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ON A QUESTION OF SADULLAEV CONCERNING BOUNDARY RELATIVE EXTREMAL FUNCTIONS

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Abstract

We study the relation between certain alternative definitions of the boundary relative extremal function. For various domains we give an affirmative answer to the question of Sadullaev whether these extremal functions are equal.

1. Introduction

Let $D \subset \mathbb{C}^n$ be a smoothly bounded domain, $A \subset \partial D$, and let $\mathrm{PSH}(D)^-$ stand for the family of non-positive plurisubharmonic functions on D. For $u \in \mathrm{PSH}(D)^-$ as usual

$$u^*(z) = \limsup_{\zeta \to z, \zeta \in D} u(\zeta) \qquad (z \in \overline{D}).$$

Sadullaev studied the first three of the following boundary extremal functions. For $z \in D$, consider

(1) $\omega_1(z, A, D) = \omega^c(z, A, D) = \sup\big\{u(z) : u \in \mathrm{PSH}(D)^- \cap C(\overline{D}), u|A \leq -1\big\}$,

(2) $\omega_2(z, A, D) = \omega(z, A, D) = \sup\big\{u(z) : u \in \mathrm{PSH}(D)^-, u^*|A \leq -1\big\}$,

(3) $\omega_3(z, A, D) = \omega^n(z, A, D) = \sup\big\{u(z) : u \in \mathrm{PSH}(D)^-, \limsup_{z \to \zeta, z \in n_\zeta} u(z) \leq -1 \text{ for } \zeta \in A\big\}$, where n_ζ is the inward normal to ∂D at ζ,

(4) $\omega^R(z, A, D) = \sup\big\{u(z) : u \in \mathrm{PSH}(D)^-, \limsup_{r \to 1^-} u(rz) \leq -1, z \in A\big\}$, if D is strongly star shaped with respect to the origin.

Actually, smoothness is needed only to define ω^n. It is clear that

$$\omega_1(\,\cdot\,, A, D) \leq \omega_2(\,\cdot\,, A, D) \leq \omega_3(\,\cdot\,, A, D).$$

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This paper is motivated by the following question (Problem 27.4 in [10]): suppose $A \subset \partial D$ is closed, for what i, j is $\omega_i^*(z, A, D) \equiv \omega_j^*(z, A, D)$?

The answer apparently depends on the geometry and convexity properties of D and the choice of the compact set $A \subset \partial D$. For instance we showed in [2] that Sadullaev's question has a positive answer when D is a smooth pseudoconvex Reinhardt domain and A is multi-circular. The result in [2] exploits the relation between relative extremal functions and convex functions in a Reinhardt domain.

In the present paper we answer in Section 3 the question affirmatively for ellipsoidal domains D_H, which are biholomorphic to the unit ball via a linear transformation. Here we exploit an idea of Wikström [11] and use Edwards' duality theorem. In Section 4 we show equality for circular sets A in the boundary of circular, strongly star shaped domains D. We attempted to use the version of Edwards' theorem in [6] and found that their result is not correct. In the appendix we give two pertaining counterexamples.

We denote the open unit disc in $\mathbb{C}$ by $\mathbb{D}$, its boundary by $\mathbb{T}$ and the unit ball in $\mathbb{C}^n$ $(n \geq 2)$ by $\mathbb{B}$. Some basic properties of the boundary relative extremal function are given in [2], [4], [8], [10], [3] ([4] appeared as [5] but the preprint is more relevant). Depending on the way the boundary is approached, plurisubharmonic function may have different boundary values. Wikström considered the compact set $A = \mathbb{T} \times \{0\}$ and the function $u \in \mathrm{PSH}(\mathbb{B})$:

$$u(z) = \log \frac{|z_2|^2}{1 - |z_1|^2}.$$

He showed that $u^* = 0$ on A. The radial limit of u, $u^R = -\infty$ on A and the non-tangential limit of u, $u^\alpha = \log(1 - 1/2\alpha)$ on A [11, Example 5.5]. We recall the definition of u^α. If $\alpha > 1$ and $z_0 \in \partial\mathbb{B}$ we put

$$D_\alpha(z_0) = \big\{ z \in \mathbb{B} : |1 - \langle z, z_0 \rangle| < \alpha(1 - |z|^2) \big\},$$

$$u^\alpha(z_0) = \limsup_{z \to z_0, z \in D_\alpha(z_0)} u(z).$$

2. Notation and definitions

Let $D = \{\rho < 0\}$ be a domain in $\mathbb{C}^n$ with C^1-boundary and defining function ρ. For $z \in \overline{D}$ and $t \in \mathbb{R}$, let

$$n(z, t) = z - t\left(\frac{\partial \rho}{\partial \bar{z}_1}(z), \ldots, \frac{\partial \rho}{\partial \bar{z}_n}(z) \right).$$

If $z \in \partial D$ the normal line n_z passing through z is parametrized by $\{n(z, t) : t \in \mathbb{R}\}$.

Let $u\colon D \to \mathbb{R} \cup \{-\infty\}$ be bounded from above and $z \in \partial D$ we define u^n at z as

$$u^n(z) = \limsup_{t \downarrow 0} u \circ n(z, t).$$

Extend u^n to $\overline{D}$ by setting $u^n(z) = u(z)$ if $z \in D$. Recall that D is called *strongly star shaped with respect to the origin* if $r\overline{D} \subset D$ for $r \in]0, 1[$. If D is strongly star shaped with respect to the origin, then for $z \in \partial D$ set $u^R(z) = \limsup_{r\uparrow 1} u(zr)$. Extend u^R to $\overline{D}$ by setting $u^R(z) = u(z)$ if $z \in D$. Throughout the paper by strongly star shaped we mean strongly star shaped with respect to the origin. Let $M(D)$ be the set of Borel probability measures with compact support on $\overline{D}$. For $z \in \overline{D}$ we consider four classes of positive measures:

(1) $J_z = J_z(\overline{D}) = \{\mu \in M(D) : u(z) \le \int_{\overline{D}} u \, d\mu$ for all $u \in \mathrm{PSH}(D) \cap \mathrm{USC}(\overline{D})\}$,

(2) $J_z^c = J_z^c(\overline{D}) = \{\mu \in M(D) : u(z) \le \int_{\overline{D}} u \, d\mu$ for all $u \in \mathrm{PSH}(D) \cap C(\overline{D})\}$,

(3) $J_z^n = J_z^n(\overline{D}) = \{\mu \in M(D) : u^n(z) \le \int_{\overline{D}} u^n \, d\mu$ for all $u \in \mathrm{PSH}(D)$, $\sup_{\overline{D}} u^n < \infty\}$, and

(4) $J_z^R = J_z^R(\overline{D}) = \{\mu \in M(D) : u^R(z) \le \int_{\overline{D}} u^R \, d\mu$ for all $u \in \mathrm{PSH}(D)$, $\sup_{\overline{D}} u^R < \infty\}$, in the case when D is strongly star shaped with respect to the origin.

Clearly for $z \in D$, $J_z^n, J_z^R \subset J_z \subset J_z^c$. Wikström studied these classes and proved that $J = J^c = J^R$ on D if D is strongly star shaped, see [11, Proposition 5.4]. If $U \subset \overline{D}$, χ_U denotes the characteristic function of U.

3. Applications of Wikström's results

We use equalities between different classes of Jensen measures to prove the equivalence of different definitions. This is done by applying Edwards' theorem to the convex cone $\mathrm{PSH}(D) \cap \mathrm{USC}(\overline{D})$ and the associated Jensen measures J_z.

PROPOSITION 3.1. *Let $D \subset \mathbb{C}^n$ be a bounded domain with C^1−boundary, $A \subset \partial D$ compact. If $J_z = J_z^n$ for all $z \in D$ then*

$$\omega(z, A, D) = \omega^n(z, A, D).$$

PROOF. We know that $\omega(\cdot, A, D) \le \omega^n(\cdot, A, D)$. Let us prove that $\omega^n(\cdot, A, D) \le \omega(\cdot, A, D)$. Let u be in the family defining ω^n.

Set $g = -\chi_A$. Note that $u^n \le g$ on $\overline{D}$. For $z \in D$ one has

$$u^n(z) \le \inf\left\{\int g\,d\mu : \mu \in J_z^n\right\} = \inf\left\{\int g\,d\mu : \mu \in J_z\right\},$$

since $J_z = J_z^n$. Because g is lower semicontinuous on $\overline{D}$, Edwards' theorem (Cor. 2.2 in [11]) gives

$$\begin{aligned} u^n(z) &\le \inf\left\{\int g\,d\mu : \mu \in J_z\right\} \\ &= \sup\left\{v(z) : v \in \mathrm{PSH}(D) \cap \mathrm{USC}(\overline{D}),\ v \le g\right\} \le \omega(z, A, D). \end{aligned}$$

As u was taken arbitrarily in the family defining ω^n, we infer that $\omega^n(z, A, D) \le \omega(z, A, D)$ for all $z \in D$. Thus $\omega(\cdot, A, D) = \omega^n(\cdot, A, D)$.

REMARK 3.2. Notice that the dual of J^n i.e. $\{u^n : u \in \mathrm{PSH}(D), \sup_D u < +\infty\}$ is not a convex cone, indeed if u and v are bounded plurisubharmonic functions in D, we do not have in general $(u + v)^n = u^n + v^n$ in $\overline{D}$, cf. [12, Definition 2.2 and below]. We use the class J^n only to obtain an inequality.

The proof above applies to the next two propositions.

PROPOSITION 3.3. *Let $D \subset \mathbb{C}^n$ be a bounded strongly star shaped domain and $A \subset \partial D$ compact. If $J_z = J_z^R$ for all $z \in D$ then*

$$\omega(z, A, D) = \omega^R(z, A, D).$$

PROPOSITION 3.4. *Let $D \subset \mathbb{C}^n$ be a bounded domain and $A \subset \partial D$ compact. If $J_z = J_z^c$ for all $z \in D$ then $\omega(z, A, D) = \omega^c(z, A, D)$ for $z \in D$.*

For $z \in \overline{D}$ define

$$J_z^* = J_z^*(\overline{D}) = \left\{\mu \in M(D) : u^*(z) \le \int_{\overline{D}} u^*\,d\mu \quad \text{for all } u \in \mathrm{PSH}(D), \sup_D u < +\infty\right\}.$$

Note that in [11] the author worked with J^* but this class does not represent a convex cone, see [1] for details. Here we work with J instead of J^* for two reasons: firstly $\mathrm{PSH}(D) \cap \mathrm{USC}(\overline{D})$ is a convex cone so Edwards' theorem can be applied, secondly for $z \in D$ and g lower semicontinuous we have

$$\inf\left\{\int g\,d\mu : \mu \in J_z^*\right\} = \inf\left\{\int g\,d\mu : \mu \in J_z\right\},$$

thus the results in [11] remain valid for J_z if z is an interior point.

COROLLARY 3.5. *If D is B-regular or if D is strongly star shaped with respect to the origin or if D is a polydisc then $\omega(z, A, D) = \omega^c(z, A, D)$ for $z \in D$.*

PROOF. In these domains $J_z = J_z^c$ for $z \in D$ see [11, Thm. 4.10, Thm. 4.11, Cor. 4.3]. Then Proposition 3.4 gives the result.

For H a positive definite hermitian $n \times n$-matrix, let $\rho_H(z) = \bar{z}^T Hz$ on $\mathbb{C}^n$ and set $D_H = \{z \in \mathbb{C}^n : \rho_H(z) < 1\}$.

PROPOSITION 3.6. *On D_H we have $J_z^n = J_z^c = J_z$ for all $z \in D_H$.*

PROOF. Set $D = D_H$. Let $z \in D$. Because for $u \in \mathrm{PSH}(D) \cap C(\overline{D})$, $u = u^n$ on $\overline{D}$, we have $J_z^n \subset J_z^c$. Let $\mu \in J_z^c$ and $u \in \mathrm{PSH}(D) \cap \mathrm{USC}(\overline{D})$. Let $0 < r < 1$. Observe that in case of D_H the map $n(\cdot, r)$ is holomorphic and maps $\overline{D}$ into D. Set $u_r(y) = u \circ n(y, r)$, $y \in \overline{D}$. Then u_r is plurisubharmonic in a neighborhood of $\overline{D}$, hence u_r can be approximated monotonically from above by functions in $\mathrm{PSH}(D) \cap C(\overline{D})$. By the monotone convergence theorem $u_r(z) \leq \int u_r \, d\mu$ for all $r \in]0, 1[$. By Fatou's lemma

$$\limsup_{r\to 0} u_r(z) \leq \limsup_{r\to 0} \int_{\overline{D}} u_r(y)\, d\mu.$$

For $y \in D$ one has $\limsup_{r\to 0} u_r(y) = u^n(y)$. Because the set $[0, 1]$ is not thin at 0, see Theorem 2.7.2 in [7], we have

$$u^n(z) = u(z) = \limsup_{r\to 0} u_r(z) \leq \int \limsup_{r\to 0} u_r(y)\, d\mu \leq \int_{\overline{D}} u^n(y)\, d\mu.$$

Thus $\mu \in J_z^n$ it follows that $J_z^c \subset J_z^n$. Hence $J_z^c = J_z^n \subset J_z \subset J_z^c$.

The unit ball, i.e. the case where $H = \mathrm{Id}$, was done in [11]. Our proof is a slight modification of Wikström's.

THEOREM 3.7. *For all $z \in D_H$ one has $\omega(z, A, D_H) = \omega^n(z, A, D_H) = \omega^R(z, A, D_H) = \omega^c(z, A, D_H)$ for all $A \subset \partial D_H$ compact.*

PROOF. By Proposition 3.6 $J^c = J^n = J$ and by Proposition 3.1 and Proposition 3.4 $\omega^c = \omega^n = \omega$. As D_H is strongly star shaped with respect to the origin, $J = J^R$ see Prop. 5.4 in [11] and by Proposition 3.3 above, the equality $\omega = \omega^R$ follows.

4. Circular sets

Our goal in this section is to generalize Theorem 2.11 in [2] and solve Sadullaev's problem for circular sets in circular, strongly star shaped, (hence balanced) domains.

THEOREM 4.1. *Let D be a bounded smooth circular domain that is strongly star shaped with respect to the origin and let $A \subset \partial D$ be a circular compact set. Then*

$$\omega^n(\,\cdot\,, A, D) \le \omega^R(\,\cdot\,, A, D) = \omega^c(\,\cdot\,, A, D).$$

In particular,

$$\omega_1(z, A, D) = \omega_2(z, A, D) = \omega_3(z, A, D).$$

PROOF. Let u be in the family defining $\omega^n(\,\cdot\,, A, D)$. Let ρ be a smooth defining function for D such that for all θ and y in a neighborhood of $\overline{D}$ we have $\rho(y) = \rho(e^{i\theta} y)$. For $0 < t < 1$ consider the function

$$v_t(z, w) = u(n(w, t)z), \qquad (w \in \overline{D},\ z \in \mathbb{C}, |z| \le 1).$$

For fixed t and w, the function $v_t(\,\cdot\,, w)$ is subharmonic on the (closed) unit disc. Observe that $n(w, t)e^{i\theta} = n(e^{i\theta} w, t)$, so that for each $w \in A$ and all θ

$$\limsup_{t \downarrow 0} v_t(e^{i\theta}, w) \le -1.$$

Hence for all $|z| \le 1$, $\limsup_{t\downarrow 0} v_t(z, w) \le -1$. It follows that $u(wz) \le -1$ for $w \in A$ and $|z| \le 1$. We infer that u belongs to the family defining $\omega^R(\,\cdot\,, A, D)$ and the inequality is proved.

Now suppose that u belongs to the family defining $\omega^R(\,\cdot\,, A, D)$. Then $u(wz) \le -1$ for $w \in A$ and $|z| < 1$. Therefore, for $0 < r < 1$ $u_r(w) = u(rw)$ is a plurisubharmonic function in a neighborhood of $\overline{D}$ that is less that -1 on A. Now u_r can be approximated from above on $\overline{D}$ by a decreasing sequence $\{v_j\}$ of continuous PSH-functions. By Dini's theorem, for every $\epsilon > 0$ there is a j_0 so that $v_j \le -1 + \epsilon$ on A hence also on a neighborhood of A. It follows that $u_r \le \omega^c(\,\cdot\,, A, D)$, and then also $u \le \omega^c(\,\cdot\,, A, D)$.

Appendix

We attempted to apply the non-compact version of Edwards' duality theorem stated in [6] to prove equalities for boundary extremal functions. However, we noticed that this version of Edwards' theorem as stated, does not hold. This appendix contains some counterexamples.

Let $D \subset \mathbb{C}^n$ be a bounded set and $F \subset C(D)$ be a convex cone containing constants. $\mathbb{B}(D)$ denotes the set of Borel probability measures with compact support in D. For $z \in D$ set

$$J_z^F(D) = \left\{\mu \in \mathbb{B}(D) : \operatorname{supp}\mu \subset D, u(z) \leq \int_D u\, d\mu \text{ for all } u \in F\right\}.$$

In case D is a bounded domain we make use of the notation in Section 2, and for $z \in \overline{D}$, we set $J_z^c = J_z^c(\overline{D})$ and $J_z = J_z(\overline{D})$. Let $g\colon D \to \mathbb{R}$ and define

$$Sg(z) = \sup\{u(z) : u \in F, u \leq g\}$$

and

$$Ig(z) = \inf\left\{\int_D g\, d\mu : \mu \in J_z^F(D)\right\}.$$

The following theorem is due to Edwards, see [9, Theorem 2.1].

THEOREM 4.2 (Edwards). *Assume that D is compact and g is a bounded Borel function on D, then $Sg(z) \leq Ig(z)$. If g is lower semicontinuous, then $Sg = Ig$.*

Edwards' theorem is very delicate. For instance if the kernel g is merely upper semicontinuous, the theorem may fail, see [9], [6]. We will show that the theorem may also fail if the set D is not compact, contrary to the following theorem, which was formulated in ([6, Thm. 1.3]).

THEOREM 4.3 ([6]). *Let D be a locally compact Hausdorff space countable at infinity. If $g \in C(D)$ then either*

$$Sg(z) = \inf\left\{\int_D g\, d\mu : \mu \in J_z^F(D)\right\}$$

or $Sg \equiv -\infty$.

However, this result does not hold if D is open.

COUNTEREXAMPLE 4.4. For the sake of finding a contradiction, assume that Theorem 4.3 holds for all open sets D' i.e.

$$\sup\{u(z) : u \in F, u \leq g\} = \inf\left\{\int_{D'} g\, d\mu : \mu \in J_z^F(D')\right\}, \tag{1}$$

where $z \in D'$, $g \in C(D')$, $F \subset C(D')$ is a convex cone containing the constants.

Let D be a bounded open ball and $V \subset\subset D$ be an open ball. Define

$$u_{D,V}(z) = \sup\{u(z) : u \in \mathrm{PSH}(D), u \le -\chi_V\}.$$

Let $u \in \mathrm{PSH}(D)^-$ so that the set $\{u = -\infty\}$ is dense in V. For $m > 0$ set $U_m = \{\frac{1}{m}u < -1\} \cap V$, and $F = \mathrm{PSH}(D) \cap C(\overline{D})$. Observe that the function $g_m = -\chi_{U_m}$ is continuous in the open set $D \setminus \partial U_m$ and that F is a convex cone in $C(D \setminus \partial U_m)$ containing the constants. By (1) we obtain for $z \in D \setminus \partial U_m$ the following equality (we take for D' the set $D \setminus \partial U_m$)

$$\inf\left\{\int_{D\setminus\partial U_m} g_m\, d\mu : \mu \in J_z^F(D \setminus \partial U_m)\right\} = \sup\{v : v \in F, v \le g_m\}$$

on $D \setminus \partial U_m$. If $v \in F$ and $v \le g_m$, then $v \le -\chi_V$ because $\overline{U_m} = \overline{V}$ implies $v \le u_{D,V}$, hence

$$\inf\left\{\int_{D\setminus\partial U_m} g_m\, d\mu : \mu \in J_z^F(D \setminus \partial U_m)\right\}$$
$$= \sup\{v : v \in F, v \le g_m \text{ on } D \setminus \partial U_m\} \le u_{D,V}.$$

As $J_z^F(D \setminus \partial U_m) \subset J_z^c$ we have on $D \setminus \partial U_m$

$$\inf\left\{\int_{\overline{D}} g_m\, d\mu : \mu \in J_z^c\right\} \le \inf\left\{\int_{D\setminus\partial U_m} g_m\, d\mu : \mu \in J_z^F(D\setminus\partial U_m)\right\} \le u_{D,V}.$$

Because D is a ball, by [11, Cor. 4.3] $J_z = J_z^c$. It follows that

$$\inf\left\{\int_{\overline{D}} g_m\, d\mu : \mu \in J_z\right\} = \inf\left\{\int_D g_m\, d\mu : \mu \in J_z^c\right\} \le u_{D,V}$$

on $D \setminus \partial U_m$. Now $\frac{1}{m}u$ is plurisubharmonic and $\frac{1}{m}u \le g_m$, hence for all $m > 0$ one has

$$\frac{1}{m}u(z) \le \inf\left\{\int_{\overline{D}} g_m\, d\mu : \mu \in J_z\right\} \le u_{D,V}(z) \quad \text{for } z \in D \setminus \partial U_m.$$

As $D \setminus \overline{V} \subset D \setminus \partial U_m$ we have for all $m > 0$ that

$$\frac{1}{m}u \le u_{D,V} \quad \text{on } D \setminus \overline{V}.$$

This is impossible since

$$0 \equiv \left(\sup_m \frac{1}{m}u\right)^* \le u_{D,V} < 0 \quad \text{on } D \setminus \overline{V}.$$

The conclusion is that equality (1) is false in open sets D'.

Next we prove that the version of Edwards' theorem stated in Theorem 4.2 does not hold for (open) B-regular domains, i.e. connected open sets.

COUNTEREXAMPLE 4.5. Let D be a bounded B-regular domain and $V \subset \partial D$ be relatively open. Then $\overline{V}$ is not b-pluripolar, see Propositions 3.5 and 2.4 in [2]. There exists a countable $L \subset D$ so that $\overline{L} = L \cup \overline{V}$ is compact in $\overline{D}$ cf. [2, Lemma 4.3]. Set $g = -\chi_L$ and $F = \mathrm{PSH}(D) \cap C(\overline{D})$. As L is non-empty and does not have any accumulation point in D, g is lower semicontinuous in D. If Theorem 4.2 would hold for D we would get for $z \in D$

$$\inf\left\{\int_D g\,d\mu : \mu \in J_z^F(D)\right\} = \sup\left\{u(z) : u \in F, u \leq g\right\} \leq \omega(z, V, D),$$

$$\inf\left\{\int_D g\,d\mu : \mu \in J_z^c\right\} \leq \inf\left\{\int_D g\,d\mu : \mu \in J_z^F(D)\right\} \leq \omega(z, V, D),$$

because $J_z^F(D) \subset J_z^c$,

$$\inf\left\{\int_D g\,d\mu : \mu \in J_z\right\} = \inf\left\{\int_D g\,d\mu : \mu \in J_z^c\right\} \leq \omega(z, V, D),$$

because $J_z = J_z^c$, and

$$\sup\left\{u(z), u \in \mathrm{PSH}(D), u \leq g\right\} \leq \inf\left\{\int_D g\,d\mu : \mu \in J_z\right\} \leq \omega(z, V, D).$$

Finally, because L is countable and therefore pluripolar, we would get

$$0 = \left(\sup\left\{u(z) : u \in \mathrm{PSH}(D), u \leq g\right\}\right)^* \leq \omega(z, V, D).$$

This is impossible since V is not b-pluripolar. The conclusion is that Edwards' theorem does not hold in D.

REMARK 4.6. Approximating g by continuous functions one can show that Theorem 4.3 does not hold in B-regular domains.

These counterexamples make it unlikely that a useful non-compact version of Edwards' theorem can be found. We have not been able to pinpoint the problematic points in ([6, Thm. 1.3]).

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