bounding hyperplanes of $\mathscr{K}$.

4. THEOREM 1. *The unit ball $\|L\|\leqq 1$ in $\mathscr{L}$ is the Euclidean closure $\overline{\mathscr{K}}$ of $\mathscr{K}$.*

PROOF. Let $\mathscr{C}$ be the unit ball $\|L\|\leqq 1$. If $\|P\|=1$ then $|L_t(P)|=|P(t)|\leqq 1$. Hence, by (0.6), $\|L_t\|\leqq 1$ for all $t\in T$. Hence $\mathscr{K}$, and so $\overline{\mathscr{K}}$, $\subset\mathscr{C}$. Conversely, let $\|L\|\leqq 1$. If $\mathscr{H}$ is any hyperplane of support for $\mathscr{K}$, then $\|P_{\mathscr{H}}\|=1$ and

$$|L(P_{\mathscr{H}})| = \left|\sum_1^n A_k y_k\right| \leqq 1 .$$

Hence $L\subset\mathscr{H}_+$. It follows that $\mathscr{C}\subset\mathscr{S}$, the intersection of all such $\mathscr{H}_+$. By the Corollary in Section 2, $\mathscr{C}\subset\overline{\mathscr{K}}$.

REMARK. *The unit-sphere $\|L\|=1$ in $\mathscr{L}$ is the frontier $\mathscr{K}_f$ of $\mathscr{K}$.*

We give two corollaries to Theorem 1 the first of which allows a simple geometrical determination of $\|L\|$, for any given L, by means of $\mathscr{K}$. It should be noted that with Φ also $\mathscr{K}$ is *given* to us by the given $\varphi_k(t)$ and T.

It is clear that, if $L \neq O$, then there exists exactly one $\varrho>0$ such that ϱL belongs to the unit sphere, that is, to $\mathscr{K}_f$. Hence $\varrho\|L\|=1$, or

COROLLARY 1. *If $L\neq O$ and $\varrho\mathscr{L}\in\mathscr{K}_f$, $\varrho>0$, then*

(1.12) $$\|L\| = 1/\varrho .$$

In the classical cases mentioned in the introduction, an algebraic identification of $\mathscr{K}$ would be required to use this corollary for an actual algebraic determination of $\|L\|$. To my knowledge, no such identification is known.

COROLLARY 2. *If $L \neq O$, then the maximal polynomials P of L as defined in* (0.7) *are exactly those $P_{\mathscr{H}}$ for which $\mathscr{H}$ is any of the hyperplanes of support for $\mathscr{K}$ that pass through $L/\|L\|$.*

PROOF. We may assume that $\|L\|=1$, so that $L \in \mathscr{K}_{\mathscr{f}}$. If $\mathscr{H}$ is any hyperplane of support through L, then $\|P_{\mathscr{H}}\|=1$ and by (1.9),

$$L(P_{\mathscr{H}}) = \sum_{k=1}^{n} A_k y_k = 1 .$$

Hence $P_{\mathscr{H}}$ is maximal for L. Conversely, let P be maximal for L. Then $\|P\|=1$, and $\mathscr{H}_P$ is a hyperplane of support. Also

$$L(P) = \sum_{k=1}^{n} A_k y_k = 1 ,$$

so that $L \in \mathscr{H}_P$.

5. We are interested in those L, if any, for which $L \in \mathscr{K} \cap \mathscr{K}_{\mathscr{f}}$. Clearly $\|L\|=1$ for such an L.

THEOREM 2. *Let $L \in \mathscr{K} \cap \mathscr{K}_{\mathscr{f}}$. Then L determines, for every ε with $|\varepsilon|=1$, a (possibly empty) subset T_ε of T, of which one at least is not empty, such that every maximal polynomial P and every representation* (1.8) *of L satisfies*

$$\text{(i)}\quad \tau_i \in \textstyle\bigcup_\varepsilon T_\varepsilon .$$

$$(1.13)\qquad \text{(ii)}\quad \textit{If } \tau_i \in T_\varepsilon, \textit{ then } P(\tau_i) = \varepsilon^{-1} .$$

$$\text{(iii)}\quad \mu_i P(\tau_i) > 0 .$$

In particular, every maximal polynomial P of L attains (*in modulus*) *its norm.*

PROOF. Consider *any* maximal polynomial P and *any* representation (1.8) of L. Then $\|P\|=1$ and

$$(1.14)\qquad \begin{aligned} 1 = L(P) &= \sum_{k=1}^{n} A_k y_k = \sum_{k=1}^{n} A_k \sum_{i=1}^{m} \mu_i \varphi_k(\tau_i) \\ &= \sum_{i=1}^{m} \mu_i P(\tau_i) \leqq \sum_{i=1}^{m} |\mu_i|\, |P(\tau_i)| \leqq \sum_{i=1}^{m} |\mu_i| = 1 . \end{aligned}$$

Hence we must have equality throughout which implies

$$|P(\tau_i)| = 1, \qquad \mu_i P(\tau_i) > 0 .$$

Keeping the representation (1.8) fixed, let $\arg \mu_i = \arg \varepsilon_i$, say. Then we must have $P(\tau_i) = \varepsilon_i^{-1}$ for *every* P.

THEOREM 3. *Let $\mathscr{H}$ be any hyperplane of support for $\mathscr{K}$ and let $P_{\mathscr{H}}$*

be the polynomial corresponding to it. Then $\mathscr{K}\cap\mathscr{H}$ *is the convex hull of all points* εL_τ, $|\varepsilon|=1$, *if any, that lie in* $\mathscr{H}$; *or equivalently, for which* $P_{\mathscr{H}}(\tau)=\varepsilon^{-1}$.

PROOF. (i) If $L\in\mathscr{K}\cap\mathscr{H}$ then $L\in\mathscr{K}\cap\mathscr{K}_f$. By Corollary 2 of Theorem 1, $P_{\mathscr{H}}$ is maximal for L. Hence, by Theorem 2, L admits some representation

$$L=\sum_{i=1}^{m}\mu_i L_{\tau_i}=\sum_{i=1}^{m}|\mu_i|\,\varepsilon_i L_{\tau_i}, \tag{1.15}$$

$$|\varepsilon_i|=1,\qquad \sum_{i=1}^{m}|\mu_i|=1,\qquad P_{\mathscr{H}}(\tau_i)=\varepsilon_i^{-1}.$$

It follows that $L\in\hat{\mathscr{C}}$, the convex hull of all εL_τ with $P_{\mathscr{H}}(\tau)=\varepsilon^{-1}$.

(ii) Conversely, let $L\in\hat{\mathscr{C}}$. Then certainly $L\subset\mathscr{K}$ so that $\|L\|\leqq 1$. Also L is of the form (1.15) so that

$$L(P_{\mathscr{H}})=\sum_{i=1}^{m}\mu_i L_{\tau_i}(P_{\mathscr{H}})=\sum_{i=1}^{m}\mu_i P_{\mathscr{H}}(\tau_i)=\sum_{i=1}^{m}|\mu_i|=1\,.$$

Hence $\|L\|=1$ and so $L\in\mathscr{K}_f$ since $\|P_{\mathscr{H}}\|=1$. Thus $P_{\mathscr{H}}$ is maximal for L so that, again by the Corollary 2 of Theorem 1, $L\in\mathscr{H}$.

(iii) Let $\mathscr{C}^*$ be the convex hull of all εL_τ, $|\varepsilon|=1$, that lie in $\mathscr{H}$. By (i), every such $\varepsilon L_\tau\in\hat{\mathscr{C}}$. Hence $\mathscr{C}^*\subset\hat{\mathscr{C}}$. On the other hand, every εL_τ with $P_{\mathscr{H}}(\tau)=\varepsilon^{-1}$ lies in $\mathscr{H}$ since $\varepsilon L_\tau(P_{\mathscr{H}})=\varepsilon P_{\mathscr{H}}(\tau)=1$ so that $P_{\mathscr{H}}$ is maximal for εL_τ. Hence $\hat{\mathscr{C}}\subset\mathscr{C}^*$.

COROLLARY 1. $\mathscr{K}$ *is open if, and only if, no polynomial* p (*other than* $p\equiv 0$) *attains* (*in modulus*) *its norm.*

PROOF. We may assume that $\|p\|=1$ so that the corresponding $\mathscr{H}_p$ is a hyperplane of support for $\mathscr{K}$.

If $|p(t)|<1$ for all $t\in T$, then the above hull $\hat{\mathscr{C}}$ and thus $\mathscr{K}\cap\mathscr{H}_p$ is empty. If this is true for every such p, then $\mathscr{K}$ will be open. Conversely, if K is open, then $\mathscr{K}\cap\mathscr{H}$ and $\hat{\mathscr{C}}$ are empty for every hyperplane of support. Hence $|p(t)|<1$ for every p and every t.

COROLLARY 2. *If* $\mathscr{K}$ *is closed, then every polynomial* p *attains* (*in modulus*) *its norm.*

For $p\equiv 0$ this is always true. Otherwise, let $\|p\|=1$. If $\mathscr{K}$ is closed then $\mathscr{K}\cap\mathscr{H}_p$ is not empty. Hence p attains its norm.

It should be noted that the converse of Corollary 2 is not true. A simple example (in the real case) is:

$$n=2;\qquad \varphi_1(t)=\tfrac{1}{2}(1+\cos t),\qquad \varphi_2(t)=\sin t;\qquad T=(-\tfrac{1}{2}\pi,\tfrac{1}{2}\pi]\,.$$

We also note, in this connection, that, clearly, $\mathscr{K}$ is closed if, and only if, $\bar{\Phi} \subset \mathscr{K}$ where $\bar{\Phi}$ is the Euclidean closure of Φ.

6. Every L admits representations of the form

$$L = \sum_{i=1}^{\infty} \mu_i L_{t_i}; \quad \text{or,} \quad y_k = \sum_{i=1}^{\infty} \mu_i \varphi_k(t_i), \quad 1 \leqq k \leqq n , \tag{1.16}$$

where the sums are supposed to converge and *the μ_i may be zero.* In fact, by (0.3), L admits even "finite" such representations.

THEOREM 4.

$$\|L\| = \inf \sum_{i=1}^{\infty} |\mu_i| , \tag{1.17}$$

where the inf *is taken with respect to all possible representations* (1.16) *In fact, it suffices to admit only finite representations.*

REMARK. The right hand side of (1.17) is always a semi-norm for L; it is a norm if, and only if, the $\varphi_k(t)$ are bounded on T.

PROOF OF THEOREM 4. For $L = O$ this is clear. Otherwise, by (1.16),

$$\|L\| \leqq \sum_{i=1}^{\infty} |\mu_i| \, \|L_{t_i}\| \leqq \sum_{i=1}^{\infty} |\mu_i| , \tag{i}$$

since $\|L_t\| \leqq 1$. Hence $\|L\| \leqq \inf \sum |\mu_i|$.

(ii) Let $0 < \varrho < 1/\|L\|$. By (1.12), $\varrho L \in \mathscr{K}$ and thus, by (1.8), has a finite representation with $\sum_1^m |\mu_i| = 1$. Hence L has such a representation with $\sum_1^m |\mu_i| = 1/\varrho$. As $\varrho \uparrow 1/\|L\|$, we see that $\inf \sum |\mu_i| \leqq \|L\|$, even for finite representations.

7. A representation (1.16) is said to be *minimal* for L if

$$\|L\| = \sum_{i=1}^{\infty} |\mu_i| . \tag{1.18}$$

$L = O$ has, clearly, minimal representations with all $\mu_i = 0$ and the t_i arbitrary in them. Otherwise, L may, or may not, have minimal representations, and there may be many. If L has a minimal representation then, clearly, every λL has also one.

THEOREM 5. *Let $L \neq O$.*

(i) *L has a minimal representation if, and only if, $L/\|L\| \in \mathscr{K}$. It has then also a finite minimal representation.*

(ii) *Let* (1.16), *with $\underline{\mu}_i$ for μ_i, be any minimal representation for L, with all $\underline{\mu}_i \neq 0$. Then the $t_i \in \bigcup_\varepsilon T_\varepsilon$, as defined in Theorem 2 for $L/\|L\|$, and*

$$\arg \underline{\mu}_i = \arg \varepsilon \qquad \text{if} \quad t_i \in T_\varepsilon; \tag{1.19}$$

that is, for given $t \in T_\varepsilon$ *and for any* $\underline{\mu} \neq 0$ *associated with* t *in a minimal representation of* L, $\arg \underline{\mu}$ *is fully determined by* L.

PROOF. We may assume that $\|L\| = 1$ so that $L \in \mathscr{K}_f$.

(ia) Let $L \in \mathscr{K}$. Then it has a finite representation (1.8) with $\sum_1^m |\mu_i| = 1$. This representation is minimal.

(ib) Suppose that L has a minimal representation. Then putting $\underline{\mu}_i = |\mu_i| \varepsilon_i$, $|\varepsilon_i| = 1$, we have

$$y_k = \sum_{i=1}^{\infty} |\underline{\mu}_i| \varepsilon_i \varphi_k(t_i) = \sum_{i=1}^{\infty} |\underline{\mu}_i| a_{i,k}, \qquad 1 \leqq k \leqq n ,$$

say, with $\sum_{i=1}^{\infty} |\mu_i| = 1$. Hence

$$\text{(1.20)} \qquad \begin{aligned} \operatorname{Re} y_k &= \sum_{i=1}^{\infty} |\underline{\mu}_i| \operatorname{Re} a_{i,k} \\ \operatorname{Im} y_k &= \sum_{i=1}^{\infty} |\underline{\mu}_i| \operatorname{Im} a_{i,k} \end{aligned} , \qquad 1 \leqq k \leqq n, \quad \sum_{i=1}^{\infty} |\underline{\mu}_i| = 1 .$$

Now it is known [5, p. 10] that this finite system of $2n$ real linear equations in an infinity of positive "unknowns" $|\underline{\mu}_i|$ has also a finite solution

$$\text{(1.21)} \qquad \begin{aligned} \operatorname{Re} y_k &= \sum_{i=1}^{m} \varrho_i \operatorname{Re} a_{ik} \\ \operatorname{Im} y_k &= \sum_{i=1}^{m} \varrho_i \operatorname{Im} a_{ik} \end{aligned} , \qquad 1 \leqq k \leqq n, \quad \varrho_i \geqq 0, \quad \sum_{i=1}^{m} \varrho_i = 1 .$$

Hence also

$$\text{(1.22)} \qquad y_k = \sum_{i=1}^{m} \varrho_i a_{ik} = \sum_{i=1}^{m} \mu_i^* \varphi_k(t_i), \qquad 1 \leqq k \leqq n, \quad \sum_{i=1}^{m} |\mu_i^*| = 1 ,$$

where $\mu_i^* = \varrho_i \varepsilon_i$. Hence $L \in \mathscr{K}$.

(ii) Let P be *any* maximal polynomial and (1.16) be *any* minimal representation of L. Then [compare (1.14)]

$$\text{(1.23)} \qquad \begin{aligned} 1 = L(P) &= \sum_{k=1}^{n} A_k y_k = \sum_{i=1}^{\infty} \underline{\mu}_i \sum_{k=1}^{n} A_k \varphi_k(t_i) \\ &= \sum_{i=1}^{\infty} \underline{\mu}_i P(t_i) \leqq \sum_{i=1}^{\infty} |\underline{\mu}_i P(t_i)| \\ &\leqq \sum_{i=1}^{\infty} |\underline{\mu}_i| = 1 , \end{aligned}$$

from which follows

$$\text{(1.24)} \qquad |P(t_i)| = 1, \qquad \underline{\mu}_i P(t_i) > 0 ,$$

if $\underline{\mu}_i \neq 0$. This, by Theorem 2, completes the proof.

We note that, since $|P(t_i)| = |L_{t_i}(P)|$, all $\|L_{t_i}\| = 1$. *Hence L can have an infinite minimal representation only if infinitely many $L_t \in \mathscr{K} \cap \mathscr{K}_f$.*

We also note that Theorem 3 determines all L in a given hyperplane $\mathscr{H}$ of support for $\mathscr{K}$ that have minimal representations.

COROLLARY 1. *$\mathscr{K}$ is open if, and only if, no $L \neq O$ has a minimal representation.*

COROLLARY 2. *$\mathscr{K}$ is closed if, and only if, every L has a minimal representation.*

One should compare these corollaries (in particular, Corollary 2) with those of Theorem 3.

In the case of the *real space* $\mathscr{P}$ all our results remain true with ε restricted to the values ± 1. Thus, in Theorem 2, L determines two subsets T_+ and T_- of T, of which one at least is not empty, such that, for *every* representation (1.8) of L, and *every* maximal polynomial P

(i) $\tau_i \in T_+ \cup T_-$,

(ii) $\tau_i \in T_+$ implies $P(\tau_i) = 1$, $\mu_i > 0$;
$\tau_i \in T_-$ implies $P(\tau_i) = -1$, $\mu_i < 0$.

Consider the classical example of $p(t) = \sum_0^m a_k t^k$ with $|p(t)| \leqq 1$ on $T = [-1, 1]$. Here $\mathscr{K}$ is closed. For the L of $\mathscr{K}_f$ each maximal polynomial P reaches its norm 1. Unless $P \equiv \pm 1$, the number of points where $|P(\tau)| = 1$ in $[-1, 1]$ is at most $m+1$ $(= n)$. Hence the representation (1.8) for an L with some maximal $P \not\equiv \pm 1$ is *uniquely* determined, the μ_i automatically satisfying (ii) above. A similar result holds for bounded trigonometrical polynomials.

II. Finite systems of linear equations in an infinity of unknowns

1. We are given a column-finite matrix $[a_{ik}]$, $i = 1, 2, 3, \ldots$, $1 \leqq k \leqq n$, of rank n and we assume that all

$$|a_{ik}| \leqq 1 . \tag{2.1}$$

Consider the finite system of linear equations

$$\sum_{i=1}^{\infty} a_{ik} \mu_i = y_k, \qquad 1 \leqq k \leqq n , \tag{2.2}$$

in the unknowns μ_i. The series are supposed to converge. There is the real and the complex case where all the occurring numbers are real, or complex, respectively. We discuss the complex case; the results in the real case are analoguous.

It is clear that, for any given numbers y_k, there exist solutions of (2.2), even finite ones with at most n of the $\mu_i \neq 0$. We are interested in the possible *minimum solutions* $\{\underline{\mu}_i\}$ of (2.2), that is, solutions for which

$$\sum_{i=1}^{\infty} |\underline{\mu}_i| = \inf \sum_{i=1}^{\infty} |\mu_i| , \tag{2.3}$$

where the inf is taken with respect to all possible solutions.

2. If we put $\varphi_k(i) = a_{ik}$ for $i = 1, 2, \ldots,\ 1 \leqq k \leqq n$, then the assumptions of our theory are satisfied. T is the set of all i, $|\varphi_k(t)| \leqq 1$ on T, and the φ_k are linearly independent on T. Any solution of the system (2.2) gives a representation (1.16) of the linear functional $L = \{y_k\}$. Polynomials $p(t)$ are sequences defined as linear combinations

$$p(i) = \sum_{k=1}^{n} A_k a_{ik}, \qquad i = 1, 2, \ldots , \tag{2.4}$$

of the columns of the matrix $[a_{ik}]$; and $\|p\| = \sup_i |p(i)|$. The spotting functionals are the column points

$$L_i = \{a_{ik}\}_{k=1}^{n} , \tag{2.5}$$

and $\mathscr{K}$ is the symmetric convex hull of this *given* sequence $\{L_i\}$ of points in $\mathscr{L}$. Its points are of the form

$$L = \sum_{j=1}^{m} \mu_j L_{ij}, \qquad \mu_j \neq 0, \quad \sum_{j=1}^{m} |\mu_j| = 1 . \tag{2.6}$$

3. We shall say that two complex sequences $\{\alpha_i\}$ and $\{\beta_i\}$ have *equal argument* if

$$\arg \alpha_i = \arg \beta_i \qquad \text{whenever} \quad \alpha_i \neq 0,\ \beta_i \neq 0 . \tag{2.7}$$

We now restate Theorem 5 as follows:

THEOREM 6. *Suppose that not all* $y_k = 0$.

(i) *If* $L = \{y_k\}$ *then the system* (2.2) *has a minimal solution if, and only if,* $L/\|L\| \in \mathscr{K}$. *It has then also a finite minimal solution.*

(ii) *All minimal solutions of* (2.2) *have equal argument.*

The two corollaries of Theorem 5 can be restated as

THEOREM 7.

(i) $\mathscr{K}$ *is open if, and only if, no system* (2.2), *with* $\sum_1^n |y_k|^2 > 0$, *has minimal solutions.*

(ii) $\mathscr{K}$ *is closed if, and only if, every system* (2.2) *has minimal solutions.*

4. Of particular interest is the case where each row of the matrix $[a_{ik}]$ converges; say,

$$a_{ik} \to \lambda_k \quad \text{as } i \to \infty, \quad 1 \leqq k \leqq n; \qquad \text{or,} \quad L_i \to \Lambda = \{\lambda_k\}. \tag{2.8}$$

We shall say that a sequence $\{\alpha_i\}$ is *directed* if

(2.9) *either* (i) $\alpha_i = 0$ eventually;

or (ii) $\arg \alpha_i \to \theta$, $0 \leqq \theta < 2\pi$ for all $\alpha_i \neq 0$.

Two sequences $\{\alpha_i\}$ and $\{\beta_i\}$ are *equally directed* if (a) each is directed (b) both have equal argument (c) both have the same θ of (2.9) if they are both in case (ii) there.

THEOREM 8. *Suppose that $L_i \to \Lambda$. Then the system* (2.2) *has a minimal solution, for every choice of the y_k, if and only if, $\Lambda \in \mathscr{K}$.*

Moreover, every minimal solution is directed and all are equally directed.

PROOF. First, $\mathscr{K}$ is closed if, and only if, $\Lambda \in \mathscr{K}$. Hence Theorem 7 (ii), proves the first part of our statement.

If all $y_k = 0$, then all $\underline{\mu}_i = 0$ for a minimal solution. This unique minimal solution is directed.

Otherwise, all minimal solutions $\{\underline{\mu}_i\}$ *have* equal argument, by Theorem 6 (ii). Let P be a maximal polynomial for $L = \{y_k\}$. Then $\|P\| = 1$ and

$$L(P) = \sum_{k=1}^{n} A_k y_k = \sum_{i=1}^{\infty} \underline{\mu}_i \sum_{k=1}^{n} A_k a_{ik} = \sum_{i=1}^{\infty} \underline{\mu}_i P(i) = \|L\|. \tag{2.10}$$

Here are two possibilities. Either $\{\underline{\mu}_i\}$ is finite and hence directed; or there are infinitely many $\underline{\mu}_i \neq 0$. Then for these $\underline{\mu}_i$, by (1.19),

$$P(i) = e^{-i\theta_i}, \qquad \arg \underline{\mu}_i = \theta_i, \tag{2.11}$$

say. Since $P(i) = L_i(P) \to \Lambda(P) = e^{-i\theta}$, say, we conclude that $\arg \underline{\mu}_i \to \theta$ for $\underline{\mu}_i \neq 0$. Hence $\{\underline{\mu}_i\}$ is directed. Again by (1.19), the θ_i are fully determined by L, so that all minimal solutions $\{\underline{\mu}_i\}$ are equally directed.

COROLLARY. *If Λ is an interior point of $\mathscr{K}$ then all minimal solutions $\{\underline{\mu}_i\}$ of* (2.2) *are finite. In fact, there exists an i_0 such that $\underline{\mu}_i = 0$ for $i \geqq i_0$ for all minimal solutions.*

This follows immediately from the remark after (1.24).

In the case $\Lambda = O$, I had proved this corollary before [4, p. 98]. In this case $\mathscr{P}$ is a subspace of the space c_0 of all sequences $\{\alpha_i\}$ that tend to zero, normed by the uniform norm. One knows the form of a general linear functional on c_0, and a simple application of the Hahn–Banach extension theorem yields the above result.

The present method is much more elementary, more general, and more revealing.

We add two remarks:

(a) Suppose that $L_i \to \Lambda$ where $\Lambda \notin \mathscr{K}$. We add to our matrix a zero column $a_{0k} = \lambda_k$, when our system becomes

$$\mu_0 \lambda_k + \sum_{i=1}^{\infty} \mu_i a_{ik} = y_k, \quad 1 \leqq k \leqq n; \qquad |a_{ik}| \leqq 1 . \tag{2.12}$$

Since also $|\lambda_k| \leqq 1$, Theorem 8 becomes applicable. Note, however, that a minimal solution has now to take the number μ_0 into account (see [4]).

(b) Let Λ be any limiting point of the L_i, and let $L_{i_j} \to \Lambda$, say. We can now apply Theorem 8, or the above remark to it, by restricting the i to the i_j. In general, this will diminish $\mathscr{K}$, and minimal solutions are now restricted to those for which $\underline{\mu}_i = 0$ whenever $i \neq i_j$ for all j.

5. In the real case all $\mu_i \neq 0$ are either positive or negative. *In Theorem* 6 *all minimal solutions* $\{\underline{\mu}_i\}$ *are simultaneously, for the same* i, *either non-negative or non-positive. In Theorem* 8, *they are all, for* $i \geqq i_0$, *of the same sign* (see [4]).

III. The finite moment problem.

1. We consider the finite moment problem

$$y_k = \int \varphi_k(t)\, d\mu(t), \qquad 1 \leqq k \leqq n . \tag{3.1}$$

Here the integrals are Lebesgue–Stieltjes integrals over the N-dimensional real Euclidean space $T = \mathscr{E}_N$, the $\mu(t)$ are distribution functions of bounded variation over T, the $\varphi_k(t)$ are arbitrary functions, linearly independent, and satisfying $|\varphi_k(t)| \leqq 1$ on T.

For given y_k, $\mu(t)$ is a solution of (3.1) if the integrals (3.1) converge. This implies that the $\varphi_k(t)$ will have to be measurable with respect to the "signed" measure μ generated by the solution $\mu(t)$. Solutions $\mu(t)$ that generate the same measure μ are considered as equivalent. There is the real and the complex case where all the numbers and functions involved are real, or complex, respectively. We shall deal with the complex case. The real case is much simpler and has analoguous results. That there exists solutions for every choice of the y_k is clear. There exist, by (0.3), even finite (atomic) solutions of the form

$$y_k = \sum_{i=1}^{m} \mu_i \varphi_k(t_i) , \tag{3.2}$$

where $\mu(t)$ is a "stepfunction" having "signed mass" μ_i at the points t_i only.

2. We require some properties of complex Lebesgue-Stieltjes integrals. For these we refer to [1, chapter III].

(i) Let the point t in $\mathscr{E}_N$ have the coordinates x_i, $1 \leqq i \leqq N$, and let $(J]$ denote an interval $a_i < x_i \leqq b_i$, $i = 1, 2, \ldots, N$, closed to the right. D denotes a "division" $\mathscr{E}_N = \bigcup_1^\infty (J_k]$ in disjoint such intervals. Similarly, if $\tau = \{\xi_i\}$, $\mathscr{E}_\tau = (x_i \leqq \xi_i]$, then D_τ denotes a division $\mathscr{E}_\tau = \bigcup_1^\infty (J_l]$ of disjoint such intervals. Let

$$\Delta_\mu(J] = \sum_\lambda (-1)^{\nu(\lambda)} \mu(\lambda) , \tag{3.3}$$

where the sum is extended over all 2^N corners λ of the closure of $(J]$, and $\nu(\lambda)$ is the number of a_i amongst the co-ordinates of λ. Next, one puts

$$\sigma_\mu(D) = \sum_1^\infty |\Delta_\mu(J_k]|, \qquad \sigma_\mu(D_\tau) = \sum_1^\infty |\Delta_\mu(J_l]| , \tag{3.4}$$

and

$$v(\infty) = \sup \sigma_\mu(D), \qquad v(\tau) = \sup \sigma_\mu(D_\tau), \tag{3.5}$$

where the sup is taken with respect to all possible divisions D, or D_τ, respectively.

The function $\mu(t)$ is of bounded variation on $\mathscr{E}_N$ if $v(\infty) < \infty$; $v(\infty)$ is then the total variation of $\mu(t)$ over $\mathscr{E}_N$. An equivalent definition of bounded variation is that both $\operatorname{Re}\mu(t)$ and $\operatorname{Im}\mu(t)$ should be of bounded variation in the (more familiar) corresponding sense. The function $v(\tau)$ "increases" with τ; that is, $v(\tau_1) \geqq v(\tau_2)$ if $\mathscr{E}_{\tau_1} \supset \mathscr{E}_{\tau_2}$. If $v(\tau)$ is "continuous to the right", that is, if $v(\tau)$ generates a Lebesgue–Stieltjes (non-negative) measure v over $\mathscr{E}_N$, then $\mu(t)$ is said to be a Lebesgue–Stieltjes distribution function for a signed measure μ, defined, in the usual way, via (3.3).

(ii) The integral $\int f(t)\, d\mu(t)$ can be defined as a combination of four real Lebesgue–Stieltjes integrals on splitting both $f(t)$ and $\mu(t)$ in real and imaginary parts. By the Radon–Nikodým theorem [1, p. 181], there exists, given $\mu(t)$, a function $\chi_\mu(t)$ such that

$$|\chi_\mu(t)| = 1 \quad \text{p.p. } (v) \tag{3.6}$$

and so that

$$\int f(t)\, d\mu(t) = \int f(t)\, \chi_\mu(t)\, dv(t) \tag{3.7}$$

for *every* $f(t)$ integrable with respect to $\mu(t)$. Also

$$\left| \int f(t)\, d\mu(t) \right| \leqq \int |f(t)|\, dv(t) . \tag{3.8}$$

3. We write the system (3.1) in the form

$$L = \int L_t \, d\mu(t) . \tag{3.9}$$

Let

$$\|L\|_* = \inf \int dv(t) , \tag{3.10}$$

where the inf is extended over all possible representations (3.9) of L.

THEOREM 9.

$$\|L\|_* = \|L\| . \tag{3.11}$$

PROOF. By (3.2) and (1.17), $\|L\|_* \leqq \|L\|$. On the other hand, let P be a maximal polynomial for L (with respect to $\|L\|$). Then $\|P\| = 1$ and, by (3.8)

$$\|L\| = L(P) = \int P(t)\, d\mu(t) \leqq \int |P(t)|\, dv(t) \leqq \int dv(t) .$$

Hence $\|L\| \leqq \|L\|_*$.

4. For a given L we *may* have *minimal representations* (2.3), with

$$L = \sum_1^\infty \underline{\mu}_i L_{t_i}, \qquad \sum_1 |\underline{\mu}_i| = \|L\|; \tag{3.12}$$

and we *may* have minimal solutions $\underline{\mu}(t)$ of (3.9), with

$$L = \int L_t \, d\underline{\mu}(t), \qquad \int d\underline{v}(t) = \|L\|_* = \|L\| . \tag{3.13}$$

THEOREM 10. *L has a minimal solution if, and only if, it has a minimal representation.*

PROOF. (i) If L has a minimal representation, then it has, by Theorem 5, also a finite one. This is also a minimal solution by a stepfunction.

(ii) Conversely, we may assume that $\|L\| = 1$ and that L has a minimal solution $\underline{\mu}(t)$. By (3.6) and (3.7), we have

$$L = \int L_t \chi_{\underline{\mu}}(t)\, d\underline{v}(t), \qquad |\chi_{\underline{\mu}}(t)| = 1 \text{ p.p. } (\underline{v}), \qquad \int d\underline{v}(t) = 1 . \tag{3.14}$$

Hence

$$\mathrm{Re} L = \int \mathrm{Re}\big(L_t \chi_{\underline{\mu}}(t)\big)\, d\underline{v}(t), \qquad \mathrm{Im} L = \int \mathrm{Im}\big(L_t \chi_{\underline{\mu}}(t)\big)\, d\underline{v}(t) , \tag{3.15}$$

or equivalently, for $1 \leqq k \leqq n$,

$$\begin{aligned} \mathrm{Re}\, y_k &= \int \mathrm{Re}\big(\varphi_k(t)\, \chi_{\underline{\mu}}(t)\big)\, d\underline{v}(t) , \\ \mathrm{Im}\, y_k &= \int \mathrm{Im}\big(\varphi_k(t)\, \chi_{\underline{\mu}}(t)\big)\, d\underline{v}(t) . \end{aligned} \tag{3.16}$$

Thus the $\operatorname{Re} y_k$, $\operatorname{Im} y_k$ are $2n$ real moments with respect to the $2n$ real functions $\operatorname{Re}(\varphi_k(t)\chi_{\underline{\mu}}(t))$, $\operatorname{Im}(\varphi_k(t)\chi_{\underline{\mu}}(t))$, and with mass $\int d\underline{v}(t)=1$. It is known (see [5, p. 5]) that this implies the existence of a finite representation

$$(3.17)\qquad \begin{aligned} \operatorname{Re} y_k &= \sum_{i=1}^{m} \varrho_i \operatorname{Re}(\varphi_k(t_i)\chi_{\underline{\mu}}(t_i)) , \\ \operatorname{Im} y_k &= \sum_{i=1}^{m} \varrho_i \operatorname{Im}(\varphi_k(t_i)\chi_{\underline{\mu}}(t_i)) , \end{aligned}$$

$1 \leq k \leq n$, $\varrho_i > 0$, $\sum_{i=1}^{m} \varrho_i = 1$; or

$$(3.18)\qquad y_k = \sum_{i=1}^{m} \varrho_i \varphi_k(t_i)\chi_{\underline{\mu}}(t_i) = \sum_{i=1}^{m} \underline{\mu}_i \varphi_k(t_i), \qquad \sum_{i=1}^{m} |\underline{\mu}_i| = 1 ,$$

where $\underline{\mu}_i = \varrho_i \chi_{\underline{\mu}}(t_i)$, since, by (3.6), we may assume all $|\chi_{\underline{\mu}}(t_i)|=1$. Now (3.18) is a (finite) minimal representation for L, and the theorem is proved.

We can now apply our general theory and obtain immediately

THEOREM 11.

(i) *If the system* (3.1) (*or* (3.9)) *has a minimal solution, then all maximal polynomials P of L attain their norm.*

(ii) *$\mathscr{K}$ is open if, and only if, no $L \neq O$ has a minimal solution*; *or if, and only if, no polynomial p* $(\neq 0)$ *attains its norm.*

(iii) *$\mathscr{K}$ is closed if, and only if, every L has a minimal solution*; *then all p attain their norms.*

5. Our final result corresponds to Theorem 5 (ii).

THEOREM 12. *Suppose that $L \neq O$.*

If L has minimal solution then it determines, for every ε with $|\varepsilon|=1$, a subset $T_L^{(\varepsilon)}$ of $\mathscr{E}_N$ of which at least one is not empty, such that, on any $T_L^{(\varepsilon)}$,

$$(3.19)\qquad P(t) = \varepsilon^{-1} \quad \text{p.p.} \quad (\underline{v})$$

and

$$(3.20)\qquad \chi_{\underline{\mu}}(t) = \varepsilon \quad \text{p.p.} \quad (\underline{v})$$

for every pair of a maximal polynomial $P(t)$ and a minimal solution $\underline{\mu}(t)$ for L.

PROOF. We may assume that $\|L\|=1$. Then, for *every* $P(t)$ and $\underline{\mu}(t)$, we have $\|P\|=1$, $L(P)=1$, and $|\chi_{\underline{\mu}}(t)|=1$ p.p. $(\underline{v})$. Now

$$(3.21)\qquad 1 = L(P) = \int P\,d\underline{\mu} = \int P\chi_\mu\,d\underline{v} \leqq \int |P\chi_\mu|\,d\underline{v} = \int |P|\,d\underline{v} \leqq \int d\underline{v} = 1\,,$$

so that there must be equality throughout. Hence

$$(3.22)\qquad \int (1-P\chi_\mu)\,d\underline{v} = \int (1-|P\chi_\mu|)\,d\underline{v} = \int (1-|P|)\,d\underline{v} = 0\,,$$

which implies

$$(3.23)\qquad |P(t)| = 1 \quad \text{p.p.} \quad (\underline{v}), \qquad P(t)\chi_\mu(t) = 1 \quad \text{p.p.} \quad (\underline{v})\,.$$

If P_0 is a fixed maximal polynomial, then, by Theorem 11(i), the set $T^{(\varepsilon)}$ where $P_0(t)=\varepsilon^{-1}$ is, for at least one ε, not empty. By (3.23), *every* $\underline{\mu}$ satisfies (3.20) there; and hence also *every* $P(t)$ satisfies (3.19) there.

6. In the case of the real moment problem (3.1) all our results remain valid. The ε in Theorem 12 are now restricted to ± 1. An equivalent result can be obtained in terms of the positive and negative variation-distributions $\underline{v}^+(t)$ and $\underline{v}^-(t)$. The use of the Radon–Nikodým theorem can then be avoided.

REFERENCES

1. N. Dunford and J. T. Schwartz, *Linear operators* I, New York, 1958.
2. S. Karlin, *Mathematical methods and theory in games, programming and economics* I-II, Reading, Mass.·London, 1959.
3. G. Pólya und G. Szegö, *Aufgaben und Lehrsätze aus der Analysis* I–II, Berlin, 1925.
4. W. W. Rogosinski, *On finite systems of linear equations with an infinity of unknowns*, Math. Zeitschr. 63 (1955), 97–108.
5. W. W. Rogosinski, *Moments of non-negative mass*, Proc. Royal Soc. London A. 245 (1958), 1–27.

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